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DEFINITE AND SEMIDEFINITE MATRICES IN A REAL SYMMETRIC MATRIX PENCIL

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Pencils that contain a definite matrix (d-pencils) have been characterized in several ways. Here d-pencils will be characterized by the property of the set $L = \{(a_i, b_i)\} \subseteqq R^2$ if S and T are simultaneously congruent to $\mathrm{diag}(a_i)$ and $\mathrm{diag}(b_i)$, respectively. This way one can describe all definite and semidefinite matrices in a d-pencil. Similarly one can characterize all pencils that contain semidefinite but no definite matrices ($s.d.$ pencils). The explicit condition on L for d-pencils is then used to reprove the theorem that two real symmetric matrices generate a d-pencil iff their associated quadratic forms do not vanish simultaneously.

DEFINITION 1. If S is symmetric we define $Q_S = \{x \in \boldsymbol{R}^n \mid x'Sx = 0\}$.

DEFINITION 2. For real symmetric (r.s.) matrices S and T one defines the *pencil* $P(S, T) = \{aS + bT \mid a, b \in \boldsymbol{R}\}$.

DEFINITION 3. (a) $P(S, T)$ is called *d-pencil* if $P(S, T)$ contains a definite matrix.

(b) $P(S, T)$ is called *s.d. pencil* if $P(S, T)$ contains a nonzero semidefinite, but no definite matrix.

The following theorem is classical:

THEOREM 0.1. *If $P(S, T)$ is a d-pencil then S and T can be diagonalized simultaneously by a real congruence transformation.*

The question whether a given pencil of r.s. matrices contains a positive definite matrix was treated in chronological order by Finsler [6], Albert [1], Reid [12], Hestenes and McShane [9], Dines [5], Calabi [4], Taussky [13], Hestenes [8], Theorem 3, and Berman [3].

Their main results are the following:

THEOREM 0.2. *Let S and T be* r.s. $n \times n$ *matrices. If $n \geqq 3$, then the following are equivalent:*

(i) *$P(S, T)$ is a d-pencil,*

(ii) *$Q_S \cap Q_T = \{0\}$, and*

(iii) trace YS = trace $YT = 0$ *for Y positive semidefinite implies* $Y = 0$.

The equivalence of (i) and (ii) was proved by Calabi [4], while Berman [3] showed the equivalence of (i) and (iii).

Before Calabi [4] only condition

$$\text{(ii}') \qquad x'Sx = 0 \quad \text{implies} \quad x'Tx > 0$$

was generally used instead of (ii). And thus only the fact that (ii′) implies (i) was proved.

In view of Theorem 0.1, to characterize d-pencils means to characterize the sets $L = \{(a_i, b_i) \mid i = 1, \cdots, n\} \subseteqq \boldsymbol{R}^2$ for which there exist $\lambda, \mu \in \boldsymbol{R}$ with $\lambda a_i + \mu b_i > 0$ for all i.

THEOREM 1.1. *$P(S, T)$ is a d-pencil iff S and T can be simultaneously diagonalized by a real congruence transformation X: $X'SX = \operatorname{diag}(a_i)$, $X'TX = \operatorname{diag}(b_i)$, and in case that there are indices i, j with $a_i a_j < 0$ we have*

$$\text{(a)} \quad \max_{a_i>0} \frac{b_i}{a_i} < \max_{a_i<0} \frac{b_1}{a_i} \ \textit{and}\ b_i < 0 \ \textit{whenever}\ a_i = 0\,,$$

or

$$\text{(b)} \quad \min_{a_i>0} \frac{b_i}{a_i} > \min_{a_i<0} \frac{b_i}{a_i} \ \textit{and}\ b_i > 0 \ \textit{whenever}\ a_i = 0\,,$$

while in case that all a_i have one sign we have

$$\text{(c)} \quad \textit{either}\ b_i < 0 \ \textit{whenever}\ a_i = 0 \ \textit{or}\ b_i > 0 \ \textit{whenever}\ a_i = 0\,.$$

Proof. If $P(S, T)$ is a d-pencil, then by Theorem 0.1 the matrices S and T can be diagonalized simultaneously by a real congruence transformation: $X'SX = \operatorname{diag}(a_i)$, $X'TX = \operatorname{diag}(b_i)$. Furthermore, there exist $\lambda, \mu \in \boldsymbol{R}$ s.t. $\lambda S + \mu T = X' \operatorname{diag}(\lambda a_i + \mu b_i)\, X$ is positive definite.

Hence the set $L = \{(a_i, b_i) \mid i = 1, \cdots, n\} \subseteqq \boldsymbol{R}^2$ must lie in an open half plane of $\boldsymbol{R}^2$ and it follows that either (a) or (b) or (c) must hold for L.

Conversely if S and T are simultaneously congruent to $\operatorname{diag}(a_i)$ and $\operatorname{diag}(b_i)$ then in case of (a) all points $(a_i, b_i) \in \boldsymbol{R}^2$ lie "below" any line L_α thru zero that has slope $\max_{a_i>0} b_i/a_i < \alpha < \max_{a_i<0} b_i/a_i$. In case of (b), they all lie "above" any such line L_α with $\min_{a_i>0} b_i/a_i > \alpha > \min_{a_i<0} b_i/a_i$ and in case of (c) they all lie to the "right" or to the "left" of any line L_α where either

$$\max_{a_i>0} \frac{b_i}{a_i} < \alpha < \infty \quad \text{or} \quad -\infty < \alpha < \min_{a_i>0} \frac{b_i}{a_i}\,.$$

Thus the set L lies in an open half plane and hence there exist $\lambda, \mu \in \boldsymbol{R}$ s.t. $\lambda a_i + \mu b_i > 0$ for all i. Thus $P(S, T)$ is a d-pencil.

Next, if $P(S, T)$ is a d-pencil we will explicitly describe the set $M^+ = \{(\lambda, \mu) \mid \lambda S + \mu T \text{ pos. def.}\} \subseteqq R^2$ and determine the positive semidefinite matrices in $P(S, T)$. The sets M^+ and its closure $M = \{(\lambda, \mu) \mid \lambda S + \mu T \text{ pos. semidef.}\}$ have been previously characterized as convex cones and this fact together with properties of quadratic forms have been used to prove the equivalence of (i) and (ii) in Theorem 0.2 (Hestenes [8]). Hestenes [8] moreover treats related questions for positively elliptic pairs of quadratic forms in Hilbert space.

THEOREM 1.2. *Let $P(S, T)$ be a d-pencil and let S, T be simultaneously congruent to* $\operatorname{diag}(a_i)$ *and* $\operatorname{diag}(b_i)$.

Then with (a), (b), *and* (c) *from Theorem* 1.1 *we have*

(1) $\lambda S + \mu T$ *is positive definite iff*

in case of (a) $\quad -\left(\max\limits_{a_i>0} \dfrac{b_i}{a_i}\right)^{-1} < \dfrac{\lambda}{\mu} < -\left(\max\limits_{a_i<0} \dfrac{b_i}{a_i}\right)^{-1}$ *and*

in case of (b) $\quad -\left(\min\limits_{a_i>0} \dfrac{b_i}{a_i}\right)^{-1} > \dfrac{\lambda}{\mu} > -\left(\min\limits_{a_i<0} \dfrac{b_i}{a_i}\right)^{-1}$ *while*

in case of (c) *if all a_i are nonnegative*

$$-\left(\max_{a_i>0} \frac{b_i}{a_i}\right)^{-1} < \frac{\lambda}{\mu} < 0 \quad \text{if } b_i < 0 \text{ whenever } a_i = 0$$

and

$$0 < \frac{\lambda}{\mu} < -\left(\min_{a_i>0} \frac{b_i}{a_i}\right)^{-1} \quad \text{if } b_i > 0 \text{ whenever } a_i = 0$$

and, if all $a_i \leqq 0$, we have

$$-\left(\max_{a_i<0} \frac{b_i}{a_i}\right)^{-1} > \frac{\lambda}{\mu} > 0 \quad \text{if } b_i < 0 \text{ whenever } a_i = 0$$

while

$$0 > \frac{\lambda}{\mu} > -\left(\min_{a_i<0} \frac{b_i}{a_i}\right)^{-1} \quad \text{if } b_i > 0 \text{ whenever } a_i = 0\,.$$

(2) $\lambda S + \lambda T$ *is positive semidefinite and not definite iff in each case respectively, λ/μ is equal to either one of the endpoints of the intervals under* (1).

Here we set

$$(0/a)^{-1} = \begin{cases} \infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0\,. \end{cases}$$

Proof. The proof follows from Theorem 1.1 and by the geometry of $\boldsymbol{R}^2$. Concerning (2) one can say that if λ/μ is an endpoint of an appropriate interval in (1), then $\lambda S + \mu T$ is positive semidefinite of rank $n - r \leqq n$ iff $r = |\{j \mid b_i/a_j = \lambda/\mu\}|$.

As corollaries we observe that

COROLLARY 1. *If in Theorem* 1.1 *the strict reverse inequalities hold in* (a) *and* (b), *or* (c), *respectively, then* $\mu S + \lambda T$ *is indefinite for any* $(\lambda, \mu) \neq (0, 0)$. *And then* $\mu S + \lambda T$ *has at least rank* 2 *for any* $(\lambda, \mu) \neq (0, 0)$.

Before characterizing *s.d.* pencil we will quote a simplified version of the canonical pair form theorem for nonsingular pairs of r.s. matrices.

DEFINITION 4. A pair of r.s. matrices S, T is called *nonsingular*, if S is nonsingular.

THEOREM 1.3. *Let S and T be a nonsingular pair of* r.s. *matrices. Let $S^{-1}T$ have real Jordan normal form* $\operatorname{diag}(J_1, \cdots, J_m)$. *Then S and T are simultaneously congruent by a real congruence transformation to* $\operatorname{diag}(\varepsilon_i E_i)$ *and* $\operatorname{diag}(\varepsilon_i E_i J_i)$ *respectively, where* $\varepsilon_i = \pm 1$ *and E_i denotes the square matric* $\begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$ *of the same size as J_i for* $i = 1, \cdots, m$.

The relevant notation has recently been described in Uhlig [15]. There it was shown that the canonical pair form just described is a finest simultaneous block diagonalization.

Note that if $J = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ is a Jordan block for $\lambda \in \boldsymbol{R}$ then for $E = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$ with $\dim E = \dim J \geqq 3$ we have

$$aE + bEJ = \begin{pmatrix} 0 & & & a + b\lambda \\ & & \cdot^{\cdot^{\cdot}} & b \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ a + b\lambda & b & & 0 \end{pmatrix}.$$

Hence $aE + bEJ$ is indefinite for all $(a, b) \neq (0, 0)$. The same holds for any J corresponding to an eigenvalue $\lambda = \alpha + \beta i \notin \boldsymbol{R}$. With this in mind we can characterize *s.d.* pencils:

THEOREM 1.4. *Let S, T be a nonsingular pair of* r.s. *matrices of dimension greater than* 2.

Then $P(S, T)$ is a s.d. pencil iff either S and T are simultaneously congruent to $\operatorname{diag}(a_i)$ *and* $\operatorname{diag}(b_i)$ *with $b_i/a_i \neq b_j/a_j$ for at least one pair of indices (i, j) and we have*

$$(\mathrm{a}')\quad \max_{a_i>0}\frac{b_i}{a_i} = \max_{a_i<0}\frac{b_i}{a_i} \quad \textit{or} \quad (\mathrm{b}')\quad \min_{a_i>0}\frac{b_i}{a_i} = \min_{a_i<0}\frac{b_i}{a_i}\ ,$$

or S and T are simultaneously congruent to

$$\operatorname{diag}(\varepsilon E, \cdots, \varepsilon E, \varepsilon_{k+1}, \cdots, \varepsilon_j, a_{j+1}, \cdots, a_n)$$

and

$$\operatorname{diag}(\varepsilon EJ, \cdots, \varepsilon EJ, \varepsilon_{k+1}\alpha, \cdots, \varepsilon_j\alpha, b_{j+1}, \cdots, b_n)\ ,$$

where $J = \begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$, $E = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$; $\varepsilon, \varepsilon_i = \pm 1$, $\alpha \in \boldsymbol{R}$, $b_l \neq \alpha a_l$ *and* ε, $b_l - \alpha a_l$ *all have the same sign for* $l = j+1, \cdots, n$.

Proof. We recall that $P(S, T)$ is a *s.d.* pencil if there is a semidefinite but no definite matrix in $P(S, T)$.

If S and T can be diagonalized simultaneously to yield $\operatorname{diag}(a_i)$ and $\operatorname{diag}(b_i)$, then $P(S, T)$ is a *s.d.* pencil iff $L = \{(a_i, b_i)\} \subseteq \boldsymbol{R}^2$ lies in a closed half plane, but in no open half plane of $\boldsymbol{R}^2$, nor on a line thru 0; which is the case iff (a′) or (b′) holds. (Note that since S is nonsingular we have $a_i \neq 0$ for all i.)

If $P(S, T)$ is a *s.d.* pencil and S and T cannot be simultaneously diagonalized, then the real Jordan normal form J_0 of $S^{-1}T$ cannot contain a Jordan block of dimension greater than two for a real eigenvalue, nor can $S^{-1}T$ have complex roots as we have observed in the sequel of Theorem 1.3. But for $J = \begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}$ and $E = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ we have $\lambda E + \mu EJ = \begin{pmatrix}0 & \lambda+\mu\alpha\\ \lambda+\mu\alpha & \mu\end{pmatrix}$ is semidefinite iff $\lambda + \mu\alpha = 0$. Hence if there are several 2-dimensional blocks in J_0 they must all correspond to the same eigenvalue $\alpha \in \boldsymbol{R}$ and they all must carry the same sign ε. The signs of the one-dimensional blocks corresponding to the same eigenvalue α cannot be specified since $\varepsilon_i(\lambda + \mu\alpha) = 0$ independent of $\varepsilon_i = \pm 1$, but for one-dimensional blocks in J_0 not corresponding to α, i.e., if $b_l \neq \alpha a_l$, then we must have that $\varepsilon\mu$ and $\lambda a_l + \mu b_l = \mu(b_l - \alpha a_l)$ all have the same sign for $l = j+1, \cdots, n$. If $\mu = 0$ we conclude that if $\lambda E + \mu EJ = \lambda E$ is semidefinite then $\lambda = 0$. Thus μ cannot be zero. And the theorem is proved, since the converse is obvious in this case.

Finally we will apply Theorem 1.1 to give a new elementary proof of the equivalence of (i) and (ii) in Theorem 0.2.

For this we need a theorem of Greub and Milnor [7, p. 256]:

THEOREM 2.1. *Let S, T be* r.s. *matrices of dimension greater than* 2. *If $Q_S \cap Q_T = \{0\}$, then S and T can be diagonalized simultaneously by a real congruence transformation.*

This Theorem was also proved by Majindar [11], Kraljevic [10], Wonenburger [16], Au-Yeung [1], and the author [14].

Proof. [(i) and (ii) of Theorem 0.2 are equivalent.] That (i) implies (ii) is obvious since for definite matrices U we have $Q_U = \{0\}$ and clearly for any $U \in P(S, T)$ we have $Q_U \supseteqq Q_S \cap Q_T$.

If (ii) holds, then by Theorem 2.1 the matrices S and T are simultaneously congruent to $D_1 = \operatorname{diag}(a_i)$ and $D_2 = \operatorname{diag}(b_i)$. We are going to show that

$$\sum a_i x_i^2 = \sum b_i x_i^2 = 0 \quad \text{implies } x_i = 0 \quad \text{for all } i$$

cannot hold unless either (a) or (b) or (c) of Theorem 1.1 is the case.

In the case that, say, all $a_i \geqq 0$ and (c) is violated, then we have $a_i = a_j = 0$ for $i \neq j$, while $b_i \geqq 0$ and $b_j \leqq 0$. With e_i denoting the ith unit vector we then have

$$0 \neq x = e_i + \sqrt{\frac{b_i}{-b_j}} \cdot e_j \in Q_{D_1} \cap Q_{D_2} \quad \text{if } b_j \neq 0 ,$$

while else $e_j \in Q_{D_1} \cap Q_{D_2}$, hence $Q_S \cap Q_T \neq \{0\}$ as well.

In the case that there are positive as well as negative a_i and if both

$$\left[\max_{a_i>0} \frac{b_i}{a_i} \geqq \max_{a_i<0} \frac{b_i}{a_i} \text{ or } b_i \geqq 0 \text{ for some } a_i = 0\right]$$

and

$$\left[\min_{a_i>0} \frac{b_i}{a_i} \leqq \min_{a_i<0} \frac{b_i}{a_i} \text{ or } b_i < 0 \text{ for some } a_i = 0\right]$$

hold, i.e., if both (a) and (b) of Theorem 1.1 are violated, then we will show that $Q = Q_{D_1} \cap Q_{D_2} \neq \{0\}$ and then the equivalence of (i) and (ii) will be proved.

We now go into subcases:

If both $b_i \geqq 0$ and $b_j \leqq 0$ for $a_i = a_j = 0$, then $Q_{D_1} \cap Q_{D_2} \neq \{0\}$ as we have just seen.

If both $\max_{a_i>0} b_i/a_i \geqq \max_{a_i<0} b_i/a_i$ and $b_i \leqq 0$ for some $a_i = 0$ hold, then we assume without loss of generality that the indices in question are 1, 2, 3, respectively, i.e., $b_1/a_1 \geqq b_2/a_2$ where $a_1 > 0$, $a_2 < 0$, and $a_3 = 0$, while $b_3 \leqq 0$.

If $b_3 = 0$, then $e_3 \in Q$. If $b_1/a_1 = b_2/a_2$, then $e_1 + (\sqrt{-a_1/a_2})\, e_2 \in Q$. Otherwise with $x = (1, \sqrt{-a_1/a_2}, \alpha, 0, \cdots, 0)$ we get

$$\sum a_i x_i^2 = a_1 - a_1 \frac{a_2}{a_2} = 0 \text{ independent of } \alpha\,,$$

and

$$\sum b_i x_i^2 = b_1 - \frac{b_2 a_1}{a_2} + b_3 \alpha^2\,.$$

Now by assumption $b_1 - (a_1/a_2)\, b_2 > 0$ and $b_3 < 0$, hence there is a real α s.t. $\sum b_i x_i^2 = 0$ as well, and consequently $Q \neq \{0\}$. If both $\min_{a_i>0} b_i/a_i \leqq \min_{a_i<0} b_i/a_i$ and $b_i \geqq 0$ for some $a_i = 0$ hold, then a similar argument applies.

Finally assume both

$$(*)\quad \max_{a_i>0} \frac{b_i}{a_i} \geqq \max_{a_i<0} \frac{b_i}{a_i} \quad \text{and} \quad \min_{a_i>0} \frac{b_i}{a_i} \leqq \min \frac{b_i}{a_i} \text{ hold}\,.$$

If in (*) there is an equal sign, say $b_1/a_1 = b_2/a_2$ with $a_1 > 0$, $a_2 < 0$, then as above $e_1 + (\sqrt{-a_1/a_2})\, e_2 \in Q$. Thus the only case to remain to be proved is if both inequalities in (*) are strict. For $n \geqq 4$ assume that the indices in question are 1, 2, 3, 4; i.e., $b_1/a_1 > b_2/a_2$ with $a_1 > 0$, $a_2 < 0$ and $b_3/a_3 < b_4/a_4$ with $a_3 > 0$, $a_4 < 0$. Then with $x = (1, (\sqrt{-a_1/a_2}), \alpha, (\sqrt{-a_3/a_4})\,\alpha, 0, \cdots, 0)$ we have

$$\sum a_i x_i^2 = a_1 - a_1 + \alpha^2(a_3 - a_3) = 0 \quad \text{independently of } \alpha\,,$$

and

$$\sum b_i x_i^2 = b_1 - \frac{a_1}{a_2} b_2 + \alpha^2\left(b_3 - b_4 \frac{a_3}{a_4}\right).$$

By assumption $b_1 - (a_1/a_2)\, b_2 > 0$ and $b_3 - b_4\,(a_3/a_4) < 0$. Thus $\alpha \in \boldsymbol{R}$ can be chosen s.t. $\sum b_i x_i^2 = 0$.

For $n = 3$ we may WLOG assume that $\min_{a_i>0} b_i/a_i = \max_{a_i>0} b_i/a_i = b_1/a_1$. Then (*) reads like $b_1/a_1 > b_2/a_2$ and $b_1/a_1 < b_3/a_3$ with $a_1 > 0$; $a_2, a_3 < 0$. Then with $x = (1, \sqrt{-a_1\alpha/a_2}, \sqrt{-a_1\beta/a_3})$ we have

$$\sum a_i x_i^2 = a_1 - a_1(\alpha + \beta) \text{ and } \sum b_i x_i^2 = b_1 - \frac{b_2 a_1}{a_2}\alpha - \frac{b_3 a_1}{a_3}\beta\,.$$

Now the linear system in α and β:

$$\alpha + \beta = 1 \qquad \frac{b_2 a_1}{a_2}\alpha + \frac{b_3 a_1}{a_3}\beta = b_1$$

can be solved for arbitrary b_1, since its determinant

$$a_1\left(\frac{b_3}{a_3} - \frac{b_2}{a_2}\right) \neq 0$$

by assumption.

This completes the proof.

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