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# IRREDUCIBLE SUMS OF SIMPLE MULTIVECTORS

HERBERT BUSEMANN AND D. EDWARD GLASSCO II

**Denoting by $V^n(F)$ the $n$-dimensional vector space over the field $F$ of characteristic 0, let $V_r^n(F)$ be the linear space of all $r$-vectors $\tilde{R}$ over $V^n(F)$ and $G_r^n(F)$ the Grassmann cone of the simple $r$-vectors $R$ in $V_r^n(F)$. The sum $\tilde{R} = \sum_{i=1}^k R_i (R_i \in G_r^n(F))$ is irreducible if $\tilde{R}$ is not the sum of fewer than $k$ elements of $G_r^n(F)$. (Duality reduces the interesting cases to $2 \leqq r \leqq n/2$.) Such sums are trivial only for $r = 2$, because $\bigwedge_{i=1}^k R_i \neq 0$ while always sufficient for irreducibility is then also necessary. Extension of $F$ does not influence irreducibility if $r = 2$ but it can for $r > 2$.**

**The sets $W_r^n(F, k)$ of those $\tilde{R}$ in $V_r^n(F)$ which are irreducible sums of $k$ terms behave as expected when $r = 2$, but have the most surprising properties for larger $r$. Although $V_3^6(F) = \bigcup_{k=1}^3 W_3^6(F, k)$ and $W_3^6(F, 3) \neq \phi$, the sets $W_3^6(\boldsymbol{R}$ or $\boldsymbol{C}, 2)$ have interior points as sets in $V_3^6(\boldsymbol{R}$ resp. $\boldsymbol{C})$ and so does $W_3^6(\boldsymbol{R}, 3)$ but $W_3^6(\boldsymbol{C}, 3)$ does not.**

The paper is based on the thesis [1] with the same title by the second author.

The smallest number $k$ for which $V_r^n(F, k) = \bigcup_{i=1}^k W_r^n(F, i)$ coincides with $V_r^n(F)$ is denoted by $N(F, n, r)$ which by duality equals $N(F, n, n - r)$. Obviously $N(F, n, r) \leqq \binom{n}{r}$. But in spite of various inequalities relating these numbers which show that $\binom{n}{r}$ is much too large, the precise value of $N(F, n, r)$ is known only in the two cases implied by the above statements: namely $N(F, n, 2) = [n/2]$ and $N(F, 6, 3) = 3$.

The values $N(\boldsymbol{C}, 7, 3) = 5$, $N(\boldsymbol{C}, 8, 3) = 7$, and $N(\boldsymbol{C}, 9, 3) = 10$ have been claimed but questioned, see Schouten [3, p. 27] and [1].

The purpose of our investigation is to elucidate why the case $r = 2$ is so much simpler than $2 < r < n - 2$. In addition to the already mentioned facts we show that $V_2^n(F, k)$ is an algebraic variety, because, if $\tilde{R}^{(i)}$ is the $i$th exterior power of $\tilde{R}$, then $\tilde{R}^{(k+1)} = 0$ is necessary and sufficient for $\tilde{R} \in V_r^n(F, k)$ when $r = 2$, but merely necessary when $r > 2$. This implies $\dim V_2^n(\boldsymbol{R}$ resp. $\boldsymbol{C}, k) < \dim V_2^n(\boldsymbol{R}$ resp. $\boldsymbol{C}, k + 1)$ for $1 \leqq k < [n/2]$ in contrast to the case $n = 6, r = 3$. In fact we show that $V_r^n(\boldsymbol{R}$ or $\boldsymbol{C}, k)$ is for $r > 2, k > 1$, and $n \geqq (k - 1)r + 3$ not even a closed set.

An irreducible representation $\tilde{R} = \sum_{i=1}^k R_i, k > 1$, is for $r = 2$ never unique, but for $r > 2$ it is (up to a permutation) if $\bigwedge_{i=1}^k R_i \neq 0$ and $k \leqq r$. The condition $k \leqq r$ is probably superfluous but enters —like $n \geqq (k - 1)r + 3$ (instead of $n \geqq r + 3$) above—because we use the Plücker relations for simple vectors which get out of hand for large $k$. A coordinate-free approach would therefore be preferable, but in many cases we were not able to devise one.

We will continue using capitals ($R$, $S$, $T$) with a tilde and with or without subscripts for general multivectors and omit the tilde only when the vectors are known or assumed to be simple.

2. **Results for general $F, n, r, k$.** The following agreement will prove convenient. $e_1, e_2, \cdots$ are used for elements of a base. If two spaces $V^m \subset V^n$ occur, then the base $e_1, \cdots, e_n$ of $V^n$ is chosen so that $e_1, \cdots, e_m$ is a base of $V^m$. We begin with some simple remarks.

(2.1) *If* $R \in G_r^n$ *then* $R = R' + S \wedge e_n$ *with* $R' \in G_r^{n-1}$ *and* $S \in G_{r-1}^{n-1}$.

For, with suitable $v_i \in V^{n-1}$ and $\beta_i$

$$R = \bigwedge_{i=1}^{r} (v_i + \beta_i e_n)$$

$$= \bigwedge_{i=1}^{r} v_i + [\sum_{i=1}^{r} (-1)^{n-i} \beta_i v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_r] \wedge e_n .$$

If the $v_i$ are dependent, the bracket reduces to one term; if not, the bracket is an $(r-1)$-vector in the $r$-space spanned by $v_1, \cdots, v_r$ and hence is simple.

We apply (2.1) to prove

(2.2) $\tilde{R} \in W_r^n(F, k)$ *if and only if* $\tilde{R} \wedge e_{n+1} \wedge \cdots \wedge e_{n+m} \in W_{r+m}^{n+m}(F, k)$.

It suffices to prove this for $m = 1$. We show if $\tilde{R} \in W_r^n(k)$ and $\tilde{R} \wedge e_{n+1} \in W_{r+1}^{n+1}(l)$, then $l = k$. Trivially $\tilde{R} \wedge e_{n+1} \in V_{r+1}^{n+1}(k)$, whence $l \leqq k$. By (2.1) and the hypothesis $\tilde{R} \wedge e_{n+1} = \sum_{i=1}^{l} R_i = \sum_{i=1}^{l} (R_i' + S_i \wedge e_{n+1})$ with $R_i \in G_{r+1}^{n+1}$, $R_i' \in G_{r+1}^n$, and $S_i \in G_r^n$. Therefore, $\Sigma R_i' = 0$ and $\tilde{R} \wedge e_{n+1} = (\Sigma S_i) \wedge e_{n+1}$, which implies $\tilde{R} = \sum_{i=1}^{l} S_i$ and $k \leqq l$.

COROLLARY 2.3. $N(F, n+1, r+1) \geqq N(F, n, r)$.

Anticipating $N(F, n, 2) = [n/2]$ we see that both equality and inequality occur. $N(2m, 2m-2) = N(2m, 2) > N(2m-1, 2) = N(2m-1, 2m-3)$. Similarly $N(2m+1, 2m-1) = N(2m, 2m-2)$. Also $N(n, r) \geqq [(n-r+2)/2]$, but this lower bound is for $r > 2$ too small to be useful.

A consequence of (2.1) is the generalization

(2.4) *If* $\tilde{R} \in V_r^n(k)$, *then* $\tilde{R} = \tilde{R}' + \tilde{S} \wedge e_n$ *with* $\tilde{R}' \in V_r^{n-1}(k)$ *and*

$\tilde{S} \in V_{r-1}^{n-1}(k)$.

By hypothesis $\tilde{R} = \sum_{i=1}^{l} R_i (l \leqq k,\ R_i \in G_r^n)$. Applying (2.1) to each $R_i$ yields $\tilde{R} = \sum_{i=1}^{l} (R_i' + S_i \wedge e_n) = \sum_{i=1}^{l} R_i' + (\sum_{i=1}^{l} S_i) \wedge e_n$ with $R_i' \in G_r^{n-1}$ and $S_i \in G_{r-1}^{n-1}$, which is the assertion.

With $k = N(F, n, r)$ we deduce from (2.4):

(2.5) $N(F, n, r) \leqq N(F, n-1, r) + N(F, n-1, r-1)$.

For $r = 2$ equality holds when $n$ is even and inequality holds when $n$ is odd.

A linear map $f\colon U^m \longrightarrow V^n$ induces a homomorphism $f^*\colon U_r^m \longrightarrow V_r^n$ given by $f^*(u_1 \wedge \cdots \wedge u_r) = f(u_1) \wedge \cdots \wedge f(u_r)$. The map $f^*$ is surjective when $f$ is. We note

(2.6) *If* $f^*(R_1) + \cdots + f^*(R_k)$ *is irreducible in* $V_r^n$, *then so is* $R_1 + \cdots + R_k$ *in* $U_r^m$.

We apply this first to the projection $f\colon V^{n+i} \longrightarrow V^n$ defined by

$$f\colon \sum_{i=1}^{n+k} \alpha^i e_i \longrightarrow \sum_{i=1}^{n} \alpha^i e_i$$

and find:

(2.7) *If* $R_i \in G_r^n(F)$ *and* $\sum_{i=1}^{k} R_i$ *is irreducible in* $V_r^n(F)$, *then it is irreducible in* $V_r^{n+k}(F)$.

Hence

(2.8) $N(F, n+1, r) \geqq N(F, n, r)$.

The case $r = 2$ shows again that both inequality and equality can occur in (2.8). Next we apply (2.6) to the map $f\colon V^{n+k} \longrightarrow V^{n+1}$ given by

$$f\colon \sum_{i=1}^{n+k} \alpha^i e_i \longrightarrow \sum_{i=1}^{n} \alpha^i e_i + (\sum_{i=n+1}^{n+k} \alpha^i) e_{n+1}$$

and find using (2.2):

(2.9) *If* $\sum_{i=1}^{k} R_i$ *is irreducible in* $V_r^n(F)$, *then* $\sum_{i=1}^{k} R_i \wedge e_{n+i}$ *is irreducible in* $V_{r+1}^{n+k}(F)$.

Two important facts will now be proved together:

THEOREM 2.10. *If* $\bigwedge_{i=1}^{k} R_i \neq 0$, *then* $\sum_{i=1}^{k} R_i$ *is irreducible. The* *converse holds only for* $r = 2$.

THEOREM 2.11. *If* $\tilde{R} \in V_r^n(F, k)$ *then* $\tilde{R}^{(k+1)} = 0$. *The converse holds only for* $r = 2$.

If $r$ is odd then $\tilde{R}^{(i)} = 0$ for any $i > 1$ so that $\tilde{R}^{(k+1)} = 0$ imposes no condition. If $r$ is even the relation $(\sum_{i=1}^k R_i)^{(k+1)} = 0$ is obvious, so that the first part of (2.11) holds. Since

(2.12) $(\sum_{i=1}^k R_i)^{(k)} = k! \bigwedge_{i=1}^k R_i$ *for even* $r$

it follows that $\sum_{i=1}^k R_i \in W_r^n(k)$ when $\bigwedge_{i=1}^k R_i \neq 0$. Applying (2.9) we see that this also holds for odd $r$.

If $\bigwedge_{i=1}^k R_i = 0$, $r = 2$, and $R_i = v_i \wedge w_i$ then one of the $v_i$ or $w_i$ depends on the rest, say $v_k = \sum_{i=1}^{k-1} \lambda_i v_i + \sum_{i=1}^k \mu_i w_i$ so that

$$\sum_{i=1}^k R_i = \sum_{i=1}^{k-1} [v_i \wedge w_i + (\lambda_i v_i + \mu_i w_i) \wedge w_k] .$$

Each bracket represents a simple vector because it is a 2-vector in the space spanned by $v_i$, $w_i$, and $w_k$.

That $\bigwedge_{i=1}^k R_i \neq 0$ is necessary for irreducibility only when $r = 2$ follows from (2.2). This establishes (2.10).

It remains only to prove the second part of (2.11). Let $r = 2$ and $\tilde{R}^{(k+1)} = 0$. Then $\tilde{R} \in W_r^n(k + i)$ with $i \geqq 1$ is impossible because (2.10) and (2.12) would imply $\tilde{R}^{(k+i)} \neq 0$. That $\tilde{R}^{(k+1)} = 0$ is not sufficient for $\tilde{R} \in V_r^n(F, k)$ is obvious for odd $r$ and follows from (2.2) for even $r > 2$.

Corollaries of (2.10) resp. (2.11) are:

(2.13) $N(F, n, 2) = [n/2]$.

(2.14) *If* $\tilde{R} \in W_2^n(F, k)$ *then also* $\tilde{R} \in W_2^n(F_0, k)$ *for any extension field* $F_0$ *of* $F$. *This is not true for* $r > 2$.

The latter means that for each $n - 2 > r > 2$ there are $\tilde{R}$, $k' < k$, $F \subset F_0$ with $\tilde{R} \in W_r^n(F, k)$ and $\tilde{R} \in W_r^n(F_0, k')$, and follows from (5.9) and (2.2). Note: The first part of (2.14) does not mean, for example, that $\tilde{R} \in V_2^n(F)$, $\tilde{R} \in W_2^n(F_0, 2)$, hence $\tilde{R} = R_1 + R_2$ with $R_i \in G_2^n(F_0)$, imply $R_i \in G_2^n(F)$, but only that $R_i' \in G_2^n(F)$ with $\tilde{R} = R_1' + R_2'$ exist, compare (4.3).

Whereas in (2.2) and (2.9) the number of summands is the same in hypothesis and assertion, it is different in the next theorem which is therefore harder to prove.

THEOREM 2.15. *Let* $\tilde{R} \in W_r^n(F, k)$, $E_i = \bigwedge_{l=1}^r e_{n+(i-1)r+l} (i = 1, \cdots, j)$, *then* $\tilde{R} + \sum_{i=1}^j E_i \in W_r^{n+rj}(F, k + j)$.

Evidently it suffices to prove this for $j = 1$, or with $E = E_1$ that $\tilde{R} + E \in W_r^{n+r}(k + 1)$. Let $\tilde{R} + E = \sum_{i=1}^m S_i$, $S_i \in G_r^{n+r}$, and denote by $S_i'$ the projection of $S_i$ on $V_r^n$. Then $S_i'$ is simple and $\tilde{R} = \sum_{i=1}^m S_i'$. Therefore, $\tilde{R} \in W_r^n(k)$ implies $m \geqq k$ and that for $m = k$ all $S_i' \neq 0$. We show that $m = k$ is impossible.

There are at least two $S_i$ which do not lie in $G_r^n$. For, $S_i \in G_r^n$, if $i > 1$, would entail $S_1 = S_1' + E$ with $S_1' \wedge E \neq 0$, but $S_1' + E$ is not simple by (2.10). Assume that $S_1$ and $S_2$ do not lie in $G_r^n$. For $w = \sum_{i=1}^{n+r} \alpha^i e_i$, put $w' = \sum_{i=1}^n \alpha^i e_i$ and $w'' = \sum_{i=n+1}^{n+r} \alpha^i e_i$.
Then

$$S_i = \bigwedge_{j=1}^r w_{ij} = \bigwedge_{j=1}^r (w_i' + w_j'')$$

and we may assume further that $w_{11}'' \neq 0$ and $w_{21}'' \neq 0$.

There are subscripts $i, j, k, l$ with $i \neq k$ such that $w_{ij}'' \wedge w_{kl}'' \neq 0$. Otherwise $w_{11}'' \wedge w_{kl}'' = 0$ for $k \neq 0$ so that $w_{kl}'' = \lambda_{kl} w_{11}''$ for $k \neq 1$. Similarly $w_{kl}'' = \mu_{kl} w_{21}''$ for $k \neq 2$, so that $w_{1l}'' = \mu_{1l}\lambda_{21} w_{11}''$.

This, with $\lambda_{11} = 1$ and $\lambda_{1l} = \mu_{1l}\lambda_{21}$, gives

$$S_i = \bigwedge_{j=1}^r (w_j' + \lambda_{ij} w_{11}'') \ .$$

But then $\Sigma S_i$ cannot produce $E$. Thus we may assume (with a possible change of notation) that $w_{11}'' \wedge w_{21}'' \neq 0$. Then $e_1 \wedge \cdots \wedge e_n \wedge w_{11} \wedge w_{21} = e_1 \wedge \cdots \wedge e_n \wedge w_{11}'' \wedge w_{21}'' \neq 0$ and there is a base $\{e_i'\}$ of $V^{n+r}$ with $e_i' = e_i$ for $i \leqq n$, $e_{n+1}' = w_{11}$, and $e_{n+2}' = w_{21}$. Then with the original $\tilde{R}$, $E, S_1, \cdots, S_k$,

$$(\tilde{R} + E) \wedge e_{n+1}' \wedge e_{n+2}' = (S_3 + \cdots + S_k) \wedge e_{n+1}' \wedge e_{n+2}' \ ,$$

i.e.,

$$\tilde{R} \wedge e_{n+1}' \wedge e_{n+2}' \in W_{r+2}^{n+r}(k - 1)$$

contradicting (2.2) and (2.7),

3. The sets $V_r^n(F_t, k)$. Let $F_t$ be a topological field. Obviously $G_r^n(F_t) = V_r^n(F_t, 1) = W_r^n(F_t, 1)$ is a closed set in $V_r^n(F_t)$. It is clear that for $k < N(F_t, n, r)$ the set $V_r^n(F_t, k)$ cannot be open, but one might expect it to be closed. This is true for $r = 2$, see below, but in general not for $r > 2$. To show the latter it is not necessary to study general $n$ and $r > 2$ because of the following:

THEOREM 3.1. *If for a topological field* $F_t$ *the set* $V_r^n(F_t, k)$ *is* *not closed in* $V_r^n(F_t)$ *then for* $m \geqq n$, $s \geqq r$, $m - s \geqq n - r$ *and* $j \geqq 0$ *the set* $V_s^{m+js}(F_t, k+j)$ *is not closed in* $V_s^{m+js}(F_t)$.

First let $j = 0$, $m \geqq n$ and $\tilde{R} \in V_r^n(F_t, k)$. By (2.7) $\tilde{R} \in V_r^m(F_t, k)$ so that the latter is not closed. For any $m$ we conclude from $\tilde{R} \in V_r^m(F_t, k)$ and (2.2) that

$$\tilde{R} \wedge e_{m+1} \wedge \cdots \wedge e_{m+h} \in V_{r+k}^{m+k}(F_t, k) .$$

Since $V_r^m(F_t, k)$ is not closed there are $\tilde{R}_\nu$ in $V_r^m(F_t, k)(\nu = 1, 2, \cdots)$ such that $\tilde{R}_\nu \to \tilde{R} \in W_r^m(F_t, k')$ with $k' > k$.
Then by (2.2)

$$\tilde{R}_\nu \wedge e_{m+1} \wedge \cdots \wedge e_{m+h} \longrightarrow \tilde{R} \wedge e_{m+1} \wedge \cdots \wedge e_{m+h} \in W_{r+h}^{m+h}(F_t, k')$$

so that $V_{r+h}^{m+h}(F_t, k)$ is not closed. This settles the case $j = 0$ or that $V_s^m(F_t, k)$ is not closed.

With the notation of (2.15) we see with the same argument

$$V_s^{m+js}(F_t, k+j) \ni \tilde{R}_\nu + \sum_{i=1}^{j} E_i \longrightarrow \tilde{R} + \sum_{i=1}^{j} E_i \in W_s^{m+js}(F_t, k'+j)$$

which proves (3.1).

In §5 it will be shown that $N(F, 6, 3) = 3$ and $V_3^6(\boldsymbol{R}$ resp. $\boldsymbol{C}, 2)$ is not closed in $V_3^6(\boldsymbol{R}$ resp. $\boldsymbol{C})$. Probably no $V_r^n(\boldsymbol{R}$ resp. $\boldsymbol{C}, k)$ with $3 \leqq r \leqq n - 3$ and $1 < k < N(\boldsymbol{R}$ resp. $\boldsymbol{C}, n, r)$ is closed, but from (3.1) we obtain (with $2 + j = k$) this best possible result only for $k = 2$.

THEOREM 3.2. *The sets* $V_r^n(\boldsymbol{R}, k)$ *and* $V_r^n(\boldsymbol{C}, k)$ *are not closed in* $V_r^n(\boldsymbol{R})$ *resp.* $V_r^n(\boldsymbol{C})$ *when* $k \geqq 2$, $r \geqq 3$, *and* $n \geqq (k-1)r + 3$.

The mentioned best result would require a direct treatment of the case $k > 2$ instead of reduction to $k = 2$. The fact that we use Plücker relations in §5, which become very involved for large $n, r, k$, is responsible for our incomplete result in the case $k > 2$.

We now discuss the case $r = 2$. The by (2.11) necessary and sufficient condition $\tilde{R}^{(k+1)} = 0$ for $\tilde{R} \in V_2^n(F, k)$ amounts to polynomial conditions on the components $\alpha^{ik}$ of $\tilde{R} = \sum_{1 \leqq i < k \leqq n} \alpha^{ik} e_i \wedge e_k$. *The set* $V_2^n(F, k)$ *is therefore an algebraic cone in* $V_2^n(F)$ and hence closed when $F$ carries a topology.

It is also clear that for $1 \leqq k < k' \leqq [n/2]$ the set $V_2^n(F, k)$ is a proper subset of $V_2^n(F, k')$ and plausible but, since we do not know whether $V_2^n(F, k)$ is an irreducible manifold, not *a priori* certain, that the *dimension in the sense of algebraic geometry* (denoted by *a-dim*) and consequently in the case of $\boldsymbol{R}$ resp. $\boldsymbol{C}$ also the *topological dimension* (=dim), of $V_2^n(F, k)$ is less than that of $V_2^n(F, k')$. That a proof is necessary may be seen from the case $r = 3$ (see §§5 and 6). In spite of $N(F, 6, 3) = 3$ the sets $W_3^6(\boldsymbol{R}$ resp. $\boldsymbol{C}, 2)$ and $W_3^6(\boldsymbol{R}, 3)$ have nonempty interiors in $V_3^6(\boldsymbol{R}$ resp. $\boldsymbol{C})$ so that

$$\dim V_3^6(\boldsymbol{R} \text{ resp. } \boldsymbol{C},\ 2) = \dim V_3^6(\boldsymbol{R} \text{ resp. } \boldsymbol{C})$$
$$\dim W_3^6(\boldsymbol{R}, 2) = \dim W_3^6(\boldsymbol{R}, 3) = \dim V_3^6(\boldsymbol{R}) = 20 .$$

But $W_3^6(\boldsymbol{C}, 3)$ has no interior points and hence by a theorem in dimension theory (see [2, p. 46])

$$\dim W_3^6(\boldsymbol{C}, 3) < \dim W_3^6(\boldsymbol{C}, 2) = \dim V_3^6(\boldsymbol{C}) = 40 .$$

Although we need only the expression for $\tilde{R}^{(k)}$ in the case $r = 2$, we give, owing to its potential usefulness, the *expression of* $\bigwedge_{i=1}^k \tilde{R}_i$ *of k different r-vectors in terms of the components of the* $\tilde{R}_i$. The rather long proof can be found in [1, p. 51].

$$\text{Put } J = \{j_1, \cdots, j_r\} \quad \text{where} \quad 1 \leqq j_1 < \cdots < j_r \leqq n,\ n \geqq kr .$$

Let $\alpha_i^J = \alpha_i^{j_1 \cdots j_r} (i = 1, \cdots, k)$ be indeterminates and define for a permutation $\pi$ of $\{1, \cdots, r\}$

$$\alpha_i^{\pi(J)} = \alpha_i^{j_{\pi(1)} \cdots j_{\pi(r)}} = \operatorname{sgn} \pi \alpha_i^J .$$

If $\{H = h_1, \cdots, h_{kr}\}$ with $1 \leqq h_1 < \cdots < h_{kr} \leqq n$ and $J_1 \cup \cdots \cup J_k = H$ (disregarding order) then $J_\nu \cap J_\mu = \phi$ for $\nu \neq \mu$ and $J_1, \cdots, J_k$ in this order is a permutation of $H$ whose sign is denoted by

$$\begin{bmatrix} J_1 \cdots J_k \\ H \end{bmatrix} .$$

We then define

$$F^H(\alpha_1, \cdots, \alpha_k) = \sum_{J_1 \cup \cdots \cup J_k = H} \begin{bmatrix} J_1 \cdots J_k \\ H \end{bmatrix} \alpha_1^{J_1} \cdots \alpha_k^{J_k}$$

where $\alpha_\nu$ stands for $\{\alpha_\nu^J : J \subset H\}$. If $\pi$ is a permutation of $\{1, \cdots, k\}$, then

$$F^H(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(k)}) = (\operatorname{sgn} \pi)^r F^H(\alpha_1, \cdots, \alpha_k) .$$

THEOREM 3.3. *If* $\tilde{R}_i = \sum_{J \subset N} \alpha_i^J e_J$ *with* $N = \{1, \cdots, n\}$, *then* $\bigwedge_{i=1}^k \tilde{R}_i = \sum_{H \subset N} F^H(\alpha_1, \cdots, \alpha_k) e_H$.

Consequently, if $Q^H(\alpha)$ originates from $F^H(\alpha_1, \cdots, \alpha_k)$ through replacing each $\alpha_i$ by the same $\alpha = \{\alpha^J\}$, then we obtain

COROLLARY 3.4. *If* $\tilde{R} = \sum_{J \subset N} \alpha^J e_J$, *then* $\tilde{R}^{(k)} = k! \sum_{H \subset N} Q^H(\alpha) e_H$.

From (3.4) one deduces with the conventions $\binom{0}{2} = \binom{1}{2} = 0$,

$$a\text{-dim}\, V_2^n(F, k) = \binom{n}{2} - \binom{n-2k}{2}, 1 \leqq k \leqq \left[\frac{n}{2}\right]$$

(see [1, p. 65]). Hence

$$\dim V_2^n(\boldsymbol{R}, k) = \binom{n}{2} - \binom{n-2k}{2}, \dim V_2^n(\boldsymbol{C}, k) = 2\left[\binom{n}{2} - \binom{n-2k}{2}\right],$$

and so

$$\dim V_2^n(\boldsymbol{R} \text{ resp. } \boldsymbol{C}, k) < \dim V_2^n(\boldsymbol{R} \text{ resp. } \boldsymbol{C}, k+1) \text{ for } 1 \leqq k < \left[\frac{n}{2}\right]$$

in contrast to the case $r = 3$.

4. **Uniqueness.** Let $R_i \in G_r^n(F)$, $(i = 1, \cdots, k)$. The sum $\sum_{i=1}^k R_i$ is called *unique* in $V_r^n(F)$ if $S_i \in G_r^n(F)(i = 1, \cdots, k)$ and $\Sigma\, R_i = \Sigma S_i$ imply that $S_{\pi(i)} = R_i (i = 1, \cdots, k)$ for a suitable permutation $\pi$ of $\{1, \cdots, k\}$.

Obviously:

(4.1) *If $\sum_{i=1}^k R_i$ is unique then it is irreducible.*

(4.2) *If $\sum_{i=1}^k R_i$ is irreducible resp. unique then so is $\sum_{i=1}^j R_i$ for $j < k$.*

The converse of (4.1) does not hold; in particular:

(4.3) *If $r = 2$, $k > 1$, $\bigwedge_{i=1}^k R_i \neq 0$ then $\sum_{i=1}^k R_i$ is not unique, i.e., no irreducible sum of 2-vectors is unique.*

Because of (4.2) it suffices to observe that

$$e_1 \wedge e_2 + e_3 \wedge e_4 = (e_1 + e_3) \wedge e_2 + e_3 \wedge (-e_2 + e_4)\,.$$

However, if $r > 2$ and $\bigwedge_{i=1}^k R_i \neq 0$, then $\sum_{i=1}^k R_i$ probably is unique. Because the Plücker relations are hard to handle for large $k$ we were able to prove only:

THEOREM 4.4. *If $r > 2$, $k \leqq r$, and $\bigwedge_{i=1}^k R_i \neq 0$, then $\sum_{i=1}^k R_i$ is unique.*

Here both the field $F$ and the dimension $n$ of the space (except that $n \geqq rk$ is, of course, implied) are deliberately omitted because they are immaterial.

First we convince ourselves that $n$ is unimportant and at the end of the proof we indicate why $F$ is.

Let $R_i = v_{(i-1)r+1} \wedge \cdots \wedge v_{ir} (i = 1, \cdots, k)$, $\bigwedge v_i \neq 0$. It suffices to prove (for a given $F$) that $\Sigma R_i$ is unique in the space $V$ spanned by $v_1, \cdots, v_{kr}$. For, let also

$$\tilde{R} = \sum_{i=1}^{k} R_i = \Sigma R_i^*,\ R_i^* \in V_r^n,\ V^n \supset V .$$

Under projection of $V^n$ on $V$ let $R_i^* \to R_i'$. Then $\Sigma R_i^* \to \Sigma R_i'$, $\tilde{R} \to \tilde{R}$, $\bigwedge R_i^* \to \bigwedge R_i'$ so that $\tilde{R} = \Sigma R_i'$ and if (4.4) holds in $V$ then $\{R_i'\}$ is a permutation of $\{R_i\}$. Therefore, $\bigwedge R_i' \neq 0$ and hence $\bigwedge R_i^* \neq 0$. If $R_i^* = v^*_{(i-1)r+1} \wedge \cdots \wedge v^*_{ir}$ then because (4.4) holds in the space spanned by $v_1^*, \cdots, v_{rk}^*$ we have $R^*_{\pi(i)} = R_i$ for a suitable permutation $\pi$ of $\{i, \cdots, k\}$.

In the proof of (4.4) we therefore assume that $n = rk$ and $\tilde{R} = \sum_{i=1}^{k} R_i$ with

$$R_i = e_{(i-1)r+1} \wedge \cdots \wedge e_{ir} = e_{L(i)} ,$$

where

$$L(\nu) = \{(\nu - 1)r + 1, \cdots, \nu r\} \quad (\nu = 1, \cdots, k) .$$

We further put

$$I = \{i_1, \cdots, i_k\} \quad \text{with} \quad 1 \leqq i_1 < \cdots < i_k \leqq rk ,$$

also

$$I(\nu) = I/\{i_\nu\},\ I(\nu, \mu) = I/\{i_\nu, i_\mu\}, \text{ etc} .$$

It will prove convenient and causes no ambiguities to use $I(\nu)$ for $\{i_1, \cdots, i_{\nu-1}, i_{\nu+1}, \cdots, i_k\}$ even if $i_\nu$ is not defined.

$\Omega$ is the set of all $I$ with $i_\nu \in L(\nu) (\nu = 1, \cdots, k)$, and $I(\nu) \in \Omega$ means $i_\mu \in L(\mu)$ for $\mu \neq \nu$.

We also use

$$e_{L(\nu)} = \bigwedge\nolimits_{i \in L(\nu)} e_i, \quad e_I = \bigwedge\nolimits_{i \in I} e_i, \text{ etc.}$$

The sign depends on the order but will prove irrelevant. Finally $E(\nu)$ and $F(\nu)$ are the spaces spanned by the $e_i$ with $i \in L(\nu)$ or $i \notin L(\nu)$ respectively.

From now on we will often use the *Plücker relations* (see [3, p. 23] and [4, p. 27]) which in our type of notation may be stated as follows:

Let $P = \{p_1, \cdots, p_r\}$, $1 \leqq p_i \leqq n$, $P(i) = P/\{p_i\}$,

$$\alpha^P = \alpha^{p_1 \cdots p_r} = \operatorname{sgn} \pi \alpha^{p_{\pi(1)} \cdots p_{\pi(r)}}$$

for a permutation $\pi$ of $\{1, \cdots, r\}$ and similarly for $Q$. The vector

$$\frac{1}{r!}\sum_P \alpha^P e_P = \sum_{1\leqq p_1<\cdots<p_r\leqq n} \alpha^{p_1\cdots p_r} e_{p_1}\wedge\cdots\wedge e_{p_r}\in V_r^n$$

is simple if and only if for any $P$, $Q$

*Plücker*: $\alpha^P\alpha^Q + \sum_{i=1}^r (-1)^i \alpha^{P(r)q_i}\alpha^{p_r Q(i)} = 0$.

We prove several lemmas beginning with

(4.5) *Let* $\tilde T = \sum \gamma^I e_I$ *and suppose* $\tilde T \wedge \tilde R = 0$. *If* $k < r$, *or* $k = r$ *but* $\tilde T$ *is simple, then* $\gamma^I \neq 0$ *only if* $I \in \Omega$. *Thus simple* $\tilde T \neq 0$ *implies* $\gamma^I \neq 0$ *for at least one* $I \in \Omega$.

If $k < r$ the assertion follows from

$$\tilde R \wedge \tilde T = \sum_{\nu=1}^k \Big[e_{L(\nu)} \wedge \Big(\sum_{I\cap L(\nu)=\phi} \gamma^I e_I\Big)\Big],$$

$$e_{L(\nu)}\wedge e_I \neq e_{L(\mu)}\wedge e_I \quad \text{for} \quad \nu\neq\mu \quad \text{and} \quad k<r,$$

and the observation that $I \cap L(\nu) = \phi$ for some $\nu$ is equivalent to $I \notin \Omega$.

If $k = r$ then the terms in $\tilde R \wedge \tilde T$ with $e_{L(1)}$ as a factor are

$$e_{L(1)} \wedge \Big[\sum_{i_1>r}\gamma^I e_I + (-1)^r \gamma^{L(1)}(e_{L(2)} + \cdots + e_{L(k)})\Big].$$

Therefore, $\gamma^I = 0$ if $i_1 > r$ (hence $I \notin \Omega$) and $\gamma^{L(\nu)} + (-1)^r\gamma^{L(1)} = 0$ for $\nu > 1$. Generally $\gamma^I = 0$ if, $I \notin \Omega$ and $I$ is no $L(\nu)$; moreover,

$$\gamma^{L(\mu)} + (-1)^r \gamma^{L(\nu)} = 0 \quad \text{if} \quad \mu \neq \nu.$$

If $r$ is even then $\gamma^{L(\nu)} = 0$ for all $\nu$ so that $\gamma^I \neq 0$ only for $I \in \Omega$.

If $r$ is odd then $\gamma^{L(\nu)} = \lambda$ for all $\nu$ and thus

$$\tilde T = \lambda \tilde R + \sum_{I\in\Omega} \gamma^I e_I.$$

We show $\tilde T$ is simple only if $\lambda = 0$ which completes the assertion.

Let $I \in \Omega$ and assume $i_1 \neq r$. Then with

$$L(1, r)i_s = \{1, \cdots, r-1, i_s\}$$

one of the Plücker relations for the simplicity of $\tilde T$ is

$$0 = \gamma^{L(1)}\gamma^I + \sum_{s=1}^r (-1)^s \gamma^{L(1,r)i_s}\gamma^{rI(s)} = \lambda\gamma^I,$$

for $\gamma^{L(1,r)i_s} = 0$ because $L(1, r) \in \bigcup_{\nu=1}^r L(\nu) \cup \Omega$ for $s > 1$, and $L(1, r)i_1$ contains a repeated index (since $i_1 \neq r$). If $i_1 = r$ just permute $L(1)$ so that $r$ is not the last element. Thus $\gamma^I = 0$ for all $I \in \Omega$, or

$\lambda = 0$. Since $\tilde{T} = \lambda\tilde{R}$ would not be simple we must have $\lambda = 0$.

Let $H = \{h_1, \cdots, h_{k-1}\}$ with $1 \leqq h_1 < \cdots < h_{k-1} \leqq rk$.

(4.6) *If* $\tilde{S} = \sum \beta^H e_H$ *and* $\tilde{R} \wedge \tilde{S}$ *is simple then, for*

$$I \in \Omega,\ \beta^{I(s)}\beta^{I(t)} = 0$$

*if* $s \neq t$.

The terms in the expansion

$$\tilde{R} \wedge \tilde{S} = \Sigma \alpha^{i_1 \cdots i_{r+k-1}} e_{i \cdots i_{r+k-1}}$$

which contain $e_I$ as a factor are given by

$$(4.7) \quad e_I \wedge [\pm \beta^{I(1)} e_{L(1,i_1)} \pm \cdots \pm \beta^{I(k)} e_{L(k,i_k)}]$$

where $L(\nu, i_\nu) = L(\nu)/\{i_\nu\}$. Consider the Plücker relation for $\tilde{R} \wedge \tilde{S}$ beginning with

$$\alpha^{I, L(s,i_s)} \alpha^{I, L(t,i_t)} = \beta^{I(s)} \beta^{I(t)} .$$

The terms not written down all vanish. The first $k - 1$ that follow vanish because the first factor has a repeated superscript. From the $(k + 1)$st term on, the last element of $L(i, i_s)$ is the first superscript of the second factor $\alpha$ which then vanishes because it does not appear in (4.7). (This requires $r \geqq 3$. The first $\alpha$ also vanishes and for a similar reason.)

The following is the decisive step in our long argument:

(4.8) *If both* $S = \Sigma \beta^H e_H$ *and* $\tilde{R} \wedge S$ *are simple and some* $\beta^{I(1)} \neq 0$ $(I(1) \in \Omega)$ *then* $\beta^{i_1 \nu_1 \cdots \nu_{k-2}} = 0$ *for* $i_1 \in L(1)$ *and any* $\nu_s (s = 1, \cdots, k - 2)$. *Briefly* $S \in F(1)_{k-1}$.

Take any $i_1 \in L(1)$ and join it to $I(1)$. This produces an $I \in \Omega$. We prove inductively.

$$\beta^{\nu_1 \cdots \nu_\lambda I(k-\lambda, \cdots, k)} = 0 \quad \text{for all} \quad \nu_s \quad \text{and} \quad \lambda \leqq k - 2 .$$

If $k = 2$ we have $\beta^{i_1} = \beta^{I(2)} = 0$ by (4.6) and $\beta^{I(1)} \neq 0$.

If $k \geqq 3$ we make

*Step* 1. Consider the Plücker relation

$$0 = \beta^{I(1)} \beta^{\nu I(k-1,k)} - \beta^{I(1,k)\nu} \beta^{i_k I(k-1,k)} + \beta^{I(1,k) i_1} \beta^{i_k \nu I(1,k-1,k)}$$
$$- \beta^{I(1,k) i_2} \beta^{i_k \nu I(2,k-1,k)} \pm \cdots \pm \beta^{I(1,k) i_{k-2}} \beta^{i_k \nu I(k-2,k-1,k)} .$$

Except for order

$$i_k I(k-1, k) = I(k-1), \quad \text{and} \quad I(1, k) i_1 = I(k)$$

so that the second and third terms vanish by (4.6). The remaining terms vanish because the sets $I(1, k)i_2, \cdots, I(1, k)i_{k-2}$ contain repeated elements. If $k = 3$ we are finished. If $k > 3$ we make

*Step* 2. Take the Plücker relation

$$\begin{aligned} 0 = {} & \beta^{I(1)} \beta^{\nu\mu I(k-2,k-1,k)} - \beta^{I(1,k)\nu} \beta^{i_k \mu I(k-2,k-1,k)} \\ & + \beta^{I(1,k)\mu} \beta^{i_k \nu I(k-2,k-1,k)} - \beta^{I(1,k) i_1} \beta^{i_k \nu\mu I(1,k-2,k-1,k)} \\ & + \beta^{I(1,k) i_2} \beta^{i_k \nu\mu I(2,k-2,k-1,k)} - \cdots \pm \beta^{I(1,k) i_{k-3}} \beta^{i_k \nu\mu I(k-3,k-2,k-1,k)} . \end{aligned}$$

Except for order

$$i_k \mu I(k-2, k-1, k) = \mu I(k-2, k-1)$$

and

$$i_k \nu I(k-2, k-1, k) = \nu I(k-2, k-1)$$

so that the second and third terms vanish by Step 1 (that $k-1, k$ are replaced by $k-2, k-1$ is immaterial since the argument of Step 1 is the same for any permutation of $\{2, \cdots, k\}$). The fourth term vanishes because of (4.6) and $I(1, k)i_1 = I(k)$. In all following terms the sets of superscripts in the first factor $\beta$ contain repeated elements and these terms vanish also. This completes the argument in case $k = 4$. If $k > 4$, the process clearly continues.

(4.9) *If both $S = \Sigma \beta^H e_H$ and $\tilde{R} \wedge S$ are simple and $\beta^{I(t)} \neq 0$ for some $I(t) \in \Omega$ then $S \in F(t)_{k-1}$. If $S = w_1 \wedge \cdots \wedge w_{k-1}$ then each $w_i \in F(t)$.*

The first part is a consequence of (4.8). The second statement follows from the general lemma.

(4.10) *If $m < n$, $V^m \subset V^n$, $v_j \in V^n$, $\bigwedge_{j=1}^{s} v_j \neq 0$ and $\bigwedge_{j=1}^{s} v_j \in V_s^m$ then $v_j \in V^m (j = 1, \cdots s)$.*

Setting $v_i = v_i' + \beta_i e_n$ $(i = 1, \cdots, s)$ in the proof of (2.1) yields the case $m = n - 1$ from which the general case follows.

THEOREM 4.11. *If $k \leqq r$, $S = \bigwedge_{i=1}^{k-1} w_i \neq 0$ and $\tilde{R} \wedge S$ is simple then $w_i \in F(t)$ for a suitable $t$.*

If $S = \Sigma \beta^H e_H$ then it suffices by (4.9) to show that $\beta^{I(t)} \neq 0$ for a suitable $I(t) \in \Omega$. Because $\tilde{R} \wedge S$ is a simple $(r + k - 1)$-vector there is a vector $v = \sum_{i=1}^{rk} \delta^i e_i$ such that $\tilde{R} \wedge S \wedge v = 0$ and $S \wedge v \neq 0$. If

$T = S \wedge v = \Sigma\gamma^J e^J$ then

$$\gamma^J = \Sigma \pm \beta^{J(t)}\delta^{j_t} .$$

By (4.5) there is at least one $I \in \Omega$ with $\gamma^I \neq 0$, hence $\beta^{I(t)} \neq 0$ for some $t$.

After these preparations we are ready to prove (4.4). First observe

(4.12) *If* $\tilde{R} = \sum_{\nu=1}^{k} v_{L(\nu)}(v_i \in V)$ *then* $\bigwedge_{i=1}^{rk} v_i \neq 0$.

Each $v_i \neq 0$ because $\sum_{i=1}^{k} e_{L(\nu)} = \tilde{R}$ is irreducible. Assme $\bigwedge v_i = 0$ and let $\{w_j\}, 1 \leqq j \leqq \lambda < kr$, be a maximal set of independent $v_i$. Since the $w_i$ span a proper subspace $V'$ of $V$, an $e_\mu$ with $w_1 \wedge \cdots \wedge w_\lambda \wedge e_\mu \neq 0$ exists, and the $w_i$ together with $e_\mu$ span a space $V''$ with $V' \subset V'' \subset V$. Now $\Sigma v_{L(\nu)}$ is irreducible in $V$ and $V'$, and therefore (see (2.2) and (2.7)) $\Sigma_{L(\nu)} \wedge e_\mu$ is irreducible in $V''$ and in $V$. But if $\mu \in L(s)$ then $\tilde{R} \wedge e_\mu = \sum_{\nu \neq s} e_{L(\nu)} \wedge e_\mu$.

(4.13) *If* $\tilde{R} = \sum_{\nu=1}^{k} v_{L(\nu)}(v_i \in V)$, $k \leqq r$, $I = \{i_1, \cdots, i_k\} \in \Omega$ *then* $v_{i_t} \in E(\pi(t))$ *for a suitable permutation* $\pi$ *of* $\{1, \cdots, k\}$.

First $v_{I(s)} \neq 0$ by (4.12). Next

$$\tilde{R} \wedge v_{I(s)} = v_{L(s)} \wedge v_{I(s)} \neq 0$$

is simple. Therefore (4.11) yields $v_{i_\nu} \in F(\pi(s))$, $(\nu \neq s)$ for a suitable number $\pi(s)$, $(1 \leqq \pi(s) \leqq k)$. We must show that $\pi(s)$ defines a permutation of $\{1, \cdots, k\}$ or that $\pi(s) \neq \pi(t)$ for $s \neq t$. Assume $\pi(s) = \pi(t)$ for some $s \neq t$. Then $v_{i_\nu} \in F(\pi(s))$ for $\nu = 1, \cdots, k$ because $I(s) \cup I(t) = I$ whence $v_I \in F(\pi(s))_k$ and $\tilde{R} \wedge v_I = e_{L(\pi(s))} \wedge v_I \neq 0$ contradicting $\tilde{R} \wedge v_I = 0$. Thus $v_{i_t} \in \bigcap_{i_t \in I(s)} F(\pi(s)) = \bigcap_{s \neq t} F(\pi(s)) = E(\pi(t))$.

This establishes the uniqueness of $\Sigma e_{L(\nu)}$. For, consider $I \in \Omega$ and put $I' = \{j_1, i_2, \cdots, i_k\}$ with $j_1 \in L(1)$. Then $I' \in \Omega$. Since $v_{i_\nu} \in E(\pi(\nu))$ for $\nu > 1$ it follows from (4.13) that $v_{j_1} \in E(\pi(1))$. Thus $v_{j_1} \in E(\pi(1))$ for all $j_1 \in L(1)$ and $v_{L(1)} = \alpha_1 e_{L(\pi(1))}$.

Generally, $v_{L(\nu)} = \alpha_\nu e_{L(\pi(\nu))}$ whence $\alpha_\nu = 1 (\nu = 1, \cdots, k)$ and uniqueness follows.

The condition $v_i \in V$ which entered the proof of (4.12) because we applied (2.2) can now be eliminated; $\Sigma e_{L'\nu)}$ retains its form after extension of the underlying field and therefore remains unique after the extension. This justifies the formulation of (4.4) which does not mention a field.

5. The case $n = 6, r = 3$. The remainder of the paper deals with the case $n = 6, r = 3$ whose importance was noted in connection

with (3.2). We first show $N(F, 6, 3) = 3$ which may be new for $F \neq C$. Our inequalities (2.3) and (2.5) give only

$$2 = N(F, 5, 2) \leqq N(F, 6, 3) \leqq N(F, 5, 3) + N(F, 5, 2) = 4 .$$

With $e_{ijk} = e_i \wedge e_j \wedge e_k$ we prove:

(5.1) $\tilde{S} = e_{145} + e_{246} + e_{356}$ *is irreducible; whence* $N(F, 6, 3) \geqq 3$.

This proof rests on the observation:

(5.2) *If* $R = \tilde{S} \wedge \sum_{i=1}^{6} \beta^i e_i$ *is simple then* $\beta^1 = \beta^2 = \beta^3 = 0$.

(The converse is trivial but not needed.) If

$$R = \sum_{1 \geqq i<j<k<l<\geqq 6} \alpha^{ijkl} e_{ijkl}$$

then

$$\sum_{1<j<k<6} \alpha^{1jk6} e_{1jk6} = e_{16} \wedge [\beta^6 e_{45} - \beta^1(e_{24} + e_{35})] .$$

Therefore one of the Plücker relations for $R$ is

$$0 = \alpha^{1624}\alpha^{1635} - \alpha^{1623}\alpha^{4165} + \alpha^{1625}\alpha^{4613} = (\beta^1)^2 .$$

Similarly, $\beta^2 = \beta^3 = 0$.

Assume $\tilde{S}$ were reducible, $\tilde{S} = v_{123} + v_{456}$ (where again $v_{ijk} = v_i \wedge v_j \wedge v_k$) with $v_i = \sum_{k=1}^{6} \beta_i^k e_k$. Then $\tilde{S} \wedge v_i$ is simple, so that by (5.2) $\beta_i^k = 0$ for $k \leqq 3$, whence

$$\tilde{S} = [\det(\beta_k^{3+i}) + \det(\beta_{3+k}^{3+i})] e_{456} ,$$

which is false because $\tilde{S} \wedge e_6 = e_{1456} \neq 0$.

To show $N(F, 6, 3) \leqq 3$ we need the lemma:

(5.3) *Given* $\tilde{R}_i \in V_2^4(F)(i = 1, \cdots, m)$ *there are* $\lambda_i \in F$ *and* $R_i \in G_2^4(F)(i = 0, \cdots m)$ *such that* $\tilde{R}_i = R_i + \lambda_i R_0(i = 1, \cdots, m)$.

If $\tilde{R}_i$ is simple then $R_i = \tilde{R}_i$, $\lambda_i = 0$ will do, so we assume that no $\tilde{R}_i$ is simple. $G_2^4$ is a quadratic cone and a hypersurface in $V_2^4(F)$. Therefore, $R_0 \in G_2^4$ exists such that the tangent hyperplane of $G_2^4$ at $R_0$ does not contain any $\tilde{R}_i$, and no line through $\tilde{R}_i$ and $R_0$ intersects $G_2^4$ (as a locus in $V_2^4$ completed to a projective space) at infinity. Then the line through $\tilde{R}_i$ and $R_0$ intersects $G_2^4$ in a second point $R_i'$ so that

$$\tilde{R}_i = (1 - \lambda_i)R_i' + \lambda_i R_0 = R_i + \lambda_i R_0 .$$

This argument does not require extending $F$ because it amounts to solving a quadratic equation of which one root is $F$.

Now let $\tilde{R} = \sum_{1 \leqq i<j<k \leqq 6} \alpha^{ijk} e_{ijk} \in V_3^6(F)$ be given. A simple calculation shows that either $\tilde{R} \in V_3^6(F, 2)$ or a base $\{\bar{e}_i\}$ exists in terms of which

$$\tilde{R} = \sum_{1 \leqq i<j \leqq 4} \beta^{ij5} \bar{e}_{ij5} + \sum_{1 \leqq i<j \leqq 4} \beta^{ij6} \bar{e}_{ij6} = \tilde{S}_1 \wedge \bar{e}_5 + \tilde{S}_2 \wedge \bar{e}_6$$

with $\tilde{S}_i \in V_2^4$. By (5.3) there are $S_i \in G_2^4$ and $\lambda_i \in F$ such that

$$\begin{aligned} \tilde{R} &= (S_1 + \lambda_1 S_0) \wedge \bar{e}_5 + (S_2 + \lambda_2 S_0) \wedge \bar{e}_6 \\ &= S_1 \wedge \bar{e}_5 + S_2 \wedge \bar{e}_6 + S_0 \wedge (\lambda_1 \bar{e}_5 + \lambda_2 \bar{e}_6) \in V_3^6(F, 3) . \end{aligned}$$

Thus:

$$N(F, 6, 3) = 3 . \tag{5.4}$$

By a similar argument we prove

$$3 \leqq N(F, 7, 3) \leqq 5 . \tag{5.5}$$

The left inequality follows from $3 = N(6, 3) \leqq N(7, 3)$, see (2.8). For the right inequality one shows (see [1, p. 90]) that either $\tilde{R} \in V_3^7(2)$ or with a suitable base $\{\bar{e}_i\}$

$$\tilde{R} = \sum_{1 \leqq i<j \leqq 4} \beta^{ij5} \bar{e}_{ij5} + \sum_{1 \leqq i<j \leqq 4} \beta^{ij6} \bar{e}_{ij6} + \sum_{1 \leqq i<j \leqq 4} \beta^{ij7} \bar{e}_{ij7} + \sum_{i=1}^{5} \beta^{i67} \bar{e}_{i67} .$$

The last sum is simple and applying (5.3) to the first three terms on the right yields $N(F, 7, 3) \leqq 5$. This method does not extend to $N(n, 3)$ with $n > 7$.

We now study a *special type* of $\tilde{R} \in V_3^6(C)$ which will confirm some of the important assertions made previously.

Let $Y$ be the set of triples

$$Y = \{123, 126, 135, 156, 234, 246, 345, 456\}$$

and suppose that the $\alpha^I$, $I \in Y$, satisfy the inequalities

$$\begin{aligned} &\alpha^{123}\alpha^{156} + \alpha^{126}\alpha^{135} \neq 0 , \qquad \alpha^{123}\alpha^{246} + \alpha^{126}\alpha^{234} \neq 0 , \\ &\alpha^{123}\alpha^{345} + \alpha^{135}\alpha^{234} \neq 0 , \qquad \alpha^{234}\alpha^{456} + \alpha^{246}\alpha^{345} \neq 0 , \\ &\alpha^{135}\alpha^{456} + \alpha^{156}\alpha^{345} \neq 0 , \qquad \alpha^{126}\alpha^{456} + \alpha^{156}\alpha^{246} \neq 0 , \end{aligned} \tag{5.6}$$

and that the roots $\lambda$, $\mu$ of

$$(\alpha^{123} x - \alpha^{234})(\alpha^{156} x - \alpha^{456}) + (\alpha^{126} x + \alpha^{246})(\alpha^{135} x + \alpha^{345}) = 0 \tag{5.7}$$

are distinct. They are different from zero.

THEOREM 5.8. *If $\tilde{R} = \sum_{I \in Y} \alpha^I e_I \in V_3^6(C)$ and the $\alpha^I$ satisfy* (5.6), *then $\tilde{R} = R_\beta + R_\gamma$, $R_\beta = \sum_{I \in Y} \beta^I e_I$, $R_\gamma = \sum_{I \in Y} \gamma^I e_I$, where $R_\beta$ and $R_\gamma$ are simple with $R_\beta \wedge R_\gamma \neq 0$. Hence the representation $R_\beta + R_\gamma$ is* *unique* (*by* (4.4)).

*If* $\lambda$, $\mu$ *are the solutions of* (5.7) *then the* $\beta^I$ *and* $\gamma^I(I \in Y)$ *are given by*

$$\beta^{1ij} = \frac{\mu\alpha^{1ij} - \alpha^{4ij}}{\mu - \lambda}, \quad \gamma^{1ij} = \frac{\alpha^{4ij} - \lambda\alpha^{1ij}}{\mu - \lambda},$$
$$\beta^{4ij} = \lambda\beta^{1ij}, \qquad \gamma^{4ij} = \mu\gamma^{1ij}.$$

*No* $\beta^I$ *or* $\gamma^I(I \in Y)$ *vanishes.*

This representation was found by using Plücker relations (see [1, pp. 98–106]), but after it is explicitly given one readily verifies that $R_\beta$ and $R_\gamma$ are simple and that $R_\beta \wedge R_\gamma \neq 0$. In fact, it is easy to factor $R_\beta$ and $R_\gamma$, see [4, p. 21]: Since $\beta^I \neq 0$ if $I \in Y$, letting $\nu = (\beta^{123})^{-2/3}$ we find

$$R_\beta = u \wedge v \wedge w = \Sigma u^i e_i \wedge \Sigma v^i e_i \wedge \Sigma w^i e_i$$

with

$$u^i = \nu\beta^{23i},\ v^i = -\nu\beta^{13i},\ w^i = \nu\beta^{12i}$$

(see also [1, p. 102]), and similarly for $R_\gamma$.

First we confirm the statement in the introduction that *irreducibility may depend on the field.*

(5.9) *If the* $\alpha^I$ *in* (5.8) *are real and* $\lambda$, $\mu$ *are not, then* $\tilde{R} \in W_3^6(\boldsymbol{C}, 2)$ *but* $\tilde{R} \in W_3^6(\boldsymbol{R}, 3)$.

For because $R_\beta + R_\gamma$ is unique, $\tilde{R} \in V_3^6(\boldsymbol{R}, 2)$ is impossible, and this with $N(\boldsymbol{R}, 6, 3) = 3$ entails the assertion.

Next we observe that the vector

$$\tilde{R}(\eta) = e_1 \wedge (e_2 + e_5) \wedge e_6 + e_1 \wedge e_3 \wedge e_5 + \eta e_2 \wedge e_4 \wedge e_6$$
$$+ (e_2 + e_5) \wedge e_3 \wedge e_4 \qquad (\eta \neq 0)$$

is a special case of (5.8) and that $\lambda$, $\mu$ are real when $\eta < 0$. Letting $\eta \to 0^-$ we find

$$\tilde{R}(0^-) = e_1 \wedge (e_2 + e_5) \wedge e_6 + e_1 \wedge e_3 \wedge e_5 + (e_2 + e_5) \wedge e_3 \wedge e_4$$

which by (5.1) lies in $W_3^6(\boldsymbol{R}$ or $\boldsymbol{C}, 3)$. Therefore:

(5.10) *The sets* $V_3^6(\boldsymbol{R}, 2)$ *resp.* $V_3^6(\boldsymbol{C}, 2)$ *are not closed in* $V_3^6(\boldsymbol{R})$ *resp.* $V_3^6(\boldsymbol{C})$.

Theorem (3.2) whose proof used (5.10) is therefore completely established.

We now prove a surprising fact for $\boldsymbol{C}$ which has no analogue for $\boldsymbol{R}$ (see (6.3)):

(5.11) *The interior of $W_3^6(\boldsymbol{C}, 3)$ as a set in $V_3^6(\boldsymbol{C})$ is empty.*

We show that if $\tilde{R} = R_1 + R_2 + R_3$, $R_i \in G_3^6(C)$, is irreducible then it is the limit of elements in $V_3^6(\boldsymbol{C}, 2)$.

$R_i$ and $R_j (i \neq j)$ have no nonvanishing 2-vector as a common factor, because $R_i + R_j$ would then be simple. Thus two cases are to be considered:

(1) $R_i \wedge R_j \neq 0$ for some $i, j$, say $R_1 \wedge R_2 \neq 0$,

(2) $R_i$ and $R_j$ have for $i \neq j$ a vector $v_k \neq 0$ (but no 2-vector $\neq 0$) as a common factor where $(i, j, k)$ is a permutation of $(1, 2, 3)$.

In the latter case the $v_i$ are either parallel or no two $v_i$ are parallel. If they were parallel we could choose $e_6$ parallel to the $v_i$ so that $\tilde{R} = \tilde{S} \wedge e_6$ with $\tilde{S} \in V_2^5(\boldsymbol{C})$, and $\Sigma R_i$ would be reducible since $N(F, 5, 2) = 2$. If no two $v_i$ are parallel then with suitable $u_i$

$$R_1 = u_1 \wedge v_2 \wedge v_3,\ R_2 = u_2 \wedge v_1 \wedge v_3,\ R_3 = u_3 \wedge v_1 \wedge v_2 .$$

The vectors $u_i$, $v_j$ are independent, for otherwise $\Sigma R_i$ would be a 3-vector in a space of dimension less than 6 and by $N(F, 5, 3) = 2$ reducible. The proof of (5.10) shows $\tilde{R}$ can be approximated by elements of $V_3^6(\boldsymbol{C}, 2)$.

In case (1) there are vectors $w_1, \cdots, w_6, v_1, v_2, v_3$ such that $w_1, w_2, w_3$ are parallel to $R_1$, $w_4, w_5, w_6$ are parallel to $R_2$, $R_3 = v_1 \wedge v_2 \wedge v_3$ and

$$v_1 = a_1 w_1 + a_4 w_4,\ v_2 = a_2 w_2 + a_5 w_5,\ v_3 = a_3 w_3 + a_6 w_6 .$$

If $\bigwedge_{i=1}^6 w_i \neq 0$ then $\tilde{R} = \sum_{I \in Y} \alpha^I w_I$ and $\prod_{i=1}^6 a_i \neq 0$ is equivalent to (5.6), so we have a special case of (5.8) (see [1, p. 83]) and hence $\tilde{R} \in V_3^6(\boldsymbol{C}, 2)$ contrary to the hypothesis. If $\bigwedge_{i=1}^6 w_i = 0$ and/or $\prod_{i=1}^6 a_i = 0$ we can choose $w_i'$ and $a_i'$ arbitrarily close to $w_i$ resp. $a_i$ such that $\bigwedge_{i=1}^6 w_i' \neq 0$ $\prod_{i=1}^6 a_i' \neq 0$ and $\lambda \neq \mu$, so that $\tilde{R}$ is the limit of elements in $V_3^6(\boldsymbol{C}, 2)$.

For $kr \leqq n$ let *$Z_r^n(F, k)$ be the set of $\tilde{R} = \sum_{i=1}^k R_i$ with $\bigwedge_{i=1}^k R_i \neq 0$.* Then $Z_r^n(F, k) \subset W_r^n(F, k)$ by (2.10).

(5.12) *$Z_r^n(\boldsymbol{R}$ resp. $\boldsymbol{C}, k)$ is dense in $V_r^n(\boldsymbol{R}$ resp. $\boldsymbol{C}, k)$.*

This is nearly obvious: If $\tilde{R} = \sum_{i=1}^j R_i \in Z_r^n(j)$, $j < k$, then $R_{j+1}, \cdots, R_k$ exist with $\bigwedge_{i=1}^k R_i \neq 0$ and

$$\tilde{R} = \lim (\tilde{R} + \delta \sum_{i=j+1}^{k} R_i) \text{ as } \delta \to 0 .$$

If $\tilde{R} = \sum_{i=1}^j R_i \in W_r^n(j)$, $j \leqq k$, $\bigwedge_{i=1}^j R_i = 0$ and $R_i = \bigwedge_{h=1}^r v_{(i-1)r+h}$ then

$w_i \to v_i$ with $\bigwedge_{i=1}^{jr} w_i \neq 0$ exist and $\sum_{i=1}^{j} \bigwedge_{h=1}^{r} w_{(i-1)r+h} \to \tilde{R}$.

Because

$$V_3^6(C) = Z_3^6(C, 2) \cup V_3^6(C, 2)/Z_3^6(C, 2) \cup W_3^6(C, 3) ,$$

(5.11, 12) show that $Z_3^6(C, 2)$ is dense in $V_3^6(C)$, so that $V_3^6(C)/Z_3^6(C, 2)$ has no interior points, and hence has dimension less than 40 (= $\dim V_3^6(C)$), see [2, p. 46]. In the next section we will see that $Z_3^6(C, 2)$ is open. Thus

(5.13) *The set* $Z_3^6(C, 2)$ *is open and dense in* $V_3^6(C)$, *hence*

$$V_3^6(C)/Z_3^6(C, 2)$$

*is closed and* $\dim Z_3^6(C, 2) = 40$, $\dim V_3^6(C)/Z_3^6(C, 2) < 40$.

Note that $W_3^6(C, 3) \subset V_3^6(C)/Z_3^6(C, 2)$ and that therefore *the closure of* $W_3^6(C, 3)$ *has dimension less than* 40.

**6. The sets $Z_3^6(R$ resp. $C, 2)$ and $W_3^6(R, 3)$.** We now prove that $Z_3^6(R$ resp. $C, 2)$ is open. Actually our next theorem provides much more information which will allow us to show that $W_3^6(R, 3)$ has a nonempty interior.

THEOREM 6.1. *Let* $F = R$ *or* $C$. *If* $R_1, R_2 \in G_3^6(F)$ *and* $R_1 \wedge R_2 \neq 0$ *then there is a neighborhood* $\tilde{U}(\tilde{R}_0)$ *of* $\tilde{R}_0 = R_1 + R_2$ *in* $V_3^6(F)$ *such that for* $\tilde{R} \in \tilde{U}(\tilde{R}_0)$ *there are simple* $R_1', R_2'$ *with* $\tilde{R} = R_1' + R_2'$. *Furthermore, given neighborhoods* $U_i(R_i)$ *of* $R_i$ *in* $G_3^6(F)$ *there is a neighborhood* $\tilde{U}'(\tilde{R}_0) \subset \tilde{U}(\tilde{R}_0)$ *such that* $\tilde{R} \in \tilde{U}'(\tilde{R}_0)$ *implies* $R_i' \in U_i(R_i)$ *and* $R_1' \wedge R_2' \neq 0$. *Consequently* $\tilde{R} \in Z_3^6(F, 2)$ *and by* (4.4) $R_1' + R_2'$ *is unique.*

Necessary for (6.1) to work is that $G_3^6 \times G_3^6$ and $V_3^6$ have the same dimension, which is the case because $a\text{-}\dim G_r^n = r(n-r) + 1$, $a\text{-}\dim V_r^n = \binom{n}{r}$ and $2[3(6-3) + 1] = 20 = \binom{6}{3}$. But this argument is far from sufficient as similar situations for other dimensions show; the structure of $G_3^6$ enters.

Since $R_1 \wedge R_2 \neq 0$ we can choose a base so that $R_1 = e_{123}$, $R_2 = e_{456}$. A neighborhood of $R_1$ on $G_3^6$ consists of the simple $R_1' = \Sigma \beta^I e_I = \sum_{1 \leq i < j < k \leq 6} \beta^{ijk} e_{ijk}$ with $\beta^{123}$ close to 1 and the remaining $\beta^I$ close to 0, so $\beta^{123} \neq 0$ may be assumed. Similarly for $R_2' = \Sigma \gamma^I e_I$ and $\gamma^{456} \neq 0$. The components of $\tilde{R}_0$ are $1, 0, \cdots, 0, 1$.

The special properties of $G_3^6$ arise from the Plücker relations which (see [1, p. 69]) with $\lambda = (\beta^{123})^{-1}$ are equivalent to

$$\beta^{ijk} = \pm \lambda \begin{vmatrix} \beta^{i\rho j} & \beta^{i\rho k} \\ \beta^{i\sigma j} & \beta^{i\sigma k} \end{vmatrix} 1 \leqq i \leqq 3,\ 4 \leqq j < k \leqq 6,\ (i, \rho, \sigma) = \pi(1, 2, 3)\ ,$$

$$\beta^{456} = \lambda^2 \begin{vmatrix} \beta^{124} & \beta^{125} & \beta^{126} \\ \beta^{134} & \beta^{135} & \beta^{136} \\ \beta^{234} & \beta^{235} & \beta^{236} \end{vmatrix} = B\ .$$

Similarly with $\mu = (\gamma^{456})^{-1}$

$$\gamma^{ijk} = \pm \mu \begin{vmatrix} \gamma^{i\rho k} & \gamma^{j\rho k} \\ \gamma^{i\sigma k} & \gamma^{j\sigma k} \end{vmatrix} 1 \leqq i < j \leqq 3,\ 4 \leqq k \leqq 6,\ (\rho, \sigma, k) = \pi(4, 5, 6)\ ,$$

$$\gamma^{123} = \mu^2 \begin{vmatrix} \gamma^{145} & \gamma^{245} & \gamma^{345} \\ \gamma^{146} & \gamma^{246} & \gamma^{346} \\ \gamma^{156} & \gamma^{256} & \gamma^{356} \end{vmatrix} = C\ .$$

If now $\Sigma\alpha^I e_I = \tilde{R} = R_1' + R_2' = \Sigma\beta^I e_I + \Sigma\gamma^I e_I$ then $\alpha^I = \beta^I + \gamma^I$, and substitution gives

$$\alpha^{123} = \beta^{123} + C\ ,$$

$$\alpha^{ijk} = \beta^{ijk} \pm \mu \begin{vmatrix} \gamma^{i\rho k} & \gamma^{j\rho k} \\ \gamma^{i\sigma k} & \gamma^{j\sigma k} \end{vmatrix} 1 \leqq i < j \leqq 3,\ 4 \leqq k \leqq 6,\ (\rho, \sigma, k) = \pi(4, 5, 6)\ ,$$

$$\alpha^{ijk} = \pm \lambda \begin{vmatrix} \beta^{i\rho j} & \beta^{j\rho k} \\ \beta^{i\sigma j} & \beta^{i\sigma k} \end{vmatrix} + \gamma^{ijk},\ 1 \leqq i \leqq 3,\ 4 \leqq j < k \leqq 6,\ (i, \rho, \sigma) = \pi(1, 2, 3),$$

$$\alpha^{456} = B + \gamma^{456}\ .$$

Thus the 20 components of $\tilde{R}$ are expressed in terms of $\beta^{123}$, the nine $\beta^I$ with $1 \leqq i \leqq 3$, $4 \leqq j < k \leqq 6$, the nine $\gamma^I$ with $1 \leqq i < j \leqq 3$, $4 \leqq k \leqq 6$, and $\gamma^{456}$. Evaluation of the functional determinant at $(1, 0, \cdots, 0, 1)$ gives the value 1. Therefore, the implicit function theorem is applicable and yields the assertion. The details of the calculation may be found in the thesis, [1, pp. 93–97].

As a corollary we have

(6.2) *The set* $Z_3^6(\boldsymbol{R}$ *resp.* $\boldsymbol{C}, 2)$ *is open in* $V_3^6(\boldsymbol{R}$ *resp.* $\boldsymbol{C})$.

But in contrast to (5.11):

(6.3) *The interior of* $W_3^6(\boldsymbol{R}, 3)$ *is not empty.*

For take any $\tilde{R}_0 = \sum_{I \in Y} \alpha^I e_I$ of (5.8) for which the $\alpha^I$ are real but $\lambda, \mu$ are not. Then

$$\tilde{R}_0 = R_\beta + R_\gamma \quad \text{with} \quad R_\beta \wedge R_\gamma \neq 0\ .$$

By (6.1) for $\tilde{R} \in \tilde{U}'(\tilde{R}_0) \subset V_3^6(\boldsymbol{C})$

$\tilde{R} = R_1' + R_2'$ with $R_1' \wedge R_2' \neq 0$, $R_1'$ close to $R_\beta$ and $R_2'$ close to $R_\gamma$ so that $R_1'$ and $R_2'$ cannot be real either. Also $R_1' + R_2'$ is unique by (4.4) and this implies as in the proof of (5.9) that

$$\tilde{R} \in W_3^6(\boldsymbol{R}, 3) \quad \text{for} \quad \tilde{R} \in \tilde{U}'(\tilde{R}_0) \cap V_3^6(\boldsymbol{R}) ,$$

where we consider $V_3^6(\boldsymbol{R})$ as a subset of $V_6^3(\boldsymbol{C})$.

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UNIVERSITY OF SOUTHERN CALIFORNIA
AND
CALIFORNIA POLYTECHNIC STATE UNIVERSITY AT SAN LUIS OBISPO