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CONTINUATION OF HOLOMORPHIC MAPPINGS, WITH VALUES IN A COMPLEX LIE GROUP

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In this paper it is shown that any holomorphic mapping of a domain X over a Stein manifold in a complex Lie group L can be continued to a holomorphic mapping of the envelope of holomorphy of X in L.

1. Let M be a complex manifold. A pair (X, ψ) of a complex manifold X and a locally biholomorphic mapping ψ of X in M is called an open set over M. Moreover, if X is connected, (X, ψ) is called a domain over M. Let (X, ψ) and (X', ψ') be open sets over M. A holomorphic mapping λ of X in X' with $\psi = \psi' \circ \lambda$ is called a mapping of (X, ψ) in (X', ψ'). Consider domains (X, ψ) and (X', ψ') over M with a mapping λ of (X, ψ) in (X', ψ') such that each connected component of X' contains that of $\lambda(X)$. Let Y be a complex manifold. Let f be a holomorphic mapping of X in Y. A holomorphic mapping f' of X' in Y with $f = f' \circ \lambda$ is called a holomorphic continuation of f to (λ, X', ψ'). Let $\mathscr{F}$ be a family of holomorphic mappings of X in Y. If any holomorphic mapping f of $\mathscr{F}$ has a holomorphic continuation to (λ, X', ψ'), (λ, X', ψ') is called a holomorphic completion of (X, ψ) with respect to the family $\mathscr{F}$. A holomorphic completion $(\tilde{\lambda}_{\mathscr{F}}, \tilde{X}_{\mathscr{F}}, \tilde{\psi}_{\mathscr{F}})$ of (X, ψ) with respet to $\mathscr{F}$ is called an envelope of holomorphy of (X, ψ) with respect to the family $\mathscr{F}$ if the following conditions are satisfied:

Let (λ', X', ψ') be another holomorphic completion of (X, ψ) with respect to $\mathscr{F}$. There is a mapping φ of (X', ψ') in $(\tilde{X}_{\mathscr{F}}, \tilde{\psi}_{\mathscr{F}})$ with $\tilde{\lambda}_{\mathscr{F}} = \varphi \circ \lambda'$ such that $(\varphi, \tilde{X}_{\mathscr{F}}, \tilde{\psi}_{\mathscr{F}})$ is a holomorphic completion of (X', ψ') with respect to the family $\mathscr{F}'$ of holomorphic continuations to (λ', X', ψ') of all holomorphic mappings of $\mathscr{F}$.

If $\mathscr{F}$ is the family of all holomorphic functions in X, the envelope of holomorphy of (X, ψ) with respect to $\mathscr{F}$ is called the envelope of holomorphy of (X, ψ). If $\mathscr{F}$ consists of only one holomorphic mapping f of X in Y, the envelope of holomorphy of (X, ψ) with respect to the family $\mathscr{F}$ is called the open set of holomorphy of f. Malgrange [5] proved the unique existence of an envelope of holomorphy, considering a connected component of the sheaf of all germs of families of holomorphic mappings with the same index set. By this construction of an envelope of holomorphy we have the following lemma.

LEMMA 1. *Let (X, ψ) be a domain over a complex manifold, Y be a complex manifold and $\mathscr{F}$ be a family of holomorphic mappings of X in Y. There is uniquely an envelope $(\tilde{\lambda}_{\mathscr{F}}, \tilde{X}_{\mathscr{F}}, \tilde{\psi}_{\mathscr{F}})$ of holomorphy of (X, ψ) with respect to $\mathscr{F}$. Moreover, let $\mathscr{G}$ be a subfamily of $\mathscr{F}$ and $(\tilde{\lambda}_{\mathscr{G}}, \tilde{X}_{\mathscr{G}}, \tilde{\psi}_{\mathscr{G}})$ be the envelope of holomorphy of (X, ψ) with respect to $\mathscr{G}$. There is a mapping φ of $(\tilde{X}_{\mathscr{F}}, \tilde{\psi}_{\mathscr{F}})$ in $(\tilde{X}_{\mathscr{G}}, \tilde{\psi}_{\mathscr{G}})$ such that $(\varphi, \tilde{X}_{\mathscr{G}}, \tilde{\psi}_{\mathscr{G}})$ is the holomorphic completion of $(\tilde{X}_{\mathscr{F}}, \tilde{\psi}_{\mathscr{F}})$ with respect to $\mathscr{G}$.*

2. We put

$$\begin{aligned} D = {} & \{z = (z_1, z_2, \cdots, z_n) \in C^n;\ |z_1| < 1 + \varepsilon,\ |z_j| < 1\ (j = 2, 3, \cdots, n)\} \\ (1) \qquad & \cup \{z = (z_1, z_2, \cdots, z_n) \in C^n;\ 1 - \varepsilon < |z_1| < 1 + \varepsilon,\ |z_j| < 1 + \varepsilon \\ & (j = 2, 3, \cdots, n)\} \end{aligned}$$

and

$$(2) \qquad E = \{z = (z_1, z_2, \cdots, z_n) \in C^n;\ |z_j| < 1 + \varepsilon\ (j = 1, 2, \cdots, n)\}$$

for a positive number ε with $\varepsilon < 1$. E is the envelope of holomorphy of D.

LEMMA 2. *Let L be a complex Lie group. For any holomorphic mapping f of D in L, there is a holomorphic mapping g of E in L such that $g = f$ in D.*

Proof. We may assume that L is connected. Let $\mathscr{H}$ be the set of all holomorphic mappings of D in L. We introduce in $\mathscr{H}$ the compact-open topology. As D is analytically contractible to a point, $\mathscr{H}$ is a connected topological group. We put

$$\begin{aligned} K(\delta) = {} & \{z \in C^n;\ |z_1| \leqq 1 + \varepsilon - \delta,\ |z_j| \leqq 1 - \delta\ (j = 2, 3, \cdots, n)\} \\ & \cup \{z \in C^n;\ 1 - \varepsilon + \delta \leqq |z_1| \leqq 1 + \varepsilon - \delta,\ |z_j| \leqq 1 + \varepsilon - \delta \\ & (j = 2, 3, \cdots, n)\} \end{aligned}$$

and

$$E(\delta) = \{z \in C^n;\ |z_j| < 1 + \varepsilon - \delta\ (j = 1, 2, \cdots, n)\}$$

for any positive number δ with $\delta < \varepsilon$. Let m be the complex dimension of L and exp be the exponential mapping of C^m in L. exp maps an open neighborhood $U = \{w \in C^m;\ |w_j| < a\ (j = 1, 2, \cdots, m)\}$ of the origin in C^m biholomorphically on an open neighborhood W of the unit element e of L. Then $\log = (\exp | U)^{-1}$ is a biholomorphic mapping of W onto U. We put $\mathscr{V}(1) = \{h \in \mathscr{H};\ h(K(\delta)) \subset W\}$. Then $\mathscr{V}(1)$ is a neighborhood of the unit element 1 of the topological group $\mathscr{H}$. Since $\mathscr{H}$ is connected, $\mathscr{H}$ is generated by $\mathscr{V}(1)$. There is a finite number s of elements $f_1, f_2, \cdots, f_{s-1}$ and f_s of $\mathscr{V}(1)$ such that $f = f_1 f_2 \cdots f_s$ in D. Each $\log f_i$ is a holomorphic mapping of $K(\delta)$ in the polydisc U. There is a holomorphic mapping G_i of $E(\delta)$ in U such that $G_i = \log f_i$ in $K(\delta) \cap E(\delta)$ for $j = 1, 2, \cdots, s$. We put $g = \exp G_1 \exp G_2 \cdots \exp G_s$ in $E(\delta)$. Then g is a holomorphic mapping of $E(\delta)$ in L such that $g = f$ in $K(\delta) \cap E(\delta)$. Since δ is arbitrary, we have Lemma 2 by the theorem of identity.

Let (X, ψ) be a domain over a Stein manifold S, L be a complex Lie group, $\mathscr{F} = \{f_i; i \in I\}$ be a family of holomorphic mappings of X in L and (λ, X', ψ') be the envelope of holomorphy of (X, ψ) with respect to $\mathscr{F}$.

LEMMA 3. *(X', ψ') is p_7-convex in the sense of Docquier-Grauert* [2].

Proof. Assume that (X', ψ') were not p_7-convex. There is a continuous mapping φ of the closure $\bar{D}$ of D, defined in (1) for a positive number ε with $\varepsilon < 1$, in $X' \cup \tilde{\partial} X'$ such that $\varphi(D) \subset X'$, $\varphi(b, c) \in \tilde{\partial} X'$ for $b \in C$ with $|b| \leqq 1 - \varepsilon$, $c = (c_2, \cdots, c_n) \in C^{n-1}$ with $|c_j| = 1$ $(j = 2, 3, \cdots, n)$ and $\psi' \circ \varphi$ can be continued to a biholomorphic mapping π of E, given in (2) for the above ε, in S. Here $\tilde{\partial} X'$ is the boundary of the domain (X', ψ') over S defined in § 3.1d of [2].

Let f_i' be a holomorphic continuation of $f_i \in \mathscr{F}$ to (λ, X', ψ') for any $i \in I$. By Lemma 2 there is a holomorphic mapping g_i of E in L such that $g_i = f_i' \circ \varphi$ in D for any $i \in I$.

We consider the sum space $X' \cup E$. Let x_1 and x_2 be, respectively, points of X' and E such that $\psi'(x_1) = \pi(x_2)$. Let U_1 and U_2 be, respectively, neighborhoods of x_1 and x_2 in X' and E such that $\psi' | U_1$ and $\pi | U_2$ are, respectively, biholomorphic mappings of U_1 and U_2 onto an open neighborhood V in S. We shall identify x_1 and x_2 if and only if $f_i' \circ (\psi' | U_1)^{-1} = g_i \circ (\pi | U_2)^{-1}$ for any $i \in I$. Let Z be the quotient space of $X' \cup E$ by this equivalence relation. Let ξ: $X' \to Z$ and η: $E \to Z$ be the canonical mappings. There is a locally topological mapping ζ of Z in S such that $\zeta \circ \xi = \psi'$ and $\zeta \circ \eta = \pi$. Since Z is a Hausdorff space, we can introduce in Z a complex structure such that (Z, ζ) is a domain over S.

There is a holomorphic mapping h_i of Z in L such that $h_i \circ \xi = f_i'$ and $h_i \circ \eta = g_i$ for any $i \in I$. Then h_i is a continuation of f_i' to (ξ, Z, ζ). Since (λ, X', ψ') is the envelope of holomorphy of X with respect to $\mathscr{F}$, there is a mapping μ of (Z, ζ) in (X', ψ'). Since $\zeta \circ \xi = \psi'$, $\xi \circ \mu$ and $\mu \circ \xi$ are, respectively, the identities of Z and X'. Hence ξ is a biholomorphic mapping of X' on Z. Since $(b, c) \in E$, we have $\varphi(b, c) = (\xi^{-1} \circ \eta)(b, c) \in X'$. This is a contradiction.

3. Let (X, ψ) be a domain over a Stein manifold S, $(\tilde{\lambda}, \tilde{X}, \tilde{\psi})$ be its envelope of holomorphy, L be a complex Lie group and f be a holomorphic mapping of X in L. Let (λ, X', ψ') be the open set of holomorphy of f. By Lemma 3 and a theorem of Docquier-Grauert [2], X' is a Stein manifold. Hence (X', ψ') is a domain of holomorphy of a holomorphic function in X. Since $(\tilde{\lambda}, \tilde{X}, \tilde{\psi})$ is the envelope of holomorphy with respect to the family of all holomorphic functions in X, there is a mapping μ of $(\tilde{X}, \tilde{\psi})$ in (X', ψ') such that $\lambda = \mu \circ \tilde{\lambda}$ by Lemma 1. Let f' be the holomorphic continuation of f to (λ, X', ψ'). Then $f' \circ \mu$ is the holomorphic continuation of f to $(\tilde{\lambda}, \tilde{X}, \tilde{\psi})$. Thus we have proved the following theorem.

THEOREM. *Let (X, ψ) be a domain over a Stein manifold S and $(\tilde{\lambda}, \tilde{X}, \tilde{\psi})$ be its envelope of holomorphy. Any holomorphic mapping of X in a complex Lie group L has a holomorphic continuation to $(\tilde{\lambda}, \tilde{X}, \tilde{\psi})$.*

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