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ON NUMERICAL RANGES OF ELEMENTS OF LOCALLY m-CONVEX ALGEBRAS

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The concept of numerical range is extended from normed algebras to locally m-convex algebras. It is shown that the approximating relations between the numerical range and the spectrum of an element are preserved in the generalization. The set of elements with bounded numerical range is characterized and the relation between boundedness of the spectrum and of the numerical range is discussed. The Vidav-Palmer theory is generalized to give a characterization of b^*-algebras by numerical range.

In a complex unital Banach algebra the numerical range of an element is a set of complex numbers which can be used to approximate the spectrum of an element. In a complex locally m-convex algebra with identity, for each element we define a set of numerical ranges and establish similar approximation to the spectrum of the element. In a normed algebra the spectrum and the numerical range of each element are bounded sets, but in a locally m-convex algebra the spectrum and the numerical ranges of an element may be unbounded. For a locally m-convex algebra with identity we characterize those elements with a bounded numerical range as an important normed subalgebra, and we discuss the relation between boundedness of the spectrum and the numerical ranges. In the normed algebra theory the study of hermitian elements, those with real numerical range, has led to the important Vidav-Palmer theory characterizing unital B^*-algebras among unital Banach algebras. We generalize the results of this theory to a characterization of b^*-algebras by numerical range.

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1. The numerical ranges of an element. For a complex normed algebra $(A, \|\cdot\|)$ with identity 1 where $\|1\| = 1$, i.e., a *complex unital normed algebra*, we define the set

$$D(A, \|\cdot\|; 1) \equiv \{f \in A' : f(1) = 1 \quad \text{and} \quad \|f\| = 1\} .$$

For each $a \in A$ we define the *numerical range of* a as the set

$$V(A, \|\cdot\|; a) \equiv \{f(a) : f \in D(A, \|\cdot\|; 1)\} ,$$

and the *numerical radius of* a as

$$v(A, \|\cdot\|; a) \equiv \sup\{|\lambda|: \lambda \in V(A, \|\cdot\|; a)\} .$$

The set $D(A, \|\cdot\|; 1)$ is a convex weak * compact subset of A' and the numerical range $V(A, \|\cdot\|; a)$ is also a convex compact subset of the complex numbers, [2, p. 16]. The properties and applications of normed algebra numerical ranges have been studied extensively and the main results are conveniently presented by F. F. Bonsall and J. Duncan in [2].

Locally m-convex algebras, i.e., l.m.c. algebras, are examined in some detail by E. A. Michael in [4]. We call a l.m.c. algebra with identity a *unital l.m.c. algebra*. It is our aim to extend the concept of numerical range from complex unital normed algebras to complex unital l.m.c. algebras. It is sufficient for our purpose to note that, for a given l.m.c. algebra A with identity 1 there exists a separating family of submultiplicative semi-norms $\{p_\alpha\}$ on A which generates the topology and is such that $p_\alpha(1) = 1$ for all α, [3, p. 7]. Given such an algebra, we denote by $P(A)$ the class of all such families of semi-norms on A, and by $(A, \{p_\alpha\})$ the algebra A with a particular family of semi-norms $\{p_\alpha\} \in P(A)$.

Given $(A, \{p_\alpha\})$, for each α let N_α denote the nullspace of p_α, A_α denote the quotient space A/N_α, and $\|\cdot\|_\alpha$ denote the norm on A_α defined by $\|x + N_\alpha\|_\alpha = p_\alpha(x)$. For each α consider the natural linear mapping $x \mapsto x_\alpha \equiv x + N_\alpha$ of A onto A_α. We note that 1_α is the identity in A_α and that $\|1_\alpha\|_\alpha = 1$ for each α. Michael has given the significant result that A is isomorphic to a subalgebra of the product of the normed algebras $(A_\alpha, \|\cdot\|_\alpha)$, Proposition 2.7, [4, p. 13]. Using this characterization of l.m.c. algebras we are able to generalize much of the numerical range theory for normed algebras directly to a theory of numerical range for l.m.c. algebras.

Given $(A, \{p_\alpha\})$, we define the set

$$D_\alpha(A, p_\alpha; 1) \equiv \{f \in A' : f(1) = 1 \text{ and } |f(x)| \leqq p_\alpha(x) \text{ for all } x \in A\} ,$$

and we write

$$D(A, \{p_\alpha\}; 1) \equiv \bigcup_\alpha \{D_\alpha(A, p_\alpha; 1)\} .$$

For each $a \in A$ we write

$$V_\alpha(A, p_\alpha; a) \equiv \{f(a): f \in D_\alpha(A, p_\alpha; 1)\} ,$$

and define the *numerical range of* a as the set

$$V(A, \{p_\alpha\}; a) \equiv \bigcup_\alpha \{V_\alpha(A, p_\alpha; a)\} .$$

To each linear functional f on (A, p_α) which annihilates N_α, we can define the linear functional F on A_α by $F(x_\alpha) = f(x)$, and to each linear functional F on A_α we can define the linear functional f on (A, p_α) by $f(x) = F(x_\alpha)$. Consequently, from the definition of the norm in A_α we see that $D_\alpha(A, p_\alpha; 1)$ is isomorphic to $D(A_\alpha, \|\cdot\|_\alpha; 1_\alpha)$, and for $a \in A$

$$V_\alpha(A, p_\alpha; a) = V(A_\alpha, \|\cdot\|_\alpha; a_\alpha) .$$

Hence, we have the numerical range of a characterized by the normed algebra numerical ranges of the a_α in that

$$V(A, \{p_\alpha\}; a) = \bigcup_\alpha \{V(A_\alpha, \|\cdot\|_\alpha; a_\alpha)\} .$$

Both $D(A, \{p_\alpha\}; 1)$ and $V(A, \{p_\alpha\}; a)$ depend upon the particular family of semi-norms $\{p_\alpha\} \in P(A)$ chosen to associate with A. It is clear that when $\{p_\alpha\}$ is a directed family, $D(A, \{p_\alpha\}; 1)$ is a convex subset of A' and the numerical range $V(A, \{p_\alpha\}; a)$ is a convex subset of the complex numbers.

For each $a \in A$ we write

$$v_\alpha(A, p_\alpha; a) \equiv \sup\{|\lambda| : \lambda \in V_\alpha(A, p_\alpha; a)\} ,$$

and we define the *numerical radius of* a as

$$v(A, \{p_\alpha\}; a) \equiv \sup\{|\lambda| : \lambda \in V(A, \{p_\alpha\}; a)\} .$$

We note that $v_\alpha(A, p_\alpha; a) \leqq p_\alpha(a)$ for each α, and we allow $v(A, \{p_\alpha\}; a) = \infty$. We have that

$$\begin{aligned} v(A, \{p_\alpha\}; a) &= \sup_\alpha v_\alpha(A, p_\alpha; a) \\ &= \sup_\alpha v(A_\alpha, \|\cdot\|_\alpha; a_\alpha) . \end{aligned}$$

It is clear that the numerical range and the numerical radius have the following properties. For $a \in A$ and λ, μ complex

$$V(A, \{p_\alpha\}; \lambda a + \mu) = \lambda V(A, \{p_\alpha\}; a) + \mu$$

and

$$v(A, \{p_\alpha\}; \lambda a + \mu) \leqq |\lambda| v(A, \{p_\alpha\}; a) + |\mu| ,$$

and for $a, b \in A$

$$V(A, \{p_\alpha\}; a + b) \subseteqq V(A, \{p_\alpha\}; a) + V(A, \{p_\alpha\}; b)$$

and

$$v(A, \{p_\alpha\}; a + b) \leqq v(A, \{p_\alpha\}; a) + v(A, \{p_\alpha\}; b) .$$

2. The numerical ranges and the spectrum. In a unital Banach algebra the numerical range of an element approximates its spectrum. We now establish similar approximating relations between the numerical ranges and the spectrum of an element in a complete unital l.m.c. algebra.

We recall that, given an algebra A with identity, for each $a \in A$, the *spectrum of* a is defined as the set

$$\sigma(A; a) \equiv \{\lambda : a - \lambda \text{ is not invertible}\} .$$

THEOREM 1. *Let A be a complete unital l.m.c. algebra. Given $(A, \{p_\alpha\})$, for each $a \in A$*

$$\sigma(A, a) \subseteqq V(A, \{p_\alpha\}; a) .$$

Proof. For each α let $\bar{A}_\alpha$ denote the completion of A_α. We have from Corollary 5.3(a), [4, p. 22] that

$$\sigma(A; a) = \bigcup_\alpha \sigma(\bar{A}_\alpha; a_\alpha) .$$

But from Theorem 2.6, [2, p. 19] we have that

$$\sigma(\bar{A}_\alpha; a) \subseteqq V(\bar{A}_\alpha, \|\cdot\|_\alpha; a_\alpha) ,$$

and from Theorem 2.4, [2, p. 16] that

$$V(\bar{A}_\alpha, \|\cdot\|_\alpha; a_\alpha) = V(A_\alpha, \|\cdot\|_\alpha; a_\alpha)$$

so it follows that

$$\begin{aligned}\sigma(A; a) &\subseteqq \bigcup_\alpha \{V(A_\alpha, \|\cdot\|_\alpha; a_\alpha)\} \\ &= V(A, \{p_\alpha\}; a) .\end{aligned}$$

THEOREM 2. *Let A be a complete unital l.m.c. algebra. For each $a \in A$*

$$\operatorname{co} \sigma(A; a) \subseteqq \bigcap \{V(A, \{p_\alpha\}; a) : \{p_\alpha\} \in P(A)\} \subseteqq \overline{\operatorname{co}}\, \sigma(A; a) .$$

Proof. From Theorem 1 we have that

$$\operatorname{co} \sigma(A; a) \subseteqq \bigcap \{V(A, \{p_\alpha\}, a); \{p_\alpha\} \in P(A)\} .$$

If $\overline{\operatorname{co}}\, \sigma(A; a)$ is not all the complex plane then, for any $\lambda \notin \overline{\operatorname{co}}\, \sigma(A; a)$ there exists an open disc D_λ center λ such that D_λ can be strictly separated from $\overline{\operatorname{co}}\, \sigma(A; a)$ by a straight line L. Since

$$\sigma(A; a) = \bigcup_\alpha \sigma(\bar{A}_\alpha; a_\alpha) ,$$

D_λ is strictly separated from $\sigma(\bar{A}_\alpha; a_\alpha)$ for any α, by the straight line L. However, for each α, $\sigma(\bar{A}_\alpha; a_\alpha)$ is a compact set so there exists an open disc $D_\alpha \supseteqq \sigma(\bar{A}_\alpha; a_\alpha)$ which is strictly separated from D_λ by the same straight line L. We have from [2, p. 23] that, for each α, there exists a norm $\|\cdot\|'_\alpha$ equivalent to $\|\cdot\|_\alpha$ on $\bar{A}_\alpha$ such that

$$\sigma(\bar{A}_\alpha; a_\alpha) \subseteqq V(\bar{A}_\alpha, \|\cdot\|'_\alpha; a_\alpha) \subseteqq D_\alpha \,.$$

Now for each α,

$$V(\bar{A}_\alpha, \|\cdot\|'_\alpha; a_\alpha) = V(A, \|\cdot\|'_\alpha; a_\alpha) \,.$$

Defining the semi-norm p'_α on A by

$$p'_\alpha(x) = \|x_\alpha\|'_\alpha \,,$$

it is clear that the family $\{p'_\alpha\} \in P(A)$, and

$$V(A, \{p'_\alpha\}; a) = \bigcup_\alpha \{V(A_\alpha, \|\cdot\|'_\alpha; a_\alpha)\} \,.$$

So D_λ is strictly separated from $V(A, \{p'_\alpha\}, a)$ by the straight line L. It follows that D_λ is strictly separated from $\bigcap \{V(A, \{p_\alpha\}; a) \colon \{p_\alpha\} \in P(A)\}$, and this implies that

$$\bigcap \{V(A, \{p_\alpha\}; a) \colon \{p_\alpha\} \in P(A)\} \subseteqq \overline{\mathrm{co}}\, \sigma(A; a) \,.$$

3. Elements with bounded numerical range. We now establish an important set of inequalities which are generalizations of an inequality from the normed algebra theory, and we use them to characterize elements with bounded numerical range.

LEMMA 1. *Let A be a unital l.m.c. algebra. Given $(A, \{p_\alpha\})$, for $a \in A$ and each α*

$$v(A, \{p_\alpha\}, a) \geqq \frac{1}{e} p_\alpha(a) \,.$$

Proof. From Theorem 4.1, [2, p. 34] we have, for each α

$$v(A_\alpha, \|\cdot\|_\alpha; a_\alpha) \geqq \frac{1}{e} \|a_\alpha\|_\alpha \,.$$

So

$$\begin{aligned} v(A, \{p_\alpha\}, a) &= \sup_\alpha v(A_\alpha, \|\cdot\|_\alpha; a_\alpha) \\ &\geqq \frac{1}{e} \|a_\alpha\|_\alpha = \frac{1}{e} p_\alpha(a) \,, \end{aligned}$$

for each α.

From the fact that every $\{p_\alpha\} \in P(A)$ is a separating family we can make the following deduction.

COROLLARY 1. *If for a given* $a \in A$, *there exists an* $(A, \{p_\alpha\})$ *such that* $V(A, \{p_\alpha\}; a) = \{0\}$ *then* $a = 0$.

We can also make a statement about elements with bounded numerical range.

COROLLARY 2. *Given* $(A, \{p_\alpha\})$, *if for* $a \in A$, $V(A, \{p_\alpha\}; a)$ *is bounded then* $\sup_\alpha p_\alpha(a) < \infty$.

For the characterization of the set of elements with bounded numerical range we also use the following lemma.

LEMMA 2. *Let A be a unital l.m.c. algebra. Given* $(A, \{p_\alpha\})$ *we have, for each* $a \in A$

$$\sup \operatorname{Re} V(A, \{p_\alpha\}; a) = \left.\begin{matrix} \inf\limits_{\lambda>0} \\ \lim\limits_{\lambda\to 0+} \end{matrix}\right\} \frac{1}{\lambda}\left\{\sup_\alpha p_\alpha(1 + \lambda a) - 1\right\} .$$

Proof. It is clear that for any $f \in D(A, \{p_\alpha\}; 1)$ and $\lambda > 0$

$$\operatorname{Re} f(a) \leqq \frac{1}{\lambda}\left\{\sup_\alpha p_\alpha(1 + \lambda a) - 1\right\}$$

and therefore,

$$\sup \operatorname{Re} V(A, \{p_\alpha\}, a) \leqq \inf_{\lambda>0} \frac{1}{\lambda}\left\{\sup_\alpha p_\alpha(1 + \lambda a) - 1\right\} . \tag{1}$$

It follows that the result holds when $V(A, \{p_\alpha\}; a)$ is unbounded. We consider the case when $V(A, \{p_\alpha\}; a)$ is bounded, and write for every α

$$\tau_\alpha \equiv \sup \operatorname{Re} V_\alpha(A_\alpha, \|\cdot\|_\alpha; a_\alpha) ,$$

and

$$\tau \equiv \sup \operatorname{Re} V(A, \{p_\alpha\}; a) .$$

Now by [2, p. 18], for every α

$$\frac{1}{\lambda}\{\| 1_\alpha + \lambda a_\alpha \|_\alpha - 1\} \leqq (1 - \lambda\tau_\alpha)^{-1}\{\tau_\alpha + \lambda \| a_\alpha^2 \|_\alpha\} ,$$

when $0 < \lambda < \| a_\alpha \|_\alpha^{-1}$. Since $V(A, \{p_\alpha\}; a)$ is bounded we have from Corollary 2 to Lemma 1 that there exists an $M > 0$ such that $M \geqq$

$\sup_\alpha p_\alpha(a)$. Then, for every α

$$\frac{1}{\lambda}\{p_\alpha(1 + \lambda a) - 1\} \leqq (1 - \lambda\tau)^{-1}\{\tau + \lambda M^2\}$$

when $0 < \lambda < 1/M$. Therefore,

$$\lim_{\lambda\to 0+} \frac{1}{\lambda}\left\{\sup_\alpha p_\alpha(1 + \lambda a) - 1\right\} \leqq \tau .$$

Together with inequality (1), this completes the proof.

Let A be a unital l.m.c. algebra. Given $(A, \{p_\alpha\})$ we can define the subalgebra

$$B \equiv \left\{x \in A\colon \sup_\alpha p_\alpha(x) < \infty\right\} .$$

Now $p(x) \equiv \sup_\alpha p_\alpha(x)$ is a norm for B since $\{p_\alpha\}$ is a separating family and we note that $1 \in B$ and $p(1) = 1$.

It can be seen from the proof of Theorem 2.3, [1, p. 32], that if A is a complete unital l.m.c. algebra then given $(A, \{p_\alpha\})$, the normed subalgebra (B, p) is complete. However, an examination of the sequence $\{x_n\}$ where $x_n = \{1, 2, 3, \cdots, n, \cdots, n, \cdots\}$, in the algebra A of Example 2 below, shows that there exists an incomplete unital l.m.c. algebra A with $\{p_\alpha\} \in P(A)$ such that (B, p) is complete.

Given $(A, \{p_\alpha\})$, we can characterize elements with bounded numerical range as elements of (B, p).

THEOREM 3. *If A is a unital l.m.c. algebra then given $(A, \{p_\alpha\})$,*

$$B = \{x \in A\colon V(A, \{p_\alpha\}; x) \text{ is bounded}\}$$

and when $\{p_\alpha\}$ is a directed family, for every $a \in B$

$$\overline{V(A, \{p_\alpha\}; a)} = V(B, p; a) .$$

Proof. If, for a given $a \in A$, $V(A, \{p_\alpha\}; a)$ is bounded then Corollary 2 to Lemma 1 implies that $a \in B$. If $a \in B$ then $\sup_\alpha p_\alpha(1 + \lambda a) = p(1 + \lambda a)$ for all λ, so

$$\sup \operatorname{Re} V(A, \{p_\alpha\}; a) = \left.\begin{matrix}\inf\limits_{\lambda>0} \\ \lim\limits_{\lambda\to 0+}\end{matrix}\right\} \frac{1}{\lambda}\{p(1 + \lambda a) - 1\}$$

$$= \sup \operatorname{Re} V(B, p; a) .$$

Hence, since $V(B, p; \lambda a) = \lambda V(B, p; a)$ and $V(A, \{p_\alpha\}; \lambda a) = \lambda V(A, \{p_\alpha\}; a)$ for all λ complex, $|\lambda| = 1$, we deduce that every $a \in B$ has bounded numerical range $V(A, \{p_\alpha\}; a)$. When $\{p_\alpha\}$ is a directed family, both numerical ranges are convex sets so we deduce from the Krein-Milman Theorem that for every $a \in B$

$$\overline{V(A, \{p_\alpha\}; a)} = V(B, p; a) .$$

The following result relates boundedness of the spectrum to boundedness of the numerical range.

THEOREM 4. *Let A be a complete unital l.m.c. algebra. For any $a \in A$, $\sigma(A; a)$ is bounded if and only if there exists an $(A, \{p_\alpha\})$ such that $V(A, \{p_\alpha\}; a)$ is bounded.*

Proof. If for $a \in A$ there exists an $(A, \{p_\alpha\})$ such that $V(A, \{p_\alpha\}; a)$ is bounded, then it follows from Theorem 1 that $\sigma(A; a)$ is bounded.

Conversely, consider $a \in A$ with $\sigma(A; a)$ bounded. There exists a disc D in the complex plane such that $\sigma(A; a) \subseteqq D$. Now $\sigma(A; a) = \bigcup_\alpha \sigma(\bar{A}_\alpha; a_\alpha)$. From [2, p. 23], for each α there exists a norm $\|\cdot\|'_\alpha$ equivalent to $\|\cdot\|_\alpha$ on $\bar{A}_\alpha$ such that

$$\sigma(\bar{A}_\alpha; a_\alpha) \subseteqq V(\bar{A}_\alpha; \|\cdot\|'_\alpha; a_\alpha) \subseteqq D .$$

For each α, defining the semi-norm p'_α on A by

$$p'_\alpha(x) = \|x_\alpha\|'_\alpha ,$$

the family $\{p'_\alpha\} \in P(A)$ and

$$\sigma(A; a) \subseteqq V(A, \{p'_\alpha\}; a) = \bigcup_\alpha V(A_\alpha, \|\cdot\|'_\alpha, a_\alpha) \subseteqq D .$$

Further to the relation between boundedness of the spectrum and the numerical range given in Theorem 4, the following example shows that there exist l.m.c. algebras A where $\sigma(A; a)$ is bounded for a given $a \in A$ but where there exists $(A, \{p_\alpha\})$ such that $V(A, \{p_\alpha\}; a)$ is unbounded.

EXAMPLE 1. Let A be the algebra of all sequences of complex numbers, $x \equiv \{\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots\}$ with pointwise definition of addition and multiplication by a scalar, but with convolution multiplication and with unit $1 \equiv \{1, 0, \cdots, 0, \cdots\}$. A sequence of submultiplicative semi-norms $\{p_n\}$ is defined on A by

$$p_n(x) = \sum_{k=1}^{n} |\lambda_k| ,$$

and the sequence satisfies $p_n(1) = 1$ for all n, and is separating. Consider $a \in A$ such that $\lambda_n \nrightarrow 0$. Then $p_n(a) \to \infty$, so by Lemma 1 $V(A, \{p_n\}; a)$ is unbounded. But $\sigma(A; a) = \{\lambda_1\}$, which is bounded.

It is worth noting that a complete unital l.m.c. algebra with a bounded numerical range property has the following property.

THEOREM 5. *If A is a complete unital l.m.c. algebra where there exists an $(A, \{p_\alpha\})$ such that $V(A, \{p_\alpha\}; a)$ is bounded for all $a \in A$, then $\sigma(A; a)$ is compact for all $a \in A$.*

Proof. We note that $B = A$ and so $\sigma(A; a) = \sigma(B; a)$, for every $a \in A$. Since A is complete it follows that (B, p) is complete and so $\sigma(B; a)$ is compact for every $a \in A$.

However, the algebra A of Example 1 has $\sigma(A; a)$ compact for all $a \in A$ but there exists an $(A, \{p_\alpha\})$ such that $V(A, \{p_\alpha\}; a)$ is not bounded for all $a \in A$.

It is known that there exist non-normable l.m.c. algebras where $\sigma(A; a)$ is compact for all $a \in A$, [4, p. 80]. The following example gives the further information that there exist non-normable l.m.c. algebras A where, for a certain $(A, \{p_\alpha\})$, $V(A, \{p_\alpha\}; a)$ is bounded for all $a \in A$.

EXAMPLE 2. Let A be the algebra l^∞ of all bounded sequences of complex numbers, $x \equiv \{\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots\}$, with pointwise definition of the algebra operations and with unit $1 \equiv \{1, 1, \cdots, 1, \cdots\}$. A sequence of submultiplicative semi-norms $\{p_n\}$ is defined on A by

$$p_n(x) = |\lambda_n| ,$$

and the sequence satisfies $p_n(1) = 1$ for all n, and is separating. Now p, defined by

$$p(x) \equiv \sup_n \{p_n(x)\}$$

is the usual l^∞-norm on A, so $B = A$ and from Theorem 3, $V(A, \{p_n\}; a)$ is bounded for all $a \in A$. However, it is clear that $(A, \{p_n\})$ is non-normable.

4. A characterization of b^*-algebras by numerical range. A *l.m.c. * algebra* A is a l.m.c. algebra with a continuous involution $*$. We let $S(A)$ denote the set $\{x \in A: x = x^*\}$, the selfadjoint elements of A. A *b^*-algebra* A is a complete l.m.c. * algebra where there exists a family $\{p_\alpha\} \in P(A)$ such that $p_\alpha(x^*x) = p_\alpha(x)^2$ for all $x \in A$ and every α, [1, p. 31].

In a unital normed algebra $(A, \|\cdot\|)$, the *set of hermitian elements* $H(A, \|\cdot\|)$ is the set of elements a with real numerical range $V(A, \|\cdot\|; a)$. For a unital l.m.c. algebra A, given $(A, \{p_\alpha\})$ we define the *set of hermitian elements* $H(A, \{p_\alpha\})$ as the set of elements a with real numerical range $V(A, \{p_\alpha\}; a)$. It is clear from the definition of the numerical range in $(A, \{p_\alpha\})$ that $a \in H(A, \{p_\alpha\})$ if and only if $a_\alpha \in H(A_\alpha, \|\cdot\|_\alpha)$ for all α.

One of the outstanding successes of the normed algebra numerical range theory is the Vidav-Palmer Theorem [2, p. 65] which characterizes unital B^* algebras as unital Banach algebras which have an hermitian decomposition. We now consider a generalization of this work to the characterization of unital b^*-algebras amongst the complete unital l.m.c. algebras.

We need the following property of the hermitian elements.

LEMMA 3. *Let A be a complete unital l.m.c. algebra. Given $(A, \{p_\alpha\})$, the set $H(A, \{p_\alpha\})$ is closed.*

Proof. Consider h a cluster point of $H(A, \{p_\alpha\})$. Then, for each α, h_α is a cluster point of $H(A_\alpha, \|\cdot\|_\alpha)$. But by Lemma 7, [5, p. 198], $H(A_\alpha, \|\cdot\|_\alpha)$ is closed in $(A_\alpha, \|\cdot\|_\alpha)$, so $h_\alpha \in H(A_\alpha, \|\cdot\|_\alpha)$. Since $V(A, \{p_\alpha\}; h) = \bigcup_\alpha V(A_\alpha, \|\cdot\|_\alpha; h_\alpha)$ we have that $h \in H(A, \{p_\alpha\})$.

THEOREM 6. *Let A be a complete unital l.m.c. algebra. Given $(A, \{p_\alpha\})$, the following statements are equivalent.*

(i) *$A = H(A, \{p_\alpha\}) + iH(A, \{p_\alpha\})$, a direct sum,*

(ii) *There is an involution $*$ on A such that A is a l.m.c. $*$ algebra where $S(A) = H(A, \{p_\alpha\})$,*

(iii) *There is an involution $*$ on A such that A is $*$ isomorphic to a $*$ subalgebra of a product of B^*-algebras $(\bar{A}_\alpha, \|\cdot\|_\alpha)$,*

(iv) *There is an involution $*$ on A such that A is a b^*-algebra,*

(v) *There is an involution on B such that (B, p) is a dense B^*-algebra.*

Proof. (i) $\Rightarrow$ (ii) Since $A = H(A, \{p_\alpha\}) + iH(A, \{p_\alpha\})$, we define the involution $*$ on A as follows: for $x = h + ik$ where $h, k \in H(A, \{p_\alpha\})$ put $x^* = h - ik$. We need to show that $*$ is continuous on A. Now for every α, $A_\alpha = H(A_\alpha, \|\cdot\|_\alpha) + iH(A_\alpha, \|\cdot\|_\alpha)$ and $*$ induces an involution $*$ on A_α where for $x_\alpha = h_\alpha + ik$ we have $x_\alpha^* = h_\alpha - ik_\alpha$. But from Lemma 5.8, [2, p. 50], since $A_\alpha \subseteq J(\bar{A}_\alpha)$, we have for every α, that $*$ is continuous on A_α and since $p_\alpha(x) = \|x_\alpha\|_\alpha$ for all $x \in A$, $*$ is continuous on A. It is clear that with this involution $*$, $S(A) = H(A, \{p_\alpha\})$.

(ii) $\Rightarrow$ (iii) Since $H(A, \{p_\alpha\}) = S(A)$ we have $A = H(A, \{p_\alpha\}) + iH(A, \{p_\alpha\})$ and so, for every α, $A_\alpha = H(A_\alpha, \|\cdot\|_\alpha) + iH(A_\alpha, \|\cdot\|_\alpha)$. But by Theorem 8.2, [2, p. 74], A_α is a pre-B^*-algebra and so $\bar{A}_\alpha$ is a B^*-algebra for every α. Our result follows from Michael's characterization of l.m.c. algebras, Proposition 2.7, [4, p. 13].

(iii) $\Rightarrow$ (iv) For every α, since $(\bar{A}_\alpha, \|\cdot\|_\alpha)$ is a B^*-algebra and $p_\alpha(x) = \|x_\alpha\|_\alpha$ for all $x \in A$, we have $p_\alpha(x^*x) = p_\alpha(x)^2$ for all $x \in A$; that is, A is a b^*-algebra.

(iv) $\Rightarrow$ (v) This is proved as Theorem 2.3, [1, p. 32].

(v) $\Rightarrow$ (i) Since (B, p) is a unital B^*-algebra, $B = H(B, p) + iH(B, p)$. For any $h, k \in H(B, p)$, we have from Theorem 3, that $h, k \in H(A, \{p_\alpha\})$. But then for every α, $h_\alpha, k_\alpha \in H(A_\alpha, \|\cdot\|_\alpha)$ and since $\|x_\alpha\|_\alpha = p_\alpha(x)$ for all $x \in A$, we have from inequality (1), [2, p. 50] that

$$p_\alpha(h) \leqq e p_\alpha(h + ik) .$$

This inequality implies that for any net $\{h_\gamma + ik_\gamma\}$ in B convergent to x in A, both $\{h_\gamma\}$ and $\{k_\gamma\}$ converge to say h and k. But by Lemma 3 the set $H(A, \{p_\alpha\})$ is closed in A so $h, k \in H(A, \{p_\alpha\})$ and $x = h + ik$. Since B is dense in A, we have $A = H(A, \{p_\alpha\}) + iH(A, \{p_\alpha\})$.

It should be noted that this theorem gives in (v) $\Rightarrow$ (iv), a converse to Theorem 2.3, [1, p. 32], and in (iv) $\Rightarrow$ (iii), a simpler proof for Theorem 2.4, [1, p. 32], by using numerical range techniques.

The following is an application of Theorem 6 and is a generalization of Theorem 7.6, [2, p. 71].

THEOREM 7. *Let Ω be a locally compact Hausdorff space and let $A \equiv \mathscr{C}(\Omega)$ be the algebra of all complex continuous functions on Ω. If A is an F-algebra under the compact-open topology, then any l.m.c. topology generated by a family of semi-norms $\{p_\alpha\}$ such that $p_\alpha(f) = p_\alpha(|f|)$ for all $f \in A$ and $p_\alpha(1) = 1$ for all α, under which A is an F-algebra, is the compact-open topology.*

Proof. We can introduce the exponential function in A, $\exp a = 1 + \sum_{n=1}^{\infty} (1/n!)a^n$, and it is clear that $(\exp a)_\alpha = \exp a_\alpha$, for every α. Now if g is a real continuous function on Ω then for real λ and for each α

$$\begin{aligned} \|\exp(i\lambda g_\alpha)\|_\alpha &= \|(\exp i\lambda g)_\alpha\|_\alpha \\ &= p_\alpha(\exp i\lambda g) \\ &= p_\alpha(|\exp i\lambda g|) \\ &= p_\alpha(1) = 1 . \end{aligned}$$

Therefore, by Lemma 5.2, [2, p. 46], $g_\alpha \in H(A_\alpha, \|\cdot\|_\alpha)$ for every α, and so $g \in H(A, \{p_\alpha\})$. We have then $A = H(A, \{p_\alpha\}) + iH(A, \{p_\alpha\})$ and by Theorem 6 we conclude that A is a b^*-algebra. Now, by Theorem 4.2, [1, p. 36], A with the compact-open topology is also a b^*-algebra. So by Theorem 3.7, [1, p. 35], the l.m.c. topology generated by $\{p_\alpha\}$ is the compact-open topology.

In the above theorem we note that A with the compact-open topology and A with the l.m.c. topology generated by the semi-norms, must both be F-algebras; one of these being an F-algebra is not sufficient. We are indebted to the referee for the following examples which illustrate this point.

EXAMPLE 3. Let $\Omega \equiv [0, 1]$ with the usual topology and let $\{K_\alpha\}$ be the set of compact countable subsets of Ω. A family of semi-norms $\{p_\alpha\}$ defined on A by

$$p_\alpha(x) = \sup_{x \in K_\alpha} \{|f(x)|\} ,$$

satisfies the conditions of the theorem except that A with this topology is not an F-algebra, [4, Example 3.8, p. 19]. However, A with the compact-open topology is a Banach algebra, so it is clear that the topology generated by the family of semi-norms is not the compact-open topology.

EXAMPLE 4. Let Ω be the set of ordinal numbers smaller than the first uncountable ordinal, with the order topology. With norm p defined on A by

$$p(x) = \sup_{x \in \Omega} \{|f(x)|\} ,$$

A is a Banach algebra. However, A with the compact-open topology is not an F-algebra, [4, Example 3.7, p. 19], so it is clear that the norm topology on A is not the compact-open topology.

Note added in proof. We are indebted to Dr. R. T. Moore for pointing out, in connection with Example 2, that the following result can be deduced from Theorem 3 by the Open Mapping Theorem.

THEOREM. *A unital F-algebra A is normable if and only if there exists an $(A, \{p_\alpha\})$ such that $V(A, \{p_\alpha\}; a)$ is bounded for all $a \in A$.*

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