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MATRIX SUMMABILITY OF A CLASS OF DERIVED FOURIER SERIES

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Let f be L-integrable and periodic with period 2π, and let

$$\sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx) \tag{1.1}$$

be the derived Fourier series of the function f with partial sums $s'_n(x)$. We write

$$\psi_x(t) = f(x+t) - f(x-t);$$
$$g_x(t) = \frac{\psi_x(t)}{4 \sin t/2}.$$

In this paper, the following theorems are established.

THEOREM 1. **Let $A = (a_{mn})$ be a regular infinite matrix of real numbers. Then, for every $x \in [-\pi, \pi]$ for which $g_x(t)$ is of bounded variation on $[0, \pi]$,**

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} s'_n(x) = g_x(0+) \tag{1.2}$$

if and only if

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} a_{mn} \sin(n+1/2)t = 0 \quad \text{for all} \quad t \in [0, \pi]. \tag{1.3}$$

THEOREM 2. **Let $A = (a_{mn})$ be an almost regular infinite matrix of real numbers. Then, for each $x \in [-\pi, \pi]$ for which $g_x(t)$ is of bounded variation on $[0, \pi]$,**

$$\lim_{p\to\infty} \frac{1}{p} \sum_{j=0}^{p-1} t'_{m+j}(x) = g_x(0+)$$

uniformly in m if and only if

$$\lim_{p\to\infty} \frac{1}{p} \sum_{j=0}^{p-1} \sum_{n=1}^{\infty} a_{m+j,n} \sin(n+1/2)t = 0 \quad \text{for all} \quad t \in [0, \pi],$$

uniformly in m, where

$$t'_m(x) = \sum_{n=1}^{\infty} a_{mn} s'_n(x).$$

2. *Proof of Theorem* 1. We have

$$\begin{aligned} s'_n(x) &= \frac{1}{\pi} \int_0^{\pi} \psi_x(t) \Big(\sum_{k=1}^{n} k \sin kt \Big) dt \\ &= -\frac{1}{\pi} \int_0^{\pi} \psi_x(t) \frac{d}{dt} \Big[\frac{\sin(n+1/2)t}{2 \sin t/2} \Big] dt \\ &= I_n + \frac{2}{\pi} \int_0^{\pi} \sin(n+1/2)t\, dg_x(t), \end{aligned} \tag{2.1}$$

where

$$I_n = \frac{1}{\pi}\int_0^{\pi} g_x(t)\frac{\sin(n+1/2)t}{\tan t/2}dt; \tag{2.2}$$

$$\sum_{n=1}^{\infty} a_{mn}s_n'(x) = \sum_{n=1}^{\infty} a_{mn}I_n + \frac{2}{\pi}\int_0^{\pi} L_m(t)dg_x(t)\ , \tag{2.3}$$

where

$$L_m(t) = \sum_{n=1}^{\infty} a_{mn}\sin(n+1/2)t\ . \tag{2.4}$$

Since $g_x(t)$ is of bounded variation on $[0, \pi]$ and tends to $g_x(0+)$ as $t \to 0$, $g_x(t)\cos t/2$ has the same properties; so, by Jordan's convergence criterion for Fourier series,

$$I_n \longrightarrow g_x(0+) \quad \text{as} \quad n \longrightarrow \infty\ . \tag{2.5}$$

By the regularity of our method of summation, it follows that

$$\lim_{m\to\infty}\sum_{n=1}^{\infty} a_{mn}I_n = g_x(0+)\ . \tag{2.6}$$

Hence we have to show that, if (1.3) holds, then

$$\lim_{m\to\infty}\int_0^{\pi} L_m(t)dg_x(t) = 0\ , \tag{2.7}$$

and conversely.

By a theorem on the weak convergence of sequences in the Banach space of all continuous functions defined on a finite closed interval (see Banach [1], pp. 134–135), it follows that (2.7) holds if and only if

$$|L_m(t)| \leqq K \quad \text{for all} \quad m \quad \text{and for all} \quad t \in [0, \pi] \tag{2.8}$$

and (1.3) holds, where K is a constant.

Since (2.8) is automatically satisfied by one of the regularity conditions on A, it follows that (2.7) holds if and only if (1.3) holds. Thus the proof of the theorem is completed.

REMARKS. (a) We observe that, for each $g_0(t)$ of bounded variation on $[0, \pi]$, we have a corresponding odd function $f \in L\ [-\pi, \pi]$ given by

$$f(t) = 1/2\psi_0(t) = 2g_0(t)\cdot\sin t/2 \quad \text{on} \quad [0, \pi]\ .$$

(b) If $a_{mn} = 1/m$ for $n \leqq m$ and zero for $n > m$, then the condition (1.3) is obviously satisfied.

3. Note. A bounded sequence $\{s_n\}$ is said to be almost convergent to s if

$$\lim_{p\to\infty}\frac{s_n + s_{n+1} + \cdots + s_{n+p-1}}{p} = s \tag{3.1}$$

uniformly in n (see Lorentz [4]).

It is easy to see that a convergent sequence is almost convergent and the limits are the same.

Let $A = (a_{mn})$ be an infinite matrix of real numbers. A bounded sequence $\{s_n\}$ is said to be almost A-summable to s if the A-transform of $\{s_n\}$ is almost convergent to s, and the matrix A is said to be almost regular if $s_n \to s$ implies that the sequence $\{t_m\}$ of the A-transforms of $\{s_n\}$ is almost convergent to s.

Necessary and sufficient conditions for the matrix A to be almost regular are as follows (see King [3]):

$$\sup_{p\geq 1}\left(\sum_{n=1}^{\infty}\frac{1}{p}\left|\sum_{j=m}^{m+p-1} a_{jn}\right|\right) < M(m = 1, 2, \cdots; M = a \quad \text{constant})\,; \tag{3.2}$$

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=m}^{m+p-1} a_{jn} = 0 \quad \text{uniformly in } m \quad (n = 1, 2, \cdots)\,; \tag{3.3}$$

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=m}^{m+p-1}\sum_{n=1}^{\infty} a_{jn} = 1 \quad \text{uniformly in } m\,. \tag{3.4}$$

We establish the following

THEOREM 2. *Let $A = (a_{mn})$ be an almost regular infinite matrix of real numbers. Then, for every $x \in [-\pi, \pi]$ for which $g_x(t)$ is of bounded variation on $[0, \pi]$,*

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=0}^{p-1} t'_{m+j}(x) = g_x(0+) \tag{3.5}$$

uniformly in m if and only if

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=0}^{p-1}\sum_{n-1}^{\infty} a_{m+j,n}\sin(n + 1/2)t = 0 \quad \text{for all} \quad t \in [0, \pi]\,, \tag{3.6}$$

uniformly in m, where

$$t'_m(x) = \sum_{n=1}^{\infty} a_{mn}s'_n(x)\,,$$

$s'_n(x)$ being the partial sum of the derived Fourier series (1.1) *of f.*

Proof. We have, by (2.1),

$$
\begin{aligned}
\frac{1}{p}\sum_{j=0}^{p-1} t'_{m+j}(x) &= \frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=1}^{\infty} a_{m+j,n}s'_n(x) \\
(3.7)\qquad &= \frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=1}^{\infty} a_{m+j,n}I_n + \frac{2}{\pi}\int_0^{\pi}\left[\frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=1}^{\infty} a_{m+j,n}\sin{(n+1/2)t}\right]dg_x(t) \\
&= J_1 + J_2 ,
\end{aligned}
$$

say.

By (2.5), A being almost regular,

$$(3.8)\qquad J_1 \longrightarrow g_x(0+) \quad \text{uniformly in} \quad m \quad \text{as} \quad p \longrightarrow \infty .$$

So we have to show that (3.6) holds if and only if

$$J_2 \longrightarrow 0 \quad \text{uniformly in} \quad m \quad \text{as} \quad p \longrightarrow \infty .$$

Now,

$$
\begin{aligned}
&\left|\frac{1}{p}\sum_{j=0}^{p-1}\sum_{n=1}^{\infty} a_{m+j,n}\sin{(n+1/2)t}\right| \\
(3.9)\qquad &= \left|\frac{1}{p}\sum_{n=1}^{\infty}\sin{(n+1/2)t}\sum_{j=0}^{p-1} a_{m+j,n}\right| \\
&\leqq \frac{1}{p}\sum_{n=1}^{\infty}\left|\sin{(n+1/2)t}\right|\cdot\left|\sum_{j=0}^{p-1} a_{m+j,n}\right| \\
&\leqq \sum_{n=1}^{\infty}\frac{1}{p}\left|\sum_{j=0}^{p-1} a_{m+j,n}\right| < M \text{ for all } p \text{ and } m, \text{ by (3.2)} .
\end{aligned}
$$

Hence the remainder of the proof is similar to that of Theorem 1.

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