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PRIMITIVE GROUP RINGS

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Let $K[G]$ denote the group ring of G over the field K. Until recently it had been an open question as to whether $K[G]$ could be primitive, that is have a faithful irreducible module, if $G \neq \langle 1 \rangle$. An affirmative answer has just been given in the important paper of E. Formanek and R. L. Snider where a large number of examples of primitive group rings were exhibited. In this paper we continue this study.

LEMMA 1. *Let F be an algebraic field extension of K and let A be a torsion free abelian group. If I is a nonzero ideal in $F[A]$ then $I \cap K[A] \neq 0$.*

Proof. Clearly $I \cap F'[A] \neq 0$ where F' is some finitely generated and hence finite field extension of K. Therefore, it suffices to assume that $F = F'$ or equivalently that F/K is finite. Now $F[A]$ and $K[A]$ are both integral domains since A is torsion free abelian and $F[A]$ is a finitely generated free $K[A]$-module. It then follows that every nonzero element of $F[A]$ satisfies an integral polynomial over $K[A]$ with nonzero constant term. Thus if $\alpha \in I, \alpha \neq 0$ then

$$\alpha^n + \beta_{n-1}\alpha^{n-1} + \cdots + \beta_1\alpha + \beta_0 = 0$$

for some $\beta_i \in K[A]$, $\beta_0 \neq 0$. Since $\alpha \in I$ implies immediately that $\beta_0 \in I \cap K[A]$, the result follows.

THEOREM 2. *Suppose $K[G]$ is primitive and let F be a field extension of K. Suppose that either F/K is algebraic or $\Delta(G) = \langle 1 \rangle$. Then $F[G]$ is primitive.*

Proof. Let $V = K[G]/M$ be the faithful irreducible $K[G]$-module with M a maximal right ideal. Since $F[G]$ is free over $K[G]$ we have $MF[G[\neq F[G]$. Thus we can choose a maximal right ideal N of $F[G]$ with $N \supseteqq MF[G]$. Clearly $N \cap K[G] = M$ and thus if W is the irreducible $F[G]$-module $W = F[G]/N$, then $W_{K[G]} \supseteqq V$ where $W_{K[G]}$ is of course W viewed as a $K[G]$-module. Therefore, $K[G]$ acts faithfully on W. Let I be the primitive ideal of $F[G]$ corresponding to W. Then I is also a prime ideal and $I \cap K[G] = 0$.

We assume that $I \neq 0$ and derive a contradiction. By a result of Martha Smith (Corollary 7.6 of [4]), we must have $I \cap C \neq 0$ where C is the center of $F[G]$. Now $C \subseteqq F[\Delta(G)]$ so $I \cap F[\Delta(G)] \neq 0$. This shows that $\Delta(G) \neq \langle 1 \rangle$ and hence by assumption F/K is algebraic.

Now $K[G]$ must be prime so by Connell's theorem $\Delta(G)$ is torsion free abelian. Thus by Lemma 1, $(I \cap F[\Delta(G)]) \cap K[\Delta(G)] \neq 0$, a contradiction. Therefore, $I = 0$, W is a faithful irreducible $F[G]$-module and $F[G]$ is primitive.

The next result shows that the assumption $\Delta(G) = \langle 1 \rangle$ is really necessary in the above.

THEOREM 3. *Suppose $K[G]$ is primitive and the cardinality of $|K|$ is larger than the cardinality of $|G|$. Then $\Delta(G) = \langle 1 \rangle$.*

Proof. By Theorem 1 we can clearly assume that K is algebraically closed. Let D be the commuting ring for the faithful irreducible module so that D is a division algebra over K. By the density theorem, D is a homomorphic image of a subalgebra of $K[G]$ so

$$\dim_K D \leqq \dim_K K[G] = \text{card}\,|G| < \text{card}\,|K|\;.$$

We show that $D = K$. Thus let $d \in D - K$ and consider the elements $\{(d - a)^{-1} \mid a \in K\}$. This set has cardinality larger than the dimension of D so we have a linear dependence

$$\sum_1^n b_i(d - a_i)^{-1} = 0$$

with $b_i, a_i \in K$, the a_i distinct and $b_i \neq 0$. Note that all such terms commute so multiplying by $\prod_1^n (d - a_i)$ we obtain a nontrivial polynomial over K satisfied by d. But K is algebraically closed so $d \in K$, a contradiction. Hence $D = K$. Let $x \in \Delta(G)$ and let α be the class sum in $K[G]$ of the conjugacy class of x. Then $\alpha \in D = K$ so clearly $x = 1$ and $\Delta(G) = \langle 1 \rangle$.

We now work towards an extension of the Formanek-Snider theorem on locally finite groups. The following is well known.

LEMMA 4. *Let G be a finite group and let V be an irreducible $K[G]$-module. Then $K[G]$ has a minimal right ideal $I \cong V$.*

Proof. If $\alpha = \Sigma a_x x \in K[G]$, we let $\operatorname{tr} \alpha = a_1$ be the coefficient of 1. Then we know that $\operatorname{tr} \alpha\beta$ is a bilinear form on $K[G]$. Let M be a maximal right ideal with $K[G]/M \cong V$. Since $K[G]$ is a finite dimensional vector space there exists $\alpha \in K[G]$, $\alpha \neq 0$ with $\operatorname{tr}(\alpha M) = 0$. But αM is a right ideal so this implies that $\alpha M = 0$. Set $I = \alpha K[G]$ and consider the map $K[G] \to I$ given by $\beta \to \alpha\beta$. This is a $K[G]$-homomorphism onto $I \neq 0$ and M is in the kernel. Since M is maximal we have $V \cong K[G]/M \cong I$.

LEMMA 5. *Let K be a field and let K_0 be its prime subfield. Let G be a locally finite group. Then $JK[G] = 0$ if and only if $JK_0[G] = 0$.*

Proof. Suppose $JK_0[G] = 0$. Then Theorem 18.2 of [4] implies easily that $JK[G] = 0$. Now suppose that $JK[G] = 0$ and let $\alpha \in K_0[G]$, $\alpha \neq 0$. Since $JK[G] = 0$ it follows that there exists $\beta \in K[G]$ with $\alpha\beta$ not nilpotent. Since G is locally finite we can a finite subgroup H of G with $\alpha, \beta \in K[H]$. Suppose $\alpha K_0[H]$ is a nil ideal in $K_0[H]$. Then since H is finite this ideal is nilpotent and thus $(\alpha K_0[H])K$ is a nilpotent ideal in $K[H]$. Since $\alpha\beta$ is in this ideal we have a contradiction. Hence $\alpha K_0[H]$ is not nil so $\alpha \notin JK_0[H]$. Since $K_0[H] \cap JK_0[G] \subseteq JK_0[H]$ we conclude that $\alpha \notin JK_0[G]$ so $JK_0[G] = 0$.

LEMMA 6. *Suppose $H_1 \subseteq H_2 \subseteq \cdots$ is an ascending chain of subgroups of G with $G = \bigcup_1^\infty H_i$. For each i, let V_i be an irreducible $K[H_i]$-module and suppose that*

(a) *$V_i \subseteq V_{i+1}$ as $K[H_i]$-modules.*

(b) *$K[H_i]$ acts faithfully on V_{i+1}.*

Then $V = \bigcup_1^\infty V_i$ is faithful irreducible $K[G]$-module and hence $K[G]$ is primitive.

Proof. It is easy to see that V is a $K[G]$-module. Moreover, since each $K[H_i]$ acts irreducibly on V_i, $K[G]$ acts irreducibly on V. Finally let $\alpha \in K[G], \alpha \neq 0$. Then for some $i, \alpha \in K[H_i]$. Since $K[H_i]$ acts faithfully on V_{i+1} we have $V_{i+1}\alpha \neq 0$ so $V\alpha \neq 0$ and $K[G]$ acts faithfully.

The following is a generalization of Theorem 2 of [1] and its proof is a modification of the original.

THEOREM 7. *Let G be a locally finite countable group and suppose that $\Delta(G) = \langle 1 \rangle$. Let K be a field and assume that $JK[G] = 0$. Then $K[G]$ is primitive.*

Proof. If char $K = 0$ then this follows from [1] so we assume here that char $K = p > 0$. Then K_0, the prime subfield of K, is $GF(p)$ and by Lemma 5, $JK_0[G] = 0$. Furthermore by Theorem 2, since $\Delta(G) = \langle 1 \rangle$, it suffices to show that $K_0[G]$ is primitive. Thus we may assume that $K = K_0 = GF(p)$. Write $G = \{g_1, g_2, g_3, \cdots\}$.

We will define an ascending chain of finite subgroups H_i of G and irreducible $K[H_i]$-modules V_i satisfying

(a) $g_i \in H_i$

(b) $H_i \subseteq H_{i+1}$, $V_i \subseteq V_{i+1}$ as $K[H_i]$-modules

(c) $K[H_i]$ acts faithfully on V_{i+1}.

We start by taking $H_1 = \langle g_1 \rangle$ and V_1, the principal $K[H_1]$-module. Let us assume that H_i, V_i are given. Now K is a finite field and H_i is finite so $K[H_i]$ is finite and let $\alpha_1, \alpha_2, \cdots, \alpha_r$ be its finitely many nonzero elements. Since $\Delta(G) = \langle 1 \rangle$, $K[G]$ is prime and hence

$$\alpha_1 K[G]\alpha_2 K[G] \cdots \alpha_{r-1} K[G]\alpha_r \neq 0 .$$

Therefore we can choose group elements $x_1, x_2, \cdots, x_{r-1}$ with

$$\alpha = \alpha_1 x_1 \alpha_2 x_2 \cdots \alpha_{r-1} x_{r-1} \alpha_r \neq 0 .$$

Now $JK[G] = 0$ so $\alpha K[G]$ is not a nil ideal and hence there exists $\beta \in K[G]$ with $\alpha\beta$ not nilpotent. Let H_{i+1} be the subgroup of G generated by $H_i, g_{i+1}, x_1, x_2, \cdots, x_{r-1}$ and the support of β. Then H_{i+1} is a finitely generated and hence finite subgroup of G. Moreover, $g_{i+1} \in H_{i+1}$ and $H_i \subseteqq H_{i+1}$. Now $\alpha\beta \in K[H_{i+1}]$ is not nilpotent so there exists an irreducible $K[H_{i+1}]$-module V_{i+1} with $V_{i+1}\alpha\beta \neq 0$. By definition of α and the fact that $K[H_i] = \{0, \alpha_1, \alpha_2, \cdots, \alpha_r\}$ we see that $K[H_i]$ acts faithfully on V_{i+1}. Finally let I be a minimal right ideal of $K[H_i]$ which affords V_i by Lemma 4. Since $K[H_i]$ acts faithfully on V_{i+1} there exists $v \in V_{i+1}$ with $vI \neq 0$. Then clearly as $K[H_i]$-modules $V_{i+1} \supseteqq vI \cong V_i$. We have therefore shown that the sequence H_i, V_i exists.

Since $g_i \in H_i$ we have $G = \bigcup_1^\infty H_i$ and Lemma 6 applies. We conclude that $V = \bigcup_1^\infty V_i$ is a faithful irreducible $K[G]$-module. Hence $K[G]$ is primitive and the result follows.

In view of Formanek's result in [2], we have immediately

COROLLARY 8. *Let S_∞ denote the countably infinite symmetric group. If K is any field, then $K[S_\infty]$ is primitive.*

We now discuss a method of constructing faithful irreducible modules starting from normal torsion free abelian subgroups. Let K be a field and let A be a torsion free abelian group. Then $K[A]$ is an integral domain with quotient field $F = K[A]^{-1}K[A]$. We will study the group ring $F[G]$.

THEOREM 9. *Let K be a field and let G be a group with a normal torsion free abelian subgroup A. Let F be a field isomorphic to $K[A]^{-1}K[A]$. If $A \cap \Delta(G) = \langle 1 \rangle$, then $F[G]$ has an irreducible module V on which $F[A]$ acts faithfully.*

Proof. Let $\bar{G}$ be a group isomorphic to G with isomorphism $G \to \bar{G}$ given by $x \to \bar{x}$. Then $\bar{A} \cong A$ and we let $F = K[\bar{A}]^{-1}K[\bar{A}]$. Let λ denote the natural homomorphism of A into F defined by $\lambda(a) = \bar{a} \in K[\bar{A}] \subseteqq F$. Then λ extends to Λ, an F-homomorphism, $\Lambda: F[A] \to F$ given by

$$\Lambda(\Sigma f_i a_i) = \Sigma f_i \lambda(a_i) = \Sigma f_i \bar{a}_i .$$

If M denotes the kernel of Λ, then M is a maximal right ideal of $F[A]$. Now $F[G]$ is free over $F[A]$ so $MF[G] \neq F[G]$ and we can choose N to be a maximal right ideal of $F[G]$ with $N \supseteqq MF[G]$. We set $V = F[G]/N$ so that V is an irreducible $F[G]$-module.

Let I be the kernel of the action of $F[G]$ on V. Then I is a two-sided ideal of $F[G]$ and we will show that $I \cap F[A] = 0$. Observe that $N \cap F[A] = M$ so $F[G]/N \supseteqq F[A]/M$ as $F[A]$-modules and this shows that $\Lambda(I \cap F[A]) = 0$. Moreover, since I is an ideal in $F[G]$ and $A \lhd G$ we see that $I \cap F[A]$ is invariant under conjugation by the elements of G.

Suppose by way of contradiction that $I \cap F[A] \neq 0$ and choose

$$\alpha = \sum_1^n f_i a_i \,, \quad \alpha \neq 0$$

in this ideal. By multiplying α by a suitable element of A if necessary we may assume that $1 \in \operatorname{Supp} \alpha$ so say $a_1 = 1$, $f_1 \neq 0$. Moreover, since $F = K[\bar{A}]^{-1}K[\bar{A}]$ we may further assume by rationalizing denominators that all $f_i \in K[\bar{A}]$. If $x \in G$ then

$$\alpha^{x^{-1}} = \sum_1^n f_i a_i^{x^{-1}} \in I \cap F[A]$$

so

$$0 = \Lambda(\alpha^{x^{-1}}) = \sum_1^n f_i \lambda(a_i^{x^{-1}}) = \sum_1^n f_i(\overline{x a_i x^{-1}}) \,.$$

We view the above equation as an equation in $K[\bar{G}]$. As such we have $f_i \in K[\bar{A}]$ and $\overline{x a_i x^{-1}} = \bar{x}\bar{a}_i\bar{x}^{-1}$. So

$$0 = \sum_1^n f_i \bar{x}\bar{a}_i\bar{x}^{-1} \,.$$

Multiplying this on the right by $\bar{x}$ yields

$$f_1\bar{x}\bar{a}_1 + f_2\bar{x}\bar{a}_2 + \cdots + f_n\bar{x}\bar{a}_n = 0$$

and this is a linear identity in $K[\bar{G}]$ since it holds for all $\bar{x} \in \bar{G}$. If θ denotes the natural projection $\theta\colon K[\bar{G}] \to K[\Delta(\bar{G})]$ we conclude from a slight modification of Lemma 1.3 of [4] that

$$f_1\theta(\bar{a}_1) + f_2\theta(\bar{a}_2) + \cdots + f_n\theta(\bar{a}_n) = 0 \,.$$

Now $\bar{a}_1 = 1$ so $\theta(\bar{a}_1) = 1$. Moreover, for $i > 1$, $a_i \neq 1$ so by assumption $a_i \notin \Delta(G)$ and thus $\bar{a}_i \notin \Delta(\bar{G})$ and $\theta(\bar{a}_i) = 0$. Therefore, the above yields $f_1 = 0$, a contradiction. The theorem is proved.

We can now apply this result to certain special types of solvable groups. We will need the following important theorem of A. E. Zalesskiĭ which we quote below.

PROPOSITION 10. ([6]). *Let G be a solvable group. Then G has a normal Δ-subgroup H with the following property. If K is any field and if I is a nonzero ideal of $K[G]$ then $I \cap K[H] \neq 0$.*

Let G be a polycyclic group, that is a group with a finite subnormal series with cyclic quotients. We call the number of infinite cyclic quotients which occur the rank of G. Since any two such series have a common refinement it is easy to see that the rank is well defined. If K is a field we let t.d. K denote the transcendence degree of K over its prime subfield K_0 so that t.d. K is some cardinal number.

COROLLARY 11. *Let G be a polycyclic group with $\Delta(G) = \langle 1 \rangle$ and let K be a field with*

$$\text{t.d. } K \geqq \text{rank } G .$$

Then $K[G]$ is primitive.

Proof. Let K_0 be the prime subfield of K and let H be the subgroup of G given by Proposition 10. Then $H = \Delta(H)$ and since G is polycyclic so is H with rank $H \leqq$ rank G. Now by Lemma 2.2 of [4] the finitely generated Δ-subgroup H has a finite torsion subgroup $\tilde{H}$ with $H/\tilde{H}$ torsion free abelian. Since $\tilde{H}$ is characteristic in H, it is a finite normal subgroup of G and hence $\tilde{H} = \langle 1 \rangle$ since $\Delta(G) = \langle 1 \rangle$. Thus H is torsion free abelian.

By Theorem 9, if $F \cong K_0[H]^{-1}K_0[H]$ then $F[G]$ has an irreducible module V on which $F[H]$ acts faithfully since $\Delta(G) = \langle 1 \rangle$. Thus by Proposition 10, $F[G]$ acts faithfully on V and therefore $F[G]$ is a primitive group ring. Finally we observe that since H is a finitely generated torsion free abelian group, F is just a purely transcendental extension of K_0 of transcendence degree equal to the rank of H. Thus since $K \supseteqq K_0$ and

$$\text{t.d. } K \geqq \text{rank } G \geqq \text{rank } H = \text{t.d. } F$$

we see that K contains an isomorphic copy of F. Therefore, by Theorem 2, since $\Delta(G) = \langle 1 \rangle$, $K[G]$ is primitive and result follows.

It is an interesting question as to whether the transcendence degree assumption is needed in the above. It is apparently not true that a polycyclic group G with $\Delta(G) = \langle 1 \rangle$ will have a primitive group algebra over all fields K. For example, let K be the algebraic closure of $GF(p)$ and let G be polycyclic. Then it is conjectured by Ph. Hall in [3] that all irreducible $K[G]$-modules must in fact be finite dimensional over K. It was proved in [3] that this is always the case if G is nilpotent. Recently, J. Roseblade ([5]) has proved this conjecture in general. Therefore, we conclude easily that if G is polycyclic and if K is as above, then $K[G]$ is primitive if and only if $G = \langle 1 \rangle$. Since nonidentity polycyclic groups exist with $\Delta(G) = \langle 1 \rangle$ we see that at least some assumption on the field is required.

If G is solvable we cannot get as good a result as above but we can prove

COROLLARY 12. *Let G be a torsion free solvable group with $\Delta(G) = \langle 1 \rangle$. If K is any field then there exists $F \supseteq K$ such that $F[G]$ is primitive.*

Proof. Let H be the subgroup of G given in Proposition 10. Then $H = \Delta(H)$ and G is torsion free so Lemma 2.2 of [4] implies that H is torsion free abelian. Let $F \cong K[H]^{-1}K[H] \supseteq K$. Since $\Delta(G) = \langle 1 \rangle$, Theorem 9 implies that $F[G]$ has an irreducible module V on which $F[H]$ acts faithfully. Thus by Proposition 10, $F[G]$ acts faithfully and hence $F[G]$ is primitive.

In Theorem 3 of [1], it is shown that if G and K are given then there exists $G^* \supseteq G$ such that $K[G^*]$ is primitive. In the following we give a more concrete construction of such a group G^* provided we limit K somewhat. Let Z denote the infinite cyclic group.

COROLLARY 13. *Let G be an infinite group and let K be a field with*

$$\text{t. d. } K \geqq \text{card } |G| .$$

If G^* is the Wreath product $G^* = Z \wr G$ then $K[G^*]$ is primitive.

Proof. Let K_0 denote the prime subfield of K. Observe that $G^* = AG$ where A is a normal torsion free abelian subgroup of G^* which is equal to the direct sum of copies of Z indexed by the elements of G. Since A is torsion free it is easy to see that if $x \in G^*$ and $[A: C_A(x)] < \infty$ then $x \in A$. Moreover, from this and the fact that G is infinite we get easily $\Delta(G^*) = \langle 1 \rangle$.

Let $F \cong K_0[A]^{-1}K_0[A]$. Then by Theorem 9, $F[G^*]$ has an irreducible module V on which $F[A]$ acts faithfully. By the above remarks and Lemma 21.1 of [4] we conclude that $F[G^*]$ acts faithfully and hence $F[G^*]$ is primitive. From the nature of A, it is clear that F is a purely transcendental extension of K_0 with t.d. $F = \text{card } |G|$. Thus since $K \supseteq K_0$ and t. d. $K \geqq \text{card } |G|$ we see that K contains an isomorphic copy of F. Finally $\Delta(G^*) = \langle 1 \rangle$ so Theorem 2 implies that $K[G^*]$ is primitive.

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