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THE WEAK ENVELOPE OF HOLOMORPHY FOR ALGEBRAS OF HOLOMORPHIC FUNCTIONS

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The object of this paper is to study analytic continuation of algebras of functions holomorphic on complex spaces of dimension greater than 1. Classically this has been done by putting complex structure on the maximal spectrum of the algebra so that the spectrum is a Stein space with respect to the induced algebra of holomorphic functions. Grauert has given non-pathological examples where this is not possible. In the present paper the axioms of a Stein space have been weakened and the weak envelope of holomorphy has been constructed for a certain type of algebra. In particular, if the algebra A separates points and gives local coordinates on a complex space X then the weak envelope of holomorphy for the pair, (X, A) is obtained.

1. Introduction. In this paper a complex space, unless otherwise stated, will mean a normal, connected, reduced complex space. We will let $H(X)$ denote the algebra of functions holomorphic on a complex space X. A complex space E is said to be the envelope of holomorphy of a complex space X if the following conditions are satisfied:

(1) There is a holomorphic mapping $\tau: X \rightarrow E$ such that $\tau(X)$ is open in E.

(2) The map $\tau^*: H(E) \rightarrow H(X)$ is an algebra isomorphism.

(3) The complex space E is a Stein space.

It is known that if X has an envelope of holomorphy, E, and $\tau': X \rightarrow E'$ satisfies (1) and (2) above then there is a biholomorphic mapping $\varphi: E' \rightarrow E$ such that $\tau = \varphi \circ \tau'$. Moreover, the spectrum of $H(X)$, $S(H(X))$, has the structure of a complex space such that it is biholomorphically equivalent to E in a natural way [2].

Grauert has provided an example of a complex manifold X with $H(X)$ containing local coordinates and separating points, but $S(H(X))$ contains a point no neighborhood of which has the structure of an analytic variety [1]. Thus, in order to investigate the envelope of holomorphy problem, it makes sense to either modify the notion of a complex space, as Grauert has suggested [1], or to weaken the constraints on an envelope of holomorphy. In this paper we have taken the latter route. We have weakened the restrictions (1), (2), and (3) above, while preserving the maximality property. Our results, which apply to a class of algebras which includes many algebras which are not of the form $H(X)$, can be described as follows.

Let A be an algebra of functions holomorphic on a complex space X. Let A^* be the quotient field of A and

$S_A = \{(x_1, x_2) \in X \times X | f(x_1) = f(x_2)\ \forall f \in A^*$ such that f is holomorphic at x_1 and $x_2\}$.

DEFINITION 1.1. Let A be an algebra of functions holomorphic on a complex space X. We say that A *weakly identifies* the points x_1 and x_2 in X if

1. Every $f \in A^*$ which is holomorphic at x_1 is also holomorphic at x_2 and vice versa, and $f(x_1) = f(x_2)$ for all such holomorphic quotients and 2. The dimension of S_A at (x_1, x_2) is at least the dimension of X.

The algebra A is said to *weakly separate* x_1 from x_2 if it does not weakly identify x_1 and x_2. Finally, we say that A *weakly separates points* on X if every two points from X are weakly separated by A.

DEFINITION 1.2. Let A be an algebra of functions holomorphic on an n-dimensional complex space X. We say that A *covers a variety* at $p \in X$ if there is an open neighborhood W of p and a map $F: W \to \boldsymbol{C}^m$ onto an n-dimensional subvariety of an open set in $\boldsymbol{C}^m$ such that $F = (f_1, \cdots, f_m)$ with $f_i \in A$ and F is an open mapping onto its image.

We now are able to describe the type of complex space which is our candidate for the weak envelope of holomorphy.

DEFINITION 1.3. Let A be an algebra of functions holomorphic on a complex space X. We say that X is *weakly Stein* with respect to A if

(α) The algebra A weakly separates points on X.

(β) The algebra A covers a variety at every point of X.

(γ) For every $p \in X$ the quotient field A^* contains functions holomorphic at p which give coordinates for X in a neighborhood of p

and

(δ) If there is a complex space X' containing X and an algebra $A' \subsetneqq H(X')$ satisfying (α), (β), and (γ) such that the restriction map $r: A' \to A$ is an isomorphism then $X' = X$.

Let $\mathcal{O}_p$ be the stalk of the structure sheaf on X at p. If $A \subset H(X)$ is an algebra then we define $h_x: A \to \mathcal{O}_x$ as the restriction map. We will prove a theorem which clarifies the meaning of weak separation of points. In the above notation this theorem can be stated as follows:

THEOREM 1. *Suppose* A^* *contains coordinates at* $x_1, x_2 \in X$. *Then* A *weakly separates* x_1 *from* x_2 *if and only if* $h_{x_1} \circ h_{x_2}^{-1}$ *can not be extended to a continuous isomorphism of* $\mathcal{O}_{x_2}$ *onto* $\mathcal{O}_{x_1}$.

We now wish to define a class of algebras for which the weak envelope of holomorphy exists.

DEFINITION 1.4. An algebra A of functions holomorphic on a complex space X is said to be *ample* on X if for every $p \in X$ the following two conditions hold:

(a) The algebra A covers a variety at p.

(b) There is a neighborhood W of p, a holomorphic mapping φ of W onto a complex space V, an algebra $\hat{A} \subset H(V)$ such that $\varphi^*: \hat{A} \to A$ is an algebra isomorphism and $\hat{A}^*$ contains local coordinates at each point of V.

We will prove the following theorem concerning abstract algebras.

THEOREM 2. *Let* A *be an algebra over* $\mathbf{C}$. *Then there is a (not necessarily connected) complex space* $\operatorname{Rep} A$ *called the representation space of* A, *and an algebra* $\hat{A} \subset H(\operatorname{Rep} A)$ *such that each connectivity component of* $\operatorname{Rep} A$ *is weakly Stein with respect to* $\hat{A}$. *Furthermore, there is an algebra homomorphism* $\hat{h}$ *of* A *onto* $\hat{A}$ *such that for any other algebra homomorphism* h' *of* A *onto* A' *an ample algebra on a complex space* X' *there is a unique holomorphic mapping* φ *of* X' *onto an open subset of* $\operatorname{Rep} A$ *such that for every* $f \in A$, $h'(f) = \hat{h}(f) \circ \varphi$. *The mapping* φ *is 1-1 if and only if* A' *weakly separates points on* X'.

As a corollary to the above theorem we obtain the weak envelope of holomorphy for an ample algebra of holomorphic functions:

THEOREM 3. *Let* A *be an ample algebra of functions holomorphic on a complex space* X. *Then there is a complex space* E *and an algebra* $\hat{A} \subset H(E)$ *such that*

(1) *The space* E *is weakly Stein with respect to* $\hat{A}$.

(2) *There is a holomorphic mapping* τ *of* X *onto an open subset of* E *such that* $\tau^*: \hat{A} \to A$ *is an algebra isomorphism.*

(3) *If* τ' *is a holomorphic mapping of* X *onto an open subset of a complex space* X' *and there is an ample algebra* $A' \subset H(X')$ *such that* $\tau'^*: A' \to A$ *is an isomorphism then there is a unique holomorphic mapping* φ *of* X' *onto an open subset of* E *such that for every* $f \in A$,

$(\tau^*)^{-1} \circ \varphi = (\tau'^*)^{-1}(f)$. *The map* φ *is* 1-1 *if and only if* A' *is weakly separating on* X'.

The pair $(E, \hat{A})$ in Theorem 3 is unique up to biholomorphic equivalence and is called the weak envelope of holomorphy for the pair (X, A). We note that Theorem 1 says that τ identifies x_1 and x_2 if and only if A "looks exactly the same at x_1 as it does at x_2". This, coupled with the maximality property contained in Theorem 3, shows that E is the complex space where A "lives".

The theory presented here answers a question posed to me by Royden and generalizes to higher dimensions his Riemann surface representative space [5]. The germs of this theory and some results in the 2-dimensional case can be found in [3]. I wish to thank Professor Royden for numerous helpful conversations.

2. Separation of points. A family of functions $\mathscr{F}$ on a topological space X separates points on X if for $p, q \in X$ there exists $f \in \mathscr{F}$ such that $f(p) \neq f(q)$. If one takes the algebra $H(X)$ of holomorphic functions on a complex space X and considers the induced algebra $\hat{H}$ on $S(H(X))$ then he finds that $\hat{H}$ separates points on $S(H(X))$. In a sense this is one reason that $S(H(X))$ may not have complex structure, as the level sets of $H(X)$ must be collapsed to points. Thus, our first step toward obtaining a weak envelope of holomorphy which is a complex space is to weaken the definition of separation of points.

In all that follows A is an algebra of functions holomorphic on a complex space X and A^* is its quotient field.

LEMMA 2.1. *Let*

$$S_A = \{(x_1, x_2) \in X \times X \mid f(x_1) = f(x_2)\ \forall f \in A^* \textit{ such that } f \textit{ is holomorphic at } x_1 \textit{ and } x_2\}\,.$$

Then S_A *is an analytic subvariety of* $X \times X$.

Proof. For every $f \in A^*$ we have the meromorphic correspondence $X \times X \overset{(f,f)}{-\!\!\!-\!\!\circ} P_1 \times P_1$ which has graph $G_{(f,f)}$ such that

$$\begin{tikzcd} & G_{(f,f)} \arrow[dl, "\widecheck{(f,f)}"'] \arrow[dr, "\widehat{(f,f)}"] & \\ X \times X \arrow[rr, "\circ", "(f,f)"'] & & P_1 \times P_1 \end{tikzcd}$$

is commutative, $\widecheck{(f,f)}$ is proper holomorphic and $\widehat{(f,f)}$ is holomorphic [6]. Let Δ be the diagonal of $P_1 \times P_1$. Then $\widehat{(f,f)}^{-1}[\Delta] = V$ is an analytic subvariety of $G_{(f,f)}$ and, since $\widecheck{(f, f)}$ is proper, $\widecheck{(f, f)}[V]$ is an analytic subvariety of $X \times X$. Let $L(f) = \widecheck{(f, f)}[V]$. Now $S_A = \bigcap_{f \in A^*} L(f)$. Since each $L(f)$ is an analytic subvariety of $X \times X$, S_A is an analytic subvariety of $X \times X$.

Thus we may speak of the dimension of S_A at a given point and to use it in the definition of weak separation of points, Definition 1.1. We attempt to clarify the meaning of weak separation with the following theorem.

THEOREM 1. *Let A be an algebra of functions holomorphic on a complex space X. Let $x_1, x_2 \in X$ and suppose A^* contains coordinates at x_1 and x_2. Let $h_{x_i}: A \rightarrow \mathcal{O}_{x_i}$ be the restriction maps for $i = 1, 2$. Then A weakly separates x_1 from x_2 if and only if $h_{x_1} \circ h_{x_2}^{-1}$ can not be extended to a continuous algebra isomorphism of $\mathcal{O}_{x_2}$ onto $\mathcal{O}_{x_1}$.*

Proof. Suppose $h_{x_1} \circ h_{x_2}^{-1}$ extends to a continuous isomorphism of $\mathcal{O}_{x_2}$ onto $\mathcal{O}_{x_1}$, h. Let $f, g \in h_{x_1}[A]$ and suppose $q = f/g \in \mathcal{O}_{x_1}$. Then $h(q) \in \mathcal{O}_{x_2}$ and $h(q) = h(f)/h(g)$. Furthermore, since $h(1) = 1$ and units are mapped into units, $q(x_1) = q(x_2)$. Suppose $q_1^i, \cdots, q_{n_i}^i$ give coordinates at x_i, where $q_j^i \in A^*$ and $i = 1, 2$. By the above remarks, each q_j^i is holomorphic at both x_1 and x_2. Let $Q = (q_1^1, \cdots, q_{n_1}^1, q_1^2, \cdots, q_{n_2}^2)$ and V_1 (resp. V_2) be a neighborhood of x_1 (resp. x_2) such that $Q_i = Q|_{V_i}: V_i \twoheadrightarrow W_i$ is biholomorphic for $i = 1, 2$.

Now $W_1 \cap W_2$ is a subvariety of an open set in $C^{n_1+n_2}$ which contains $Q_1(x_1) = Q_2(x_2)$. Suppose $\dim_{Q_1(x_1)} W_1 \cap W_2 \leqq \dim X - 1$. Then there is a function f holomorphic in a neighborhood of $Q_1(x_1)$ in $C^{n_1+n_2}$ such that $f \circ Q_1 \equiv 0$ and $f \circ Q_2 \not\equiv 0$. Since f is given by a convergent power series we may write $\underline{f(Q_i)}_{x_i} = \Sigma a_\nu \underline{Q}_{i\,x_i}^\nu$. Now $h(\underline{Q_2}) = \underline{Q_1}$ and h is a continuous isomorphism. Thus $h(\underline{f(Q_2)}_{x_2}) = \underline{f(Q_1)}_{x_1}$. But $\underline{f(Q_1)}_{x_1} = 0$ and h has trivial kernel. Thus $\underline{f(Q_2)}_{x_2} = 0$. This is a contradiction and therefore $\dim_{Q(x_1)} W_1 \cap W_2 = \dim X = n$.

If $\dim_{(x_1, x_2)} S_A \leqq n - 1$ then there must be $q \in A^*$ which is holomorphic at x_1 and x_2 such that if we replace Q above by (Q, q) then $\dim_{Q(x_1)} W_1 \cap W_2 \leqq n - 1$. By the same argument as above we reach a contradiction. Hence $\dim_{(x_1, x_2)} S_A \geqq n$. Since we have already shown that the first criterion for weak identification is satisfied, x_1 and x_2 are weakly identified by A.

Conversely, suppose x_1 and x_2 are weakly identified by A. Then, just as above, we have Q_1 (resp. Q_2) biholomorphic on V_1 (resp. V_2) and, as our spaces are locally irreducible, we may assume that $W_1 = W_2$. Define $\varphi: V_1 \rightarrow V_2$ by $\varphi = Q_2^{-1} \circ Q_1$ and $h: \mathcal{O}_{x_2} \rightarrow \mathcal{O}_{x_1}$ by $h(f) = f \circ \varphi$. Since φ is biholomorphic it follows that h is a continuous algebra isomorphism. It remains to show that h extends $h_{x_1} \circ h_{x_2}^{-1}$.

Let $f \in A$ be given and define f_i by $f_i = f|_{V_i}$. Then $(Q_i, f_i)\colon V_i \twoheadrightarrow W_i'$ is biholomorphic. Since x_1 and x_2 are assumed to be weakly identified by A, we may assume $W_1' = W_2'$. Now W_i' is the graph of $f_i \circ Q_i^{-1}$ on W_i and $W_1 = W_2$. Therefore, $f_1 \circ Q_1^{-1} = f_2 \circ Q_2^{-1}$. Hence $f_2 \circ \varphi = f_1$ or equivalently $h(f_2) = f_1$. Thus h extends $h_{x_1} \circ h_{x_2}^{-1}$.

3. Construction of Rep A. Our main goal is to construct envelopes of holomorphy for algebras of holomorphic functions, but in this section we consider abstract algebras over $\boldsymbol{C}$ and construct their representative spaces.

DEFINITION 3.1. Let ${}_V\mathcal{O}$ be the reduced structure sheaf for the normal complex space V and ${}_V\mathcal{O}_p$ its stalk at $p \in V$. Let A be an algebra over $\boldsymbol{C}$. A *representation* of A on V is an algebra homomorphism $\psi\colon A \to H(V)$. A *local representation* is an algebra homomorphism $\sigma\colon A \to {}_V\mathcal{O}_p$ for some $p \in V$ such that $\sigma = h_p \circ \psi$, where ψ is a representation of A on V.

Let $\sigma\colon A \to {}_V\mathcal{O}_p$ and $\rho\colon A \to {}_W\mathcal{O}_q$ be local representations. We say that σ and ρ are equivalent if there are neighborhoods of p and q respectively, V' and W', in V and W and a biholomorphic map $\varphi\colon V' \to W'$ such that $\varphi(p) = q$ and $\varphi^*(\rho) = \sigma$. It is easy to check that equivalence of local representations is an equivalence relation. If σ is a local representation of A, we will use $[\sigma]$ to denote its equivalence class and if σ and ρ are equivalent local representations then we will denote that by $\sigma \sim \rho$.

The space Rep A will be composed of equivalence classes of local representations of A, each of which satisfies certain conditions.

DEFINITION 3.2. Let $\sigma\colon A \to {}_V\mathcal{O}_p$ be a local representation of A. We say that σ is *primitive* if $\sigma = h_p \circ \psi$ such that

(1) The algebra $\psi[A]$ covers a variety at p

and

(2) There are quotients $q_1, \cdots, q_m \in \psi[A]^*$ which are holomorphic near p on V and give local coordinates at p for V.

It is easy to check that if σ is primitive and $\sigma \sim \rho$ then ρ is also primitive. Thus we define an equivalence class $[\sigma]$ to be primitive if and only if σ is primitive.

DEFINITION 3.3. Let A be an algebra over $\boldsymbol{C}$. We define the representation space of A, Rep A, as the collection of all primitive equivalence classes of local representations of A.

We now proceed toward showing that Rep A can be given, in a natural way, the structure of a reduced, normal, not necessarily connected complex space. First, we cover Rep A with a system of coordinate patches which gives it a topology. Consider $[\sigma] \in \mathrm{Rep}\, A$. Then $\sigma: A \to {}_V\mathcal{O}_p$ factors by

$$\begin{tikzcd} A \arrow[rr, "\psi"] \arrow[dr, "\sigma"'] & & H(V) \arrow[dl, "h_p"] \\ & {}_V\mathcal{O}_p & \end{tikzcd} .$$

By taking V smaller if necessary, we may assume that $[h_x \circ \psi]$ is primitive for every $x \in V$. Thus we have a map $i: V \to \mathrm{Rep}\, A$ defined by $i(x) = [h_x \circ \psi]$. Since $\psi[A]^*$ contains local coordinates at p for V, we may take V small enough for these to be coordinates for all of V. Thus i is 1–1. We give $i[V]$ the topological structure of V. Since V has the complex structure of a normal, reduced complex space and since the change of coordinates on overlapping patches is biholomorphic, Rep A has induced complex structure. In order to show that Rep A has the structure of a normal, reduced, not necessarily connected complex space it is enough to show that our topology is Hausdorff.

LEMMA 3.4. *The topology defined above for* Rep A *is a Hausdorff topology.*

Proof. Let $x_1, x_2 \in \mathrm{Rep}\, A$. We suppose that any two coordinate neighborhoods of x_1 and x_2 have a nonempty intersection. In terms of representations, we have connected subvarieties V_1 and V_2 of open sets in complex Euclidean space with $x_i \in V_i$ and representations ψ_i of A on V_i such that $[h_x \circ \psi]$ is primitive for x in V_1 or V_2. Furthermore, there are sequences $\{x_n^i\} \subset V_i$ converging to x_i such that

$$[h_{x_n^1} \circ \psi_1] = [h_{x_n^2} \circ \psi_2] .$$

In order to show that the topology is Hausdorff, we must show that $[h_{x_1} \circ \psi_1] = [h_{x_2} \circ \psi_2]$.

Let

$$U_1 = \{x \in V_1 \mid \exists y \in V_2 \quad \text{with} \quad [h_x \circ \psi_1] = [h_y \circ \psi_2]\}$$

and

$$U_2 = \{y \in V_2 \mid \exists x \in V_1 \quad \text{with} \quad [h_x \circ \psi_1] = [h_y \circ \psi_2]\} .$$

For every $x \in U_1$ there exists a $y \in U_2$, an open neighborhood U_x (resp. U_y) of x (resp. y) and a biholomorphic map $\varphi_x: U_x \twoheadrightarrow U_y$ such that $\psi_1(f) =$

$\varphi_x^*(\psi_2(f))$ on U_x for every $f \in A$. In particular $U_x \subset U_1$ (resp. $U_y \subset U_2$) and thus U_1 (resp. U_2) is open in V_1 (resp. V_2). Furthermore, we claim that the map $\varphi = \bigcup_{x \in U_1} \varphi_x$ is well-defined on U_1 and a biholomorphic map of U_1 onto U_2.

To show that φ is well-defined, consider $x \in U_{x_1} \cap U_{x_2}$. Suppose that $\varphi_{x_1}(x) = y_1$ and $\varphi_{x_2}(x) = y_2$. We have taken V_2 small enough so that there are quotients $q_1, \cdots, q_n \in \psi_2[A]^*$ which are holomorphic on V_2 and give coordinates there. Thus, if $y_1 \neq y_2$ there is a q_i (say q_1) such that $q_1(y_1) \neq q_1(y_2)$. But $q_1 = \psi_2(f)/\psi_2(g)$, where $\psi_1(f) = \varphi_{x_i}^*(\psi_2(f))$ for $i = 1, 2$ and similarly for g. Thus

$$q_1(y_2) = \frac{\varphi_{x_2}^*(\psi_2(f))}{\varphi_{x_2}^*(\psi_2(g))}(x) = \frac{\psi_1(f)}{\psi_1(g)}(x) = \frac{\varphi_{x_1}^*(\psi_2(f))}{\varphi_{x_1}^*(\psi_2(g))}(x) = q_1(y_1) .$$

Hence $y_1 = y_2$ and φ is well-defined.

To show that φ is a biholomorphic map of U_1 onto U_2, it is enough to show that it is injective. Suppose that $\varphi(a) = \varphi(b) = y$. If $a \neq b$ then there is a quotient $q \in \psi_1[A]^*$ such that $q(a) \neq q(b)$. If $q = \psi_1(f)/\psi_1(g)$ then by a similar argument to the above we find that $q(a) = q(b)$. Thus $a = b$ and φ is 1–1.

To summarize, we have a biholomorphic map $\varphi: U_1 \twoheadrightarrow U_2$ such that for every $f \in A$ $\varphi^*\psi_2(f) = \psi_1(f)$ on U_1. Further, we have assumed that $x_i \in \bar{U}_i$ for $i = 1, 2$. We will show that $x_1 \in U_1$ and $x_2 \in U_2$, thereby proving that $[h_{x_1} \circ \psi_1] = [h_{x_2} \circ \psi_2]$ and Rep A is Hausdorff.

Our first step is to show that U_2 is dense in V_2. By the primitivity assumption, there are functions $f_1, \cdots, f_n \in A$ such that $F_1 = (\psi_1(f_1), \cdots, \psi_1(f_n))$ of V_1 onto a complex subvariety of an open set in C^n, such that $\dim W = \dim V_1$. Let $F_2 = (\psi_2(f_1), \cdots, \psi_2(f_n))$. Now φ is biholomorphic, F_1 is of generic maximal rank and $\varphi^* F_2 = F_1$ on U_1. Thus F_2 is of generic maximal rank on V_2. Furthermore, $F_2^{-1}[W]$ must be a subvariety of V_2 which contains x_2. Let S_i be the singular points of V_i and J_i be the points of $V_i - S_i$ where the Jacobian of F_i vanishes, $i = 1, 2$. Let $U_1' = U_1 - S_1 - J_1$ and $U_2' = U_2 - \varphi(S_1) - \varphi(J_1)$. Observe that $x_2 \in \bar{U}_2'$. Now F_1 is locally 1–1 on U_1' and therefore F_2 is locally 1–1 on U_2'. Furthermore, $U_2' \subset F_2^{-1}[W]$. But U_2' being open in V_2 and $x_2 \in \bar{U}_2'$ implies that $\dim_{x_2} F_2^{-1}[W] = \dim_{x_2} V_2$. Thus, by taking V_2 smaller if necessary, we may assume that $F_2^{-1}[W] = V_2$.

Let $u_2 \in \bar{U}_2$ such that

(1) V_2 is nonsingular at u_2,

(2) The Jacobian of F_2 is nonzero at u_2

and

(3) $u_2 \notin F_2^{-1}[F_1[S_1 \cup J_1]]$.

By (1) and (2) there is a neighborhood Ω of u_2 contained in the non-singular points of V_2 such that $F_2|_\Omega$ is a biholomorphic map. Let $u_1 \in V_1$ be any point such that $F_1(u_1) = F_2(u_2)$. Since $F_1[V_1] = W \supset F_2[V_2]$, we are assured that such points do exist. By (3) and by taking Ω smaller if necessary, $F_1^{-1}F_2$ exists and maps Ω biholomorphically onto an open neighborhood of u_1. Now $\Omega \cap U_2 \neq \emptyset$ and

$$F_1^{-1} F_2|_{\Omega\cap U_2} = \varphi^{-1}|_{\Omega\cap U_2} .$$

By uniqueness of analytic continuation $(F_1^{-1}F_2)^*$ has the same properties as $(\varphi^{-1})^*$. Since U_2 is maximal for φ^{-1}, $\Omega \subseteqq U_2$.

Let B be the set of points in V_2 excluded by conditions (1)-(3) above. We will show that the Hausdorff dimension [4] of B, $H\dim B$, is at most $H\dim V_2 - 2$. Since S_2 and J_2 are analytic varieties of dimension at most $\dim V_2 - 1$, it is easily verified that their Hausdorff dimension is at most $H\dim V_2 - 2$. Thus, in order to show that $H\dim B \leqq H\dim V_2 - 2$, it is enough to show that $F_2^{-1}[F_1[S_1 \cup J_1]] - S_2 - J_2$ has Hausdorff dimension at most $H\dim V_2 - 2$.

Now there is a collection of open subsets of V_2, $\{A_\nu : \nu = 1, 2, \cdots\}$, such that $V_2 - J_2 - S_2 = \bigcup_{\nu=1}^\infty A_\nu$ and $F_2|_{A_\nu}$ is biholomorphic. Let μ_α be the $(H\dim\ V_2 - 2 + \alpha)$-dimensional Hausdorff measure. Hence

$$\mu_\alpha[F_2^{-1}[F_1[S_1 \cup J_1]] - S_2 - J_2] \leqq \sum_{\nu=1}^\infty \mu_\alpha[F_2^{-1}[F_1[S_1 \cup J_1]] \cap A_\nu]$$
$$<< \sum_{\nu=1}^\infty \mu_\alpha[F_1[S_1 \cup J_1] \cap F_2[A_\nu]] \leqq \sum_{\nu=1}^\infty \mu_\alpha[F_1[S_1 \cup J_1]] ,$$

where the symbol $<<$ means that the left hand side is zero if the right hand side is zero. Since $\dim (S_1 \cup J_1) \leqq \dim V_2 - 1$, $F[S_1 \cup J_1]$ is a countable union of complex varieties (almost thin) each of which has dimension at most $\dim V_2 - 1$. If $\alpha > 0$ then, by the countable subadditivity of μ_α, $\mu_\alpha[F_1[S_1 \cup J_1]] = 0$. Therefore, the above inequalities show that, for $\alpha > 0$, $\mu_\alpha[F_2^{-1}[F_1[S_1 \cup J_1]] - S_2 - J_2] = 0$. As a result $H\dim B \leqq H\dim V_2 - 2$.

We have previously shown that $\partial U_2 \subseteqq B$. Thus $H\dim \partial U_2 \leqq H\dim V_2 - 2$. Now, U_2 and $V_2 - \bar{U}_2$ are disjoint open sets separated by a set of Hausdorff codimension at least 2. This is impossible [4] unless one of the two sets is empty. Therefore, $V_2 - \bar{U}_2 = \emptyset$ and U_2 is dense in V_2.

We have shown that V_1 and V_2 can be chosen such that $\varphi\colon U_1 \twoheadrightarrow U_2$ is biholomorphic onto U_2, which is an open dense subset of V_2. Suppose $\psi_1(f)/\psi_1(g) \in \psi_1[A]^*$ is holomorphic on $V_1' \supset\supset V_1$. Now

$$\psi_2(f)/\psi_2(g) \circ \varphi = \psi_1(f)/\psi_1(g)$$

on U_1. Thus $\psi_2(f)/\psi_2(g)$ is bounded on the dense set U_2 and, since

V_2 is normal, is therefore holomorphic at x_2.

We could have performed the same proof for U_1 to show that $\varphi: U_1 \twoheadrightarrow U_2$ is biholomorphic from the open set U_1, which is dense in V_1. Thus, using the analogous argument to the above, $\psi_2(f)/\psi_2(g)$ holomorphic on $V_2' \supset\supset V_2$ implies $\psi_1(f)/\psi_1(g)$ holomorphic at x_1. We thus conclude that $\psi_1(f)/\psi_1(g)$ is holomorphic at x_1 if and only if $\psi_2(f)/\psi_2(g)$ is holomorphic at x_2.

By primitivity, there are functions $f_1^i, \cdots, f_{n_i}^i$ and $g_1^i, \cdots, g_{n_i}^i$ in A such that the quotients $\psi_i(f_1^i)/\psi_i(g_1^i), \cdots, \psi_i(f_{n_i}^i)/\psi_i(g_{n_i}^i)$ are holomorphic on V_i and give coordinates there for $i = 1, 2$. Define

$$G_i: V_i \longrightarrow \boldsymbol{C}^{n_1+n_2}$$

by

$$\begin{aligned} G_i = (\psi_i(f_1^1)/\psi_i(g_1^1), \cdots, \psi_i(f_{n_1}^1)/\Psi_i(g_{n_1}^1)\ , \\ \psi_i(f_1^2)/\psi_i(g_1^2), \cdots, \psi_i(f_{n_2}^2)/\psi_i(g_{n_2}^2)) \end{aligned}$$

for $i = 1, 2$. Then G_i is biholomorphic on V_i and $G_2 \circ \varphi = G_1$ on U_1.

Now $G_i[V_i]$ is an irreducible normal subvariety containing $G_1(x_1) = G_2(x_2)$. Furthermore,

$$G_1[V_1] \cap G_2[V_2] \supseteqq G_1[U_1] = G_2[U_2]$$

and $G_i(x_i) \in \overline{G_i[U_i]}$ for $i = 1, 2$. Therefore,

$$\dim_{G_1(x_1)} G_1[V_1] = \dim_{G_1(x_1)} G_1[V_1] \cap G_2[V_2] = \dim_{G_2(x_2)} G_2[V_2]\ .$$

Hence, by choosing V_1 and V_2 appropriately, we may assume that $G_1[V_1] = G_2[V_2]$. Thus $G_2^{-1} \circ G_1$ is a biholomorphic extension of φ to a map from V_1 to V_2. Furthermore, by the unicity of analytic continuation, this has the property $(G_2^{-1} \circ G_1)^* \psi_2(f) = \psi_1(f)$ for every $f \in A$. Thus $U_1 = V_1$, if V_1 is chosen small enough, and $[h_{x_1} \circ \psi_1] = [h_{x_2} \circ \psi_2]$.

COROLLARY 3.5. *Endowed with the topology described above,* Rep A *is a reduced, normal, not necessarily connected complex space.*

4. The main theorems.

In order to prove Theorem 2 and Theorem 3, we need several preliminary lemmas.

LEMMA 4.1. *Let V be a complex space and $A \subseteqq H(V)$ an algebra which has generic maximal rank on V. Suppose there are holomorphic maps $\varphi_i: V \longrightarrow V_i$ onto the complex spaces V_i and algebras $A_i \subseteqq H(V_i)$ such that $\varphi_i^*: A_i \longrightarrow A$ is an algebra isomorphism. Suppose further that for some $p \in V$ $[h_{\varphi_1(p)} \circ (\varphi_1^*)^{-1}]$ and $[h_{\varphi_2(p)} \circ (\varphi_2^*)^{-1}]$ are primitive. Then $[h_{\varphi_1(p)} \circ (\varphi_1^*)^{-1}] = [h_{\varphi_2(p)} \circ (\varphi_2^*)^{-1}]$.*

Proof. The maps φ_i have generic maximal rank on V. Let S be the singular points of V and J_i be the subset of $V - S$ where the Jacobian of φ_i vanishes. Clearly $U = V - S - J_1 - J_2$ is an open dense subset of V. The map $\varphi_2 \circ \varphi_1^{-1}$ is locally defined and locally biholomorphic on $\varphi_1[U]$. Now, the isomorphism of A and A^i yields the existence of a unique $f^i \in A^i$ such that $f = f^i \circ \varphi_i$ for every $f \in A$. Thus, locally on $\varphi_1[U]$, $f^1 = f^2 \circ (\varphi_2 \circ \varphi_1^{-1})$. Therefore, since U is dense, we can obtain a sequence $\{x_n\}$ in U which converges to p such that $\varphi_i(x_n) \to \varphi_i(p)$ and $[h_{\varphi_1(x_n)} \circ (\varphi_1^*)^{-1}] = [h_{\varphi_2(x_n)} \circ (\varphi_2^*)^{-1}]$. For n sufficiently large, these equivalence classes of local representations must be primitive. But Rep A is Hausdorff. Thus $[h_{\varphi_1(p)} \circ (\varphi_1^*)^{-1}] = [h_{\varphi_2(p)} \circ (\varphi_2^*)^{-1}]$.

Let A be an algebra over C and Rep A its representative space. For $[\sigma] \in \operatorname{Rep} A$, define $\hat{f}([\sigma]) = \sigma(f)(p)$, where $\psi: A \to H(V)$ and $\sigma = h_p \circ \psi$. Clearly the definition of $\hat{f}$ is independent of the particular representative of the equivalence class. If V is taken sufficiently small it is biholomorphically equivalent to a coordinate neighborhood of $[\sigma]$ in Rep A and $\hat{f}$ is defined by $\psi(f)$ on V. Thus $\hat{f}$ is holomorphic on Rep A. We define the algebra homomorphism $\hat{h}: A \to \hat{A}$ by $\hat{h}(f) = \hat{f}$.

LEMMA 4.2. *The algebra $\hat{A}$ weakly separates points on* Rep A *and $\hat{A}^*$ gives local coordinates at each $p \in$* Rep A.

Proof. It follows immediately from the definition of primitivity that $\hat{A}^*$ gives local coordinates at each $p \in \operatorname{Rep} A$.

Let $x_1, x_2 \in \operatorname{Rep} A$. Suppose that every $\hat{q} \in \hat{A}^*$ which is holomorphic at x_1 is holomorphic at x_2 and vice versa. Also assume that for such holomorphic quotients, $\hat{q}(x_1) = \hat{q}(x_2)$. Under these hypotheses, we will show that $\dim_{(x_1, x_2)} S_{\hat{A}}^{\hat{}} \geqq \dim X$ implies that $x_1 = x_2$.

Let V_1 and V_2 be coordinate neighborhoods of x_1 and x_2 respectively. Since x_i is a primitive representation of A, there are quotients $\hat{q}_1^i, \cdots, \hat{q}_{n_i}^i$ holomorphic on V_i such that $Q_i = (\hat{q}_1^i, \cdots, \hat{q}_{n_i}^i)$ is biholomorphic on V_i. Now, by our assumption, Q_1 is holomorphic on V_2 and Q_2 is holomorphic on V_1. Therefore, (Q_1, Q_2) is biholomorphic on both V_1 and V_2 such that $(Q_1, Q_2)(x_1) = (Q_1, Q_2)(x_2)$. Let $G_i = (Q_1, Q_2)|_{V_i}$. Since $\dim_{(x_1, x_2)} S_{\hat{A}}^{\hat{}} \geqq \dim X$, we may assume, by taking V_i smaller if necessary, that $G_1[V_1] = G_2[V_2]$. Define $\varphi: V_1 \to V_2$ by $\varphi = G_2^{-1} \circ G_1$, for every $f \in A$, let $\hat{f}_i = \hat{f}|_{V_i}$. For the same reasons as above,

$$(G_1, \hat{f}_1)[V_1] = (G_2, \hat{f}_2)[V_2] .$$

Thus $\hat{f}_2 \circ \varphi = \hat{f}_1$ for every $f \in A$. Therefore, since $\varphi(x_1) = x_2$, $x_1 = x_2$.

Recall that an algebra A of functions holomorphic on a complex space X is ample on X if for every $p \in X$ A covers a variety at p and there is a neighborhood W of p, a holomorphic mapping φ of W onto a complex space V and an algebra $\hat{A} \subset H(V)$ such that $\varphi^*: \hat{A} \to A$ is an algebra isomorphism and $\hat{A}^*$ contains local coordinates at each point of V.

LEMMA 4.3. *Let A be an ample algebra of functions holomorphic on a complex space X. Then there is a unique holomorphic map $\tau: X \to \operatorname{Rep} A$ such that $\tau^*: \hat{A} \to A$ is an algebra isomorphism with $\tau^{*-1} = \hat{h}$. The set $\tau[X]$ is an open subset of* $\operatorname{Rep} A$ *and the map τ is* 1–1 *if and only if A weakly separates points on X.*

Proof. Since A is ample on X, we have for each $p \in X$ the map φ in the definition of ampleness. We define τ by $\tau(p) = [h_{\varphi(p)} \circ (\varphi^*)^{-1}]$. The map τ is clearly holomorphic and $\tau^{*-1}(f) = \hat{f}$. By Lemma 4.1, τ is unique. The φ guaranteed by the ampleness of A maps onto a complex space W. Thus a biholomorphic copy of W is contained in $\tau[W]$ with $\tau(p)$ corresponding to $\varphi(p)$. Therefore, $\tau[X]$ is open. It remains to show that τ is 1–1 if and only if A weakly separates points on X.

Let $x_1, x_2 \in X$ and suppose $\tau(x_1) = \tau(x_2)$. Thus there are neighborhoods V_i of x_i and a biholomorphic map $\varphi: V_1 \twoheadrightarrow V_2$ such that $f_1 = f_2 \circ \varphi$, where $f_i = f|_{V_i}$, for every $f \in A$. In particular, both conditions for the weak identification of x_1 and x_2 by A are satisfied.

Conversely, suppose that x_1 and x_2 are weakly identified by A. Let V_i be a neighborhood of x_i so that on $W_i = \tau[V_i]$ the map $\hat{Q}_i = (\hat{q}^i_1, \cdots, \hat{q}^i_{n_i})$ is biholomorphic, where $\hat{q}^i_j \in \hat{A}^*$ and $i = 1, 2$. Since A weakly identifies x_1 and x_2, q^i_j is holomorphic at both x_1 and x_2 and $q^i_j(x_1) = q^i_j(x_2)$ for every i and j. Let $\hat{G}_i = (\hat{Q}_1, \hat{Q}_2)|_{W_i}$. Clearly $\hat{G}_i$ is biholomorphic on W_i. Now $G_i = \hat{G}_i \circ \tau$ maps V_i onto $\hat{G}_i[W_i]$ and, since $\dim_{(x_1, x_2)} S_A \geqq \dim X$, we can choose V_i appropriately so that $G_1[V_1] = G_2[V_2]$. Thus $\hat{G}_1[W_1] = \hat{G}_2[W_2]$ and we have a biholomorphic map φ: $W_1 \twoheadrightarrow W_2$ defined by $\varphi = \hat{G}_2^{-1} \circ \hat{G}_1$. For exactly the same reasons as in the proof of Lemma 4.2, $\hat{f}_1 = \hat{f}_2 \circ \varphi$, where $\hat{f}_i = \hat{f}|_{W_i}$, for every $\hat{f} \in \hat{A}$. Since $\varphi(\tau(x_1)) = \tau(x_2)$, $\tau(x_1) = \tau(x_2)$.

We now proceed with the main theorems.

THEOREM 2. *Let A be an algebra over* $\boldsymbol{C}$. *Then there is a* (*not necessarily connected*) *complex space* $\operatorname{Rep} A$, *and an algebra* $\hat{A} \subseteqq H(\operatorname{Rep} A)$ *such that each component of* $\operatorname{Rep} A$ *is weakly Stein with respect to $\hat{A}$. Furthermore, there is an algebra homomorphism $\hat{h}$ of A onto $\hat{A}$ such that for any other algebra homomorphism h' of A onto A', where A' is an ample algebra on a complex space X', there is a unique holo-*

morphic map φ of X' onto an open subset of Rep A *such that for every $f \in A$, $h'(f) = \hat{h}(f) \circ \varphi$. The map φ is* 1–1 *if and only if A' weakly separates points on X'.*

Proof. By the previous lemmas, in order to prove each component of Rep A is weakly Stein with respect to $\hat{A}$, it is enough to verify the maximality condition, (δ) in Definition 1.3. Thus we let X be a component of Rep A and suppose X is an open subset of a complex space X'. Suppose A' is an algebra of functions holomorphic on X' such that the restriction map $r: A' \to A$ is an algebra isomorphism. Finally we assume that A' and X' satisfy conditions (α), (β), and (γ) in the definition of weakly Stein. Since r is an isomorphism, Rep A = Rep A'. Now A' weakly separates points on X'. Thus Lemma 4.3 gives us a 1–1 holomorphic mapping $\tau: X' \to$ Rep A. Since τ is unique it must agree with the identity on X. But $\tau[X']$ is connected and therefore $\tau[X'] \subset X$. Hence $\tau[X'] = X$ and $X' = X$.

The algebra homomorphism $\hat{h}$ has already been constructed. Now $h': A \to A'$ is a homomorphism onto an ample algebra on X'. Thus, for every primitive local representation $[\sigma']$ of A', we obtain a primitive local representation $[\sigma' \circ h']$ of A. In this way we obtain a biholomorphic injection i: Rep $A' \to$ Rep A onto an open subset of Rep A. Now Lemma 4.3 gives us the map τ': $X' \to$ Rep A' such that $\tau(x_1') = \tau(x_2')$ if and only if x_1' and x_2' are weakly identified by A'. Let $\varphi = i \circ \tau$. The uniqueness of φ follows from Lemma 4.1. Clearly φ is 1–1 if and only if A' weakly separates points on X'. It remains to check that $h'(f) = \hat{h}(f) \circ \varphi$ for every $f \in A$, but this follows by a simple diagram chase.

THEOREM 3. *Let A be an ample algebra of functions holomorphic on a complex space X. Then there is a complex space E and an algebra $\hat{A} \subset H(E)$ such that*

(1) *The space E is weakly Stein with respect to $\hat{A}$.*

(2) *There is a holomorphic mapping τ of X onto an open subset of E such that $\tau^*: \hat{A} \to A$ is an algebra isomorphism.*

(3) *If τ' is a holomorphic mapping of X onto an open subset of a complex space X' and there is an ample algebra $A' \subset H(X')$ such that $\tau'^*: A' \to A$ is an isomorphism then there is a unique holomorphic mapping φ of X' onto an open subset of E such that for every $f \in A$, $(\tau^*)^{-1}(f) \circ \varphi = (\tau'^*)^{-1}(f)$. The map φ is* 1–1 *if and only if A' is weakly separating on X'.*

Proof. The map τ is given by Theorem 2 by letting $X = X'$ and E is taken to be the connected component of Rep A which contains $\tau[X]$. Then E is weakly Stein with respect to $\hat{A}$. Now $\tau^{*-1} = \hat{h}$

must be an isomorphism, because $\tau[X]$ is an open subset of E.

The map needed for (3) is also given by Theorem 2 and the unicity of τ implies that $\tau = \varphi \circ \tau'$ on X. Thus $\varphi[X'] \subset E$ and $(\tau^*)^{-1}(f) \circ \varphi = (\tau'^*)^{-1}(f)$.

5. Concluding remarks. It should be observed that the entire question of the existence of $\tau: X \to \operatorname{Rep} A$ has been avoided by building it into the definition of ampleness. This is a drawback of our theory, but the study of existence seems to be quite complicated. At this point in time we strongly believe that the existence question should be asked in the category of meromorphic maps instead of holomorphic maps.

Even though the existence question has been avoided here, our theory applies in many classical cases. For example suppose $A = H(X)$, where X is a reduced, irreducible normal complex space. Further, assume that $H(X)$ separates points and gives local coordinates. Then our theory applies and we obtain the weak envelope of holomorphy for the pair $(X, H(X))$. In particular, this applies to the type of space X and algebra $H(X)$ constructed by Grauert [1], where $H(X)$ is *not* a Stein algebra.

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