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ORDERS IN SIMPLE ARTINIAN RINGS ARE STRONGLY EQUIVALENT TO MATRIX RINGS

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The result indicated by the title will be proved. More specifically stated: when R is a left order in a simple artinian ring Q, there exist matrix units $\{e_{ij}\}$ for Q and an element $r \in D$, where D is the intersection of the centralizer of $\{e_{ij}\}$ with R, such that $rRr \subseteqq \sum De_{ij}$ and $\sum rDe_{ij} \subseteqq R$. The Faith-Utumi theorem is an immediate consequence of this relationship. Furthermore, if R is either a maximal order, or is subdirectly irreducible, or is hereditary, then there is a left order C in the centralizer of $\{e_{ij}\}$ which inherits the corresponding property of R and such that R is equivalent to the matrix ring $\sum Ce_{ij}$.

Introduction. A subring R of a simple artinian ring Q is a *left order* in Q if every element of Q is of the form $r^{-1}s$ for some $r, s \in R$. An *order* in Q is a right and left order. Two left orders R and R' in Q are *equivalent* if there exist units p, q, p', q' of Q with $pRq \subseteqq R'$ and $p'R'q' \subseteqq R$; one then writes $R \sim R'$. A *maximal left order* in Q is a left order in Q which is maximal in its equivalence class. It is assumed throughout that all left orders are inside a fixed simple artinian ring Q, and also that rings do not contain identity elements unless specifically indicated.

In the classical situation, by which is meant the theory of maximal orders over a Dedekind domain [2], all the maximal orders are equivalent. This remains true in the more general situation of Dedekind orders [9], and there exists in each equivalence class a matrix ring over a (not necessarily commutative) integral domain.

The first main result of this paper in § 2 shows that given a (left) order R in Q there exist matrix units $\{e_{ij}\}$ for Q with centralizer $\varDelta$ and an element $r \in D = \varDelta \cap R$ with $rRr \subseteqq \sum De_{ij}$, $r \sum De_{ij} \subseteqq R$, and $\sum De_{ij}r \subseteqq R$; as expected, D is a (left) order in $\varDelta$. Thus, in particular, R contains the matrix order $\sum rDe_{ij}$, giving the conclusion of the Faith-Utumi theorem [4]; and $R \sim \sum De_{ij}$ [11], with a somewhat stronger condition actually satisfied. The additional information enables one to consider the important special cases when R is a maximal, or a subdirectly irreducible, or a left hereditary left order. In each of these cases, a maximal left order $C \subseteqq \varDelta$ is chosen with the same property as R and with $r \sum Ce_{ij}r \subseteqq R$ and $rRr \subseteqq \sum Ce_{ij}$. These are treated in § 3-§ 5, where partial results are also obtained for simple orders. The method of proof involves only the machinery of linear algebra over Ore domains.

1. Preliminaries. The reader is assumed to be familiar with Goldie's characterization of (left) orders in simple artinian rings [5], with the definition and use of Morita contexts in this setting (cf. [1], [10]), and all attendant concepts (uniform module, essential submodule, and so on).

Throughout, R will denote a fixed (left) order in a simple artinian ring Q, M will be a fixed uniform left ideal of R, $N = \mathrm{Hom}_R(M, R)$, $E = \mathrm{End}_R M$; and, except where specifically indicated otherwise, attention will be directed to the standard Morita context (R, M, N, E) with bimodule maps $(\ ,\ ): M \otimes_E N \to R$ and $[\ ,\ ]: N \otimes_R M \to E$ defined via $(m, n) = (m)n$, $m'[n, m] = (m', n)m$ for all $m, m' \in M$, $n \in N$ (homomorphisms being written opposite scalars). Observe that $(\ ,\ )$ and $[\ ,\ ]$ are nonsingular in all four variables. The well-known results presented in this section are of fundamental importance in the sequel.

LEMMA 1.1. *$E = \mathrm{End}_R M$ is a (left) order in the division ring $\mathrm{End}_Q QM$.*

Proof. QM is a minimal left ideal of Q and is the R-injective hull of M. Hence one may regard $E = \mathrm{End}_R M$ as a subring of the division ring $\Delta = \mathrm{End}_Q QM$. Given $\varphi \in \Delta$, set $M_0 = M\varphi^{-1} \cap M$. Then $0 \neq [N, M_0\varphi] = [N, M_0]\varphi \subseteqq E\varphi \cap E$, and it follows that E is a left order in Δ.

Next suppose that R is also a right order in Q. Then one may regard $[N, M_0]$ as a right ideal of $\mathrm{End}_R M_0$ (by restricting the action of N to M_0). Moreover, $\mathrm{End}_R M_0$ is a right order in Δ because $M_0\varphi[N, M_0] \subseteqq (M, N)M_0 \subseteqq M_0$. Since $[N, M_0]$ is also a left ideal of E, it follows that E is a right order in Δ.

LEMMA 1.2. (Dual Basis Lemma) *There exist elements*

$$m_1, m_2, \cdots, m_t \in M,\ n_1, n_2, \cdots, n_t \in N,\ 0 \neq a \in E,\ r = \sum_{i,j=1}^{t} (m_i, n_i) \in R$$

satisfying:

(i) *$n_1, n_2, \cdots, n_t$ is a maximal linearly independent subset of $_E N$;*

(ii) *$[n_i, m_j] = \delta_{ij} a$ for i and j (where δ_{ij} is the Kronecker delta);*

(iii) *r is a regular element of R* (i.e., *r is a unit in Q*);

(iv) *$n_i r = a n_i$ and $r m_i = m_i a$ for each i.*

Proof. $N = \mathrm{Hom}_R(M, R)$ can be regarded as an essential E-submodule of $\hat{N} = \mathrm{Hom}_Q(QM, Q)$, and the latter is a finite dimensional vector space over $\Delta = \mathrm{End}_Q QM$. Thus $\hat{N}$ is the E-injective hull of N, and ${}_EN$ is finite dimensional and torsion-free. This being the situation, proofs of the lemma may be found in [1] and [10], except for the last assertion that $rm_i = m_i a$ for each i. To see this, it suffices to show that $[n_j, rm_i] = [n_j, m_i a]$ for each j; and this is evident since $[n_j, rm_i - m_i a] = a[n_j, m_i] - [n_j, m_i]a = \delta_{ij}(a^2 - a^2) = 0$.

2. Main results. The notation in this section continues that of § 1, and the notation now introduced will be followed consistently. All sums will be taken over the integers from 1 to t.

Observe that

$$r(m_i, n_j) = \sum_k (m_k, n_k)(m_i, n_j) = \sum_k (m_k[n_k, m_i], n_j) = (m_i a, n_j) .$$

Similarly, $(m_i n_j)r = (m_i, an_j)$, so that

$$(1) \qquad r(m_i, n_j) = (m_i, n_j)r \quad \text{for all} \quad 1 \leqq i, j \leqq t .$$

Thus defining

$$(2) \qquad e_{ij} = r^{-1}(m_i, n_j) = (m_i, n_j)r^{-1} ,$$

it is easy to check that $\{e_{ij}: 1 \leqq i, j \leqq t\}$ is a set of matrix units for Q. Set

$$\Delta = \{q \in Q: qe_{ij} = e_{ij}q \quad \text{for all} \quad 1 \leqq i, j \leqq t\} ,$$

and let $D = \Delta \cap R$.

Clearly then Δ is a division ring and $Q = \sum_{i,j} \Delta e_{ij} \cong \Delta_t$.

Let $R_0 = \{\sum_{i,j} (m_i b_{ij}, n_j): b_{ij} \in E\}$, $D_0 = \{\sum_i (m_i b, n_i): b \in E\}$. Both R_0 and D_0 are subrings of R. They are related as follows.

LEMMA 2.1. *$D_0 \subseteqq D$ and $R_0 = \sum_{i,j} D_0 e_{ij}$.*

Proof. Let $\sum_i (m_i b, n_i) \in D_0$, $b \in E$. Then for any choice of k and h,

$$\sum_i (m_i b, n_i)e_{kh} = \sum_i (m_i b, n_i)(m_k, n_h)r^{-1} = (m_k b, an_h)r^{-1} = (m_k b, n_h) ;$$

and similarly, $e_{kh}\sum_i (m_i b, n_i) = (m_k b, n_h)$. Hence $D_0 \subseteqq D$.

Now given $\sum_{i,j} (m_i b_{ij}, n_j) \in R_0$, $b_{ij} \in E$; for each $1 \leqq i, j \leqq t$, set $r_{ij} = \sum_k (m_k b_{ij}, n_k) \in D_0$. Then

$$r_{ij}e_{ij} = \sum_k(m_k b_{ij}, n_k)(m_i, n_j)r^{-1} = (m_i b_{ij}, an_j)r^{-1} = (m_i b_{ij}, n_j) .$$

Thus $R_0 = \sum_{i,j} D_0 e_{ij}$.

THEOREM 2.2. *Let R be a (left) order in Q. Then*

(i) $r \in D$, $\sum_{i,j} rDe_{ij} \subseteqq R$, *and* $\sum_{i,j} De_{ij}r \subseteqq R$.
(ii) $rRr \subseteqq R_0 \subseteqq \sum_{i,j} De_{ij}$.
(iii) $R \sim R_0 \sim \sum_{i,j} De_{ij}$.
(iv) D_0 *and* D *are equivalent* (*left*) *orders in* Δ.

Proof. That $r \in D$ is obvious from the definition of r and (2). $\sum_{i,j} rDe_{ij} = \sum_{i,j} e_{ij}rD = \sum_{i,j} (m_i, n_j)D \subseteqq R$, and similarly $\sum_{i,j}De_{ij}r \subseteqq R$.

$$rRr = \sum_i (m_i, n_i)R \sum_j (m_j, n_j) = \sum_{i,j} (m_i[n_iR, m_j], n_j) \subseteqq R_0$$
$$= \sum_{i,j} D_0e_{ij} \subseteqq \sum_{i,j} De_{ij} .$$

(iii) is a consequence of (i) and (ii). Thus in particular, $R_0 = \sum_{i,j} D_0e_{ij}$ and $\sum_{i,j} De_{ij}$ are also (left) orders in Q. This implies that D_0 and D must be (left) orders in Δ. It remains to prove that D and D_0 are equivalent. While this follows from (iii), it is useful to observe that in fact

$$rDr \subseteqq D_0 . \tag{3}$$

To see this it suffices to verify that

(4) $[n_id, m_j] = \delta_{ij}[n_1d, m_1]$ for any $1 \leqq i, j \leqq t$ and $d \in D$.

Now, $r^{-1}(m_i[n_id, m_j], n_j)r^{-1} = e_{ii}de_{jj} = de_{ii}e_{jj} = 0$ when $i \neq j$. Hence $(m_i[n_id, m_j], n_j) = 0$, and so $a[n_id, m_j]a = [n_i, m_i][n_id, m_j][n_j, m_j] = 0$, which establishes that $[n_id, m_j] = 0$ when $i \neq j$. Similarly,

$$r^{-1}(m_i([n_id, m_i] - [n_1d, m_1]), n_i) = e_{ii}de_{ii} - e_{i1}de_{1i} = 0 ,$$

from which it follows as above that $[n_id, m_i] - [n_1d, m_1] = 0$.

COROLLARY 2.3. (Faith-Utumi [4]) *Given a* (*left*) *order* R *in a simple artinian ring* Q *there exist matrix units* $\{e_{ij}\}$ *for* Q, *and a* (*left*) *order* C *in the centralizer of* $\{e_{ij}\}$ *such that* $C \subseteqq R$ *and* $\sum_{i,j} Ce_{ij} \subseteqq R$.

Proof. $C = rD$ is a right ideal of D, and hence C is a (left) order in Δ.

3. Maximal orders.

The results of the previous section facilitate a rapid treatment of maximal orders.

THEOREM 3.1. *If* R *is a maximal* (*left*) *order in* Q, *then there exists a maximal* (*left*) *order* C *in* Δ *such that* $rRr \subseteqq \sum_{i,j} Ce_{ij}$ *and* $\sum_{i,j} rCe_{ij}r \subseteqq R$.

Proof. Of course $1 \in R$, since R is a maximal left order in Q. Let C be any left order in Δ containing D and equivalent to D.

Then without loss of generality, it may be assumed that there exists $d, d' \in D$ with $dCd' \subseteqq D$. Consider $R' = R + Rr(\sum_{i,j} Ce_{ij})rR$; R' is a left order in Q because $R \subseteqq R'$ and $rRr \subseteqq \sum_{i,j} De_{ij} \subseteqq \sum_{i,j} Ce_{ij}$. Also R' is equivalent to R because

$$rdrR'rd' \subseteqq rd \textstyle\sum_{i,j} Ce_{ij}d' \subseteqq r \sum_{i,j} De_{ij} \subseteqq R \ .$$

By the maximality of R it must be the case that $R = R'$. In particular $r(\sum_{i,j} Ce_{ij})r = \sum_{i,j} rCre_{ij} \subseteqq R$. Thus $rCr \subseteqq R \cap \Delta = D$.

Hence given an arbitrary left order C in Δ with $D \subseteqq C$ and $D \sim C$, it is always the case that $rCr \subseteqq D$. This enables one to apply Zorn's Lemma to choose a maximal such C. The rest of the theorem is clear.

REMARK. It would be of interest to learn whether necessarily $C = D$ in the above theorem; especially in the case where M is a basic left ideal. The answer is not known to the author at this time.

4. Simple orders. The obvious question for simple orders with 1 is whether they are equivalent to matrix rings over simple Ore domains. The analogous question for Morita-equivalence is not as yet settled (see [3]). Unfortunately, even in the present simplified setting one encounters the same difficulties as arise for the Morita-equivalence problem. Recall that a ring is *subdirectly irreducible* if it has a unique nonzero minimal ideal. As usual the notation follows that of prior sections.

THEOREM 4.1. *If R is a subdirectly irreducible (left) order in Q, then there exists a subdirectly irreducible (left) order C in Δ such that $\sum_{i,j} rCre_{ij} \subseteqq R$ and $rRr \subseteqq \sum_{i,j} Ce_{ij}$. Moreover, if R is maximal in Q, C can be chosen maximal in Δ.*

Proof. When R is a maximal (left) order, choose C containing D as in Theorem 3.1; otherwise, take $C = D$. It remains only to verify that C is subdirectly irreducible. For this, let I be the unique minimal ideal of R, and let A be any nonzero ideal of $S = \sum_{i,j} Ce_{ij}$. Then $RrArR$ is a nonzero ideal of R, and so $I \subseteqq RrArR$. Hence $rIr \subseteqq (rRr)A(rRr) \subseteqq SAS \subseteqq A$. Since A was arbitrary, $rIr \neq 0$ is contained in the intersection of the ideals of S. Such ideals are of the form $\sum_{i,j} Be_{ij}$ for B an ideal of C; and from this it is immediate that C has a minimal ideal.

COROLLARY 4.2. *If R is a simple (left) order with 1, then there exists a subdirectly irreducible maximal (left) order C in Δ such that $\sum_{i,j} rCre_{ij} \subseteqq R$ and $rRr \subseteqq \sum_{i,j} Ce_{ij}$.*

LEMMA 4.3. *$r^{-1}D_0$ is a ring isomorphic to E under the homomorphism defined via $b \to r^{-1} \sum_i (m_i b, n_i)$ for $b \in E$.*

Proof. The verification is entirely routine once it is proved that the map is multiplicative; and for this it suffices to demonstrate that for any $b, c \in E$,

$$\sum_i (m_i b, n_i) r^{-1} \sum_j (m_j c, n_j) = \sum_i (m_i bc, n_i) \ . \tag{5}$$

To see this, choose $b_1, c_1 \in E$ with $0 \neq b_1 ab = c_1 a^2$ (this is possible because E is a left Ore domain), and then multiply the difference of both sides of the equation in (5) by the invertible element $\sum_k (m_k b_1, n_k)$ to obtain zero.

THEOREM 4.4. *Suppose that R is a simple (left) order with 1, and that R has a projective uniform left ideal. Then $r^{-1}D_0$ is a simple (left) order with 1 in Δ and $R \sim \sum_{i,j} r^{-1}D_0 e_{ij}$.*

Proof. Choose ${}_RM$ to be projective. Then by [6; Lemma 4], ${}_RM$ is finitely generated, and hence is an R-progenerator. It follows that E is simple, and then by the preceding lemma $r^{-1}D_0$ is simple. Now $D_0 \subseteqq r^{-1}D_0$, and D_0 is a (left) order by Theorem 2.2. Hence the same is true for $r^{-1}D_0$. Finally, $r\sum_{i,j} r^{-1}D_0 e_{ij} = \sum_{i,j} D_0 e_{ij} = R_0 \subseteqq R$ and $Rr = r^{-1}(rRr) \subseteqq r^{-1}R_0 = \sum_{i,j} r^{-1}D_0 e_{ij}$.

REMARK. In the situation of the preceding corollary, it has been seen that $r^{-1}D_0$ is Morita-equivalent to R. Therefore, any categorical property of R will be inherited by $r^{-1}D_0$.

5. Dedekind prime rings.

A maximal (left) hereditary (left) noetherian (left) order R in Q is called a *(left) Dedekind prime ring*. All orders in this section are assumed to contain the identity element.

THEOREM 5.1. *If R is a left hereditary (left) order in Q, then $r^{-1}D_0$ is a (left) hereditary (left) order in Δ, and $R \sim \sum_{i,j} r^{-1}D_0 e_{ij}$.*

Proof. Since a (left) hereditary left order is left noetherian by [8; Theorem 3.11], $E = \mathrm{End}_R M$ is the endomorphism ring of a finitely generated projective module over a (left) hereditary ring. By [9; Lemma 4.4], E is (left) hereditary, and then Lemma 4.3 ensures that this is true for $r^{-1}D_0$.

COROLLARY 5.2. *Suppose that R is a (left) Dedekind prime ring. Then $r^{-1}D_0$ is a (left) Dedekind prime domain, and $R \sim \sum_{i,j} r^{-1}D_0 e_{ij}$.*

Proof. It remains only to observe that $E = \mathrm{End}_R M$ is a maximal (left) order in Δ. This can be found in [7; Lemma 1.7].

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