PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 2, 1973

ON EXACT LOCALIZATION

ROBERT A. RUBIN

In this paper we consider certain aspects of exact localization (idempotent kernel functors having Property (T) in the language we shall be employing). The major result is that for commutative noetherian rings, every idempotent kernel functor has Property (T) if and only if the Krull dimension of the ring is less than or equal to 1.

1. Preliminaries. The terminology and notation in this paper are that of Goldman [1], with which familiarity is assumed. In particular, if Λ is a ring we denote by $K(\Lambda)$ (respectively $I(\Lambda)$) the set of kernel functors (respectively idempotent kernel functors on the category of left Λ-modules) belonging to Λ. If $\sigma \in K(\Lambda)$, we denote by $\mathscr{T}_\sigma$ the associated filter of left ideals; i.e., $\mathscr{T}_\sigma$ is the set of left ideals $\mathfrak{A}$ of Λ such that $\Lambda/\mathfrak{A}$ is σ-torsion, and such an $\mathfrak{A}$ is called a *σ-open* left ideal. Finally by the term "module" we mean a left module over the ring in question.

Our primary concern will be with kernel functors satisfying any of the conditions of Theorem 4.3 of [1], which we restate for easy reference.

THEOREM 1.1. *For any $\sigma \in I(\Lambda)$, the following conditions are equivalent*:

(i) $Q_\sigma(M) \approx Q_\sigma(\Lambda) \otimes_\Lambda M$ *for every module* M;

(ii) $Q_\sigma(\Lambda)i(\mathfrak{A}) = Q_\sigma(\Lambda)$ *for every* $\mathfrak{A} \in \mathscr{T}_\sigma$, *where* i *is the canonical map* $\Lambda \to Q_\sigma(\Lambda)$;

(iii) *Every* $Q_\sigma(\Lambda)$*-module is faithfully* σ*-injective as a* Λ*-module*; i.e., *given a* $Q_\sigma(\Lambda)$*-module* X, Λ*-modules* $B \subseteqq A$ *with* $\sigma(A/B) = A/B$ *and a* Λ*-homomorphism* $f: B \to X$, *there is a unique* Λ*-homomorphsim* g: $A \to X$ *extending* f;

(iv) *Every* $Q_\sigma(\Lambda)$*-module is* σ*-torsion-free as a* Λ*-module*;

(v) *The functor* Q_σ *is right exact and commutes with direct sums.*

An idempotent kernel functor satisfying any of the above conditions is said to have *Property* (T). Each of the conditions in (v) above has a useful equivalent (Theorems 4.3 and 4.4 of [1]) which we also list.

THEOREM 1.2. *For* $\sigma \in I(\Lambda)$, *the following are equivalent*:

(i) Q_σ *is a right exact functor*;

(ii) *If* $\mathfrak{A} \in \mathscr{T}_\sigma$, *if* $M \xrightarrow{\pi} M''$ *is an epimorphism of* σ*-torsion-free modules, and if* $f: \mathfrak{A} \to M''$ *is a homomorphism, then there exists* $\mathfrak{B} \in$

$\mathscr{T}_\sigma$ with $\mathfrak{B} \subseteqq \mathfrak{A}$ and $g: \mathfrak{B} \to M$ such that πg is the restriction of f to $\mathfrak{B}$.

THEOREM 1.3. *For $\sigma \in I(\Lambda)$ the following are equivalent:*

(i) *Q_σ commutes with direct sums;*

(ii) *Given any countable chain of left ideals whose union is σ-open, then some member of the chain is σ-open.*

REMARKS. (i) The condition of Theorem 1.2 (ii) is described by saying that every σ-open left ideal is *σ-projective*, while that of Theorem 1.3 (ii) is given by: σ is *noetherian*.

(ii) Since for σ with Property (T) every σ-open left ideal contains a finitely generated σ-open left ideal, it has been asked whether, for σ noetherian, every σ-open contains a finitely generated σ-open. We give an example, due to G. Bergman, that supplies the negative answer. Let G be a nondiscrete ordered group in which the identity e is not the inf of any countable set of elements greater than e (e.g., let G be an uncountable product of the integers ordered lexicographically). Let K be a field and let $S = \{g \in G \mid g > e\}$. If $R = K[S]$, the semigroup algebra of S over K, let I be the ideal of R generated by all $g \in S$. Then I is a maximal ideal and $I^2 = I$. Hence $\mathscr{T} = \{I, R\}$ defines $\mu \in I(R)$. Let $J_1 \subseteqq J_2 \subseteqq \cdots$ be a countable chain of ideals with $J_i \neq I, R$ for each i. Then for each i there is $g_i \in S$ with $e \neq g_i \notin J_i$. Since $e \neq \inf g_i$, there is $g \in S$ with $e \neq g \leqq g_i$ for each i. Then $g \notin \bigcup J_i$, and so $\bigcup J_i \notin \mathscr{T}$. Thus μ is noetherian, but I does not contain a finitely generated μ-open ideal.

In considering Property (T) we shall at times make simplifying assumptions that entail no loss of generality. The first such is that the ring be torsion-free for the kernel functor under investigation.

Let $\sigma \in K(\Lambda)$, and consider $\{\mathfrak{A}/\sigma(\Lambda) \mid \mathfrak{A} \in \mathscr{T}_\sigma \text{ and } \mathfrak{A} \supseteqq \sigma(\Lambda)\}$. It is routine to check that this defines a kernel functor of $\Lambda/\sigma(\Lambda)$, which we denote by σ_*, and that σ_* is idempotent if σ is.

PROPOSITION 1.4. *Let $\sigma \in I(\Lambda)$ and let $\sigma_* \in I(\Lambda/\sigma(\Lambda))$ be defined as above. Then σ has Property(T) if and only if σ_* has Property (T).*

Proof. It is immediate that if σ is noetherian, so is σ_*. Using the fact that for σ idempotent, $\mathfrak{A} + \sigma(\Lambda)$ σ-open implies $\mathfrak{A}$ is σ-open we see that if σ_* is noetherian, so is σ.

Now since σ-torsion-free Λ-modules may be identified with σ_*-torsion-free $\Lambda/\sigma(\Lambda)$-modules, and since whenever we have $\mathfrak{A} \xrightarrow{f} M$ with M σ-torsion-free, f factors through $\mathfrak{A} + \sigma(\Lambda)/\sigma(\Lambda) \approx \mathfrak{A}/\mathfrak{A} \cap \sigma(\Lambda)$ we see that every σ-open left ideal of Λ being σ-projective is equivalent to every σ_*-open left ideal of $\Lambda/\sigma(\Lambda)$ being σ_*-projective. Thus Theorems

1.1 (v), 1.2, and 1.3 give the result.

We now consider certain classes of rings for which it is possible to decide, using familiar concepts, when every idempotent kernel functor has Property (T). The first such class we shall consider is the class of rings for which every idempotent kernel functor has a minimal open ideal.

LEMMA 1.5. *Let $\sigma \in I(\Lambda)$ be such that $\mathscr{T}_\sigma$ has minimal elements. Then there is a unique idempotent two-sided ideal, L_σ, such that $\mathscr{T}_\sigma = \{\mathfrak{A} \subseteqq \Lambda \mid \mathfrak{A} \supseteqq L_\sigma\}$.*

Proof. Since $\mathscr{T}_\sigma$ is closed under intersections, it has a unique minimal element, a left ideal we denote by L_σ. Now the uniqueness and minimality of L_σ guarantee that $\mathscr{T}_\sigma = \{\mathfrak{A} \subseteqq \Lambda \mid \mathfrak{A} \supseteqq L_\sigma\}$. Furthermore, $\mathscr{T}_\sigma$ is closed under residual division, whence L_σ is two-sided, and under products of left ideals, from which it follows that L_σ is idempotent.

THEOREM 1.6. *Let $\sigma \in I(\Lambda)$ be noetherian, and suppose that $\mathscr{T}_\sigma$ has a unique minimal element L_σ. Then σ has Property (T) if and only if $L_\sigma + \sigma(\Lambda)/\sigma(\Lambda)$ is a projective $\Lambda/\sigma(\Lambda)$ ideal.*

Proof. Form $\sigma_* \in I(\Lambda/\sigma(\Lambda))$ as for Prop. 1.4. Then $\mathscr{T}_{\sigma*}$ has unique minimal element $L_\sigma + \sigma(\Lambda)/\sigma(\Lambda)$. Hence by Prop. 1.4 we may assume that $\sigma(\Lambda) = 0$. Suppose now that σ has Property (T). Let F be a free module mapping onto L_σ by p. Since $\sigma(\Lambda) = 0$, $\sigma(F) = \sigma(L_\sigma) = 0$. Thus by Theorems 1.1 (v) and 1.2 the identity map on L_σ splits p, and L_σ is projective. Conversely, suppose that L_σ is a projective ideal. Then clearly the condition of Theorem 1.2 (ii) holds, and together with the noetherian hypothesis, this implies that σ has Property (T).

REMARK. The example following Theorem 1.3 gives an example of an idempotent noetherian kernel functor whose filter has a unique minimal element, yet which fails to have Property (T).

COROLLARY 1.7. *Let Λ be left artinian, and suppose that every idempotent two-sided ideal of Λ is projective. Then every $\sigma \in I(\Lambda)$ has Property (T).*

Proof. If L is a projective ideal of Λ, then for any ideal I, $(L + I)/I$ is a projective ideal of Λ/I.

COROLLARY 1.8. *Let R be a commutative artinian ring. Then* *every $\sigma \in I(R)$ has Property (T).*

We now turn to the class of commutative noetherian rings. The crucial fact for these rings is that Property (T) is a local property.

If R is a commutative ring and X a multiplicative subset of R, we define μ_x via: if M is an R-module, $\mu_X(M) = \{m \in M \mid xm = 0$ for some $x \in X\}$. Then $\mu_X \in I(R)$, an ideal A of R is μ_X-open if and only if $A \cap X \neq \varnothing$, and for any module M, $Q_{\mu_X}(M) = M_X$(or $X^{-1}M$). As the next lemma shows, we may also localize kernel functors.

LEMMA 1.9. *Let R be a commutative ring, X a multiplicative subset of R, and $\rho \in I(R)$. Then $\{A_X \mid A \in \mathscr{T}_\rho\} = \mathscr{F}$ defines an idempotent kernel functor of R_X, which we denote by ρ_X.*

Proof. That $\mathscr{F}$ defines a kernel functor, ρ_X, of R_X is routine, following from the extension and contraction properties of ideals under localization. (See [5] p. 46, for instance.) For an ideal K of R_X let $K \cap R$ denote the inverse image of K under the canonical map from R to R_X. Now if $K \subseteqq L$ are ideals of R_X with L ρ_X-open, and L/K ρ_X-torsion, let $A = L \cap R$ and $B = K \cap R$. Then A is ρ-open, and A/B is ρ-torsion. Hence by [1], Theorem 2.5, B is ρ-open. But $K = B_X$, so K is ρ_X-open. Then [1], Theorem 2.5 applies again to show that ρ_X is idempotent.

PROPOSITION 1.10. *Let R be a commutative ring, X a multiplicative subset of R, and $\rho \in I(R)$ an idempotent kernel functor such that every ρ-open ideal contains a finitely generated ρ-open ideal. Then for any R-module M, $[Q_\rho(M)]_X = Q_{\rho_X}(M_X)$.*

Proof. Since an easy calculation shows that $(\rho(M))_X = \rho_X(M_X)$ (the finiteness condition is needed here) we may assume that $\rho(M) = \rho_X(M_X) = 0$. Then it is well-known that $Q_\rho(M) = \varinjlim_{A \in \mathscr{T}_\rho} \mathrm{Hom}_R(A, M)$, and we may take the A's to be finitely generated. (See [3] p. 11 for instance.) But

$$\begin{aligned}(Q_\rho(M))_X &\cong R_X \otimes_R \varinjlim_A \mathrm{Hom}_R(A, M) \cong \varinjlim_A [R_X \otimes_R \mathrm{Hom}_R(A, M)] \\ &\cong \varinjlim_A [\mathrm{Hom}_{R_X}(A_X, M_X)] \cong Q_{\rho_X}(M_X)\ ,\end{aligned}$$

since surely this limit is the same as that taken over A_X. (We have used here the commonly known facts that tensor product commutes with direct limits and Hom commutes with localization when the domain is finitely generated.)

As usual, when P is a prime ideal of R, we shall write ρ_P instead of ρ_{R-P}.

THEOREM 1.11. *Let R be a commutative noetherian ring, and let $\sigma \in I(R)$. Then σ has Property (T) if and only if for every prime ideal P of R, $\sigma_P \in I(R_P)$ has Property (T).*

Proof. Suppose σ has Property (T), and let V be an R_P-module. Then $Q_{\sigma_P}(R_P) \otimes_{R_P} V \cong [Q_\sigma(R)]_P \otimes_{R_P} V \cong (Q_\sigma(R) \otimes_R V)_P \cong (Q_\sigma(V))_P \cong Q_{\sigma_P}(V)$. Thus σ_P has Property (T). Conversely, suppose that for every prime ideal P, σ_P has Property (T), and let M be an R-module. Then there is a homomorphism $\alpha: Q_\sigma(R) \otimes_R M \to Q_\sigma(M)$ (this is just the statement that $Q_\sigma(M)$ is a $Q_\sigma(R)$-module). Then for any prime ideal P, $\alpha_P: (Q_\sigma(R) \otimes_R M)_P \to Q_\sigma(M)_P$. But

$$(Q_\sigma(R) \otimes_R M)_P \cong Q_{\sigma P}(R) \otimes_{R_P} M_P ,$$

and $Q_\sigma(M)_P \cong Q_{\sigma_P}(M_P)$ by Prop. 1.10. Since σ_P has Property (T) for every P, α_P is an isomorphism for every P, and so α is an isomorphism.

REMARK. An examination of the proof yields that σ has Property (T) if and only if σ_m has Property (T) for all maximal ideals m. Now if P is a prime ideal of R and $P \notin \mathscr{T}_\sigma$, then $\mathfrak{A} \in \mathscr{T}_\sigma$ implies $\mathfrak{A} \nsubseteq P$, and so every σ-open ideal meets $R - P$. Hence $\sigma_p = 0$; i.e., $\mathscr{T}_{\sigma_p} = \{R_p\}$. Thus σ has Property (T) if and only if σ_m has Property (T) for every maximal ideal $m \in \mathscr{T}_\sigma$.

We are now prepared to investigate the condition that every $\sigma \in I(R)$ has Property (T).

Recall that if R is a commutative ring, and P a prime ideal of R, the height of P, $ht(P)$, is defined to be the sup of the length of chains of prime ideals: $P_0 \subset P_1 \subset \cdots \subset P_r = P$. Furthermore, if I is an ideal of R, $ht(I) = \inf\{ht(P) \mid P \text{ is a prime ideal containing } I\}$. It will be convenient for our purposes to declare $ht(R) = \infty$.

NOTE. When we are dealing with integral domains, by the term "minimal prime ideal" we mean a prime ideal minimal among the collection of all nonzero prime ideals. Thus for integral domains a minimal prime ideal is the same thing as a height 1 prime ideal.

LEMMA 1.12. *Let R be an integral domain, with quotient field k, such that $R = \bigcap_p R_p$, where P runs through some collection of prime ideals all with height $\leqq$ some fixed integer n. If I is an ideal of height $> n$, then $I^{-1} = R$, where $I^{-1} = \{\alpha \in k \mid \alpha I \subseteqq R\}$.*

Proof. Let $\alpha \in I^{-1}$ and let Q be any prime ideal of height $\leqq n$. Now $\alpha I \subseteqq R$, and since $ht(I) > n$, $I \not\subset Q$. So there is $x \in I$ with $x \notin Q$.

But $\alpha x \in R \subseteq R_{Q'}$ and since $x \notin Q$, x is a unit in R_Q. Thus $\alpha = (\alpha x)x^{-1} \in R_Q$, and α is in R_Q for every prime ideal of height $\leqq n$. Since R is an intersection of such rings, $\alpha \in R$. So $I^{-1} \subseteq R$. But certainly $R \subseteq I^{-1}$ and we are done.

LEMMA 1.13. *Let R be a commutative ring, n a nonnegative integer, and $\mathscr{F} = \{I \subseteq R \mid ht(I) > n\}$. Then defines an idempotent kernel functor*; i.e., *there is $\sigma \in I(R)$ such that $\mathscr{F} = \mathscr{T}_\sigma$.*

Proof. $Ht(I) > n$ means the same thing as I is contained in no prime ideal of height $\leqq n$. Thus we have:

(i) if $I \in \mathscr{F}$, and $J \supseteq I$, then J cannot be contained in any prime ideal of height $\leqq n$, for then I would as well; so $J \in \mathscr{F}$;

(ii) if $I, J \in \mathscr{F}$ and P is a prime ideal of $ht \leqq n$, then $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$, a contradiction; thus IJ, and so $I \cap J$, $\in \mathscr{F}$;

(iii) let $I \in \mathscr{F}$ and $J \subseteq I$ be such that for any $x \in I$ there is a $K \in \mathscr{F}$ with $Kx \subseteq J$, and suppose that $J \subseteq P$ for some prime ideal P with $ht(P) \leqq n$. Now $I \not\subset P$, so there is $x \in I$ with $x \notin P$. But there is $K \in \mathscr{F}$ with $Kx \subseteq J \subseteq P$, and so $K \subseteq P$, contradicting $ht(K) > n$. Thus $J \in \mathscr{F}$. Hence $\mathscr{F}$ defines an idempotent kernel functor.

Recall that an integral domain R is called a *Krull* domain if (i) $R = \bigcap_p R_{p'}$ where P runs through the minimal prime ideals; (ii) for each minimal prime ideal P, R_p is a discrete valuation ring; and (iii) if $\alpha \in k$, the quotient field of R, α is a unit in all but finitely many of the R_p, P a minimal prime ideal. See [2, 4] for more about Krull domains.

In the following theorem we shall have need of the following well-known fact (see [2] Ex. 2. p. 83 for instance): an integral domain R is a Dedekind domain if and only if R is a Krull domain and every nonzero prime ideal of R is maximal.

THEOREM 1.14. *Let R be a Krull domain. Then every $\sigma \in I(R)$ has Property (T) if and only if R is a Dedekind domain.*

Proof. Suppose that every $\sigma \in I(R)$ has Property (T), and let $\mu \in I(R)$ be defined by $\mathscr{T}_\mu = \{I \subseteq R \mid ht(I) > 1\}$. Now for any integral domain D and any $\rho \in I(D)$, $Q_\rho(D) = \bigcup_{J \in \mathscr{T}_\rho} J^{-1}$. So $Q_\mu(R) = \bigcup_{I \in \mathscr{T}_\mu} I^{-1}$. But by Lemma 1.12, for each $I \in \mathscr{T}_\mu$, $I^{-1} = R$, and so $Q_\mu(R) = R$. But μ has Property (T), and so $Q_\mu(R)I = Q_\mu(R)$ for any $I \in \mathscr{T}_\mu$; i.e., for any $I \in \mathscr{T}_\mu$, $RI = R$. Thus $\mathscr{T}_\mu = \{R\}$, and the only ideal of height > 1 is R itself. Hence every nonzero prime is of height 1; i.e., every nonzero prime ideal of R is maximal, as so by the remark preceding the theorem, R is Dedekind.

Conversely, if R is Dedekind, R is an hereditary noetherian ring, and be the discussion on page 31 of [1] every $\sigma \in I(R)$ has Property (T).

THEOREM 1.15. *Let R be a commutative noetherian integral domain. Then every $\sigma \in I(R)$ has Property (T) if and only if every nonzero prime ideal of R is maximal*; (i.e., *Krull dimension of* $R \leqq 1$).

Proof. Suppose that every nonzero prime ideal of R is maximal. Since R is noetherian, Theorem 1.11 holds, and we may assume that R is local. If R is a field, $I(R) = \{0, \infty\}$, both of which have Property (T), and we are done. So suppose that R is not a field. Then the set of nonzero ideals defines $\mu \in I(R)$, and since $\mu = \mu_X$, where X is the set of nonzero elements of R, μ has Property (T). Now let $\sigma \in I(R)$, $\sigma \neq 0, \infty$. Then $0 \notin \mathscr{T}_\sigma$, and for some $\mathfrak{A} \neq R$, $\mathfrak{A} \in \mathscr{T}_\sigma$. Since $\mathfrak{A} \subseteqq \mathscr{M}$ and σ is idempotent, $\mathscr{M}^n \in \mathscr{T}_\sigma$ for any $n \geqq 0$. Let J be any nonzero ideal of R. Since R is noetherian, J contains a power of its radical which, due to a lack of other candidates, is $\mathscr{M}$. Thus $J \supseteqq \mathscr{M}^n$ for some n, and so $J \in \mathscr{T}_\sigma$. Hence $\sigma = \mu$, and so every $\sigma \in I(R)$ has Property (T).

Conversely suppose that every $\sigma \in I(R)$ has Property (T). Again by Theorem 1.11 we may assume that R is local (with maximal ideal $\mathscr{M}$), for the condition that every nonzero prime is maximal is itself a local property. Let S be the integral closure of R in its quotient field k. Then S is a Krull domain (p. 82 of [2]). Define $\sigma \in I(R)$ by $\mathscr{T}_\sigma = \{\mathfrak{A} \subseteqq R \mid \mathfrak{A} \supseteqq \mathscr{M}^n \text{ for some } n\}$. Then $Q_\sigma(R) = \bigcup_n \mathscr{M}^{-n}$, where $\mathscr{M}^{-n} = (\mathscr{M}^n)^{-1} = \{\alpha \in k \mid \alpha\mathscr{M}^n \subseteqq R\}$ ($Q_\sigma(R)$ is also known as the ideal transform of $\mathscr{M}$). Consider the set $\mathfrak{X}$ of prime ideals of S that contain $\overline{\mathscr{M}} = \mathscr{M}S$. By Theorem 33.10 of [4], $\mathfrak{X}$ is finite, and by Theorem 44 p. 29 of [2], $\mathfrak{X}$ consists entirely of maximal ideal of S. Write $\mathfrak{X} = \{\mathscr{N}_1, \cdots, \mathscr{N}_l, P_1, \cdots, P_r\}$ where all the ideals are distinct and the P_j's are all the minimal ideals of S that contain $\overline{\mathscr{M}}$. Since S is Krull, each S_{p_j} is a discrete valuation ring, and so $\bigcap_k P_j^k = 0$ for each j. So for each $j = 1, \cdots, r$ there is $n_j \geqq 1$ such that $\overline{\mathscr{M}} \subseteqq P_j^{n_j}$ but $\overline{\mathscr{M}} \nsubseteqq P_j^{n_j+1}$. Since the P_j's are distinct maximal ideals, $\prod_{j=1}^r P_j^{n_j} = \bigcap_{j=1}^r P_j^{n_j} \supseteqq \overline{\mathscr{M}}$. Set $A = \prod_{j=1}^r P_j^{n_j}$. Now by Exercise 7, p. 83 of [2], if P is a minimal prime ideal in a Krull domain, $PP^{-1} \supsetneqq P$. Hence for each $j = 1, \cdots, r$, P_j is invertible (i.e., $P_jP_j^{-1} = S$). So A is invertible and $AA^{-1} = S$. Consequently $\overline{\mathscr{M}} = S\overline{\mathscr{M}} = (AA^{-1})\mathscr{M} = A(A^{-1}\overline{\mathscr{M}})$. Let $B = A^{-1}\overline{\mathscr{M}}$. Then since $\overline{\mathscr{M}} \subseteqq A$, $B \subseteqq S$, and $BA = \overline{\mathscr{M}}$. Furthermore since $A^{-1} \supseteqq S$, $B \supseteqq \overline{\mathscr{M}}$. Let P be a height 1 prime ideal of S and suppose that $B \subseteqq P$. Then $\overline{\mathscr{M}} \subseteqq P^{n_{j_0}+1}$ so that $P = P_{j_0}$ for some j_0, and $\overline{\mathscr{M}} = BA \subseteqq P_{j_0}(\prod_{j=1}^r P_j^{n_j}) \subseteqq P_{j_0}$, a contradiction. Thus $ht(B) > 1$, and by Lemma 1.12, $B^{-1} = S$. Similarly $B^{-n} = S$ for all $n \geqq 1$. Now $BA \subseteqq A$, so that $(BA)^{-1} \supseteqq A^{-1}$. Let $\alpha \in (BA)^{-1}$.

Then $\alpha BA \subseteqq S$, and for any $x \in A$, $\alpha xB \subseteqq S$. So $\alpha A \subseteqq B^{-1} = S$, and hence $A^{-1} = (BA)^{-1}$. Thus $\bar{\mathscr{M}}^{-1} = (BA)^{-1} = A^{-1}$, and similarly, for any $n \geqq 1$, $\bar{\mathscr{M}}^{-n} = A^{-n}$. Now suppose that σ has Property (T). Then by Theorem 1.1 (ii) $\mathscr{M} Q_\sigma(R) = Q_\sigma(R)$, and so for some t, $1 \in \mathscr{M}\mathscr{M}^{-t}$. But $\mathscr{M}^{-t} \subseteqq \bar{\mathscr{M}}^{-t}$, and so $1 \in \bar{\mathscr{M}}\bar{\mathscr{M}}^{-t} = \bar{\mathscr{M}}(A^{-t})$. Since $S \subseteqq \bar{\mathscr{M}}A^{-t}$ and A is invertible, $A^t \subseteqq \bar{\mathscr{M}}$. Consequently there are no $\mathscr{N}_i$'s in $\mathfrak{X}$. Thus $\bar{\mathscr{M}}$ is contained only in minimal ideals of S. But then Theorem 44 p. 29 of [2] tells us that $\mathscr{M}$ is a minimal prime ideal of R, and the proof is complete.

The preceding theorem is the means by which we can determine when every idempotent kernel functor has Property (T) for an arbitrary commutative noetherian ring. The following lemma is rather interesting by itself. Recall that if P is a prime ideal in a commutative R, then $\mu_p \in I(R)$ is given by $\mathscr{T}_{\mu_p} = \{I \subseteqq R \mid I \not\subseteqq P\}$, and $Q_{\mu_p}(M) = M_p \approx R_p \otimes_R M$ for any module M.

LEMMA 1.16. *Let R be a commutative ring, and let $P_1, \cdots, P_n$ be prime ideals of R. Then $\inf_i \mu_{p_i} = \mu_X$, where $X = \bigcap_{i=1}^n (R - P_i)$ and $\inf_\alpha \sigma_\alpha$ is defined by $(\inf_\alpha \sigma_\alpha)M = \bigcap_\alpha \sigma_\alpha(M)$ M any module, or equivalently $\mathscr{T}_{\inf \sigma_\alpha} = \bigcap_\alpha \mathscr{T}_{\sigma_\alpha}$, for any $\{\sigma_\alpha\} \subseteqq K(R)$.*

Proof. Let $\sigma = \inf_i \mu_{p_i}$. If $\mathfrak{A} \in \mathscr{T}_\sigma$, then $\mathfrak{A} \in \mathscr{T}_{\mu_{p_i}}$ for each $i = 1, \cdots, n$, and so $\mathfrak{A} \not\subseteqq P_i$ for each i. But then $\mathfrak{A} \not\subseteqq \cup P_i$ (see Theorem 81 p. 55 of [2] for instance). So there is $a \in \mathfrak{A}$ such that $a \in \bigcap_{i=1}^n (R - P_i)$ thus $\mathfrak{A} \cap X \neq \varnothing$, and $\sigma \leqq \mu_X$. But if $\mathfrak{B} \in \mathscr{T}_{\mu_X}$, there is $b \in \mathfrak{B}$ with $b \notin P_i$ for each i. Then certainly $\mathfrak{B} \not\subseteqq P_i$ for each i, so that $\mu_X \leqq \sigma$. Hence $\inf_i \mu_{p_i} = \mu_X$.

COROLLARY 1.17. *If R is a commutative ring and $P_i, \cdots, P_n$ are prime ideals, then $\inf_i \mu_{p_i}$ has Property (T).*

THEOREM 1.18. *Let R be a commutative noetherian ring. Then every $\sigma \in I(R)$ has Property (T) if and only if every nonminimal prime ideal of R is maximal* (i.e., *if P is a prime ideal of R, $htP \leqq 1$*).

Proof. Since both of the conditions are local properties we may assume at the outset that R is local, with maximal ideal $\mathscr{M}$. Suppose that every nonminimal prime ideal is maximal. Since R is noetherian, a primary decomposition of 0 exists, from which we conclude that R has only finitely many minimal prime ideals. Thus the set of prime ideals of R consists of $\mathscr{M}$ and finitely many minimal primes $P_1, \cdots, P_n$. Now $\mu_{\mathscr{M}} = 0$ (i.e., $\mathscr{T}_{\mu_{\mathscr{M}}} = \{R\}$), and if $0, \infty \neq \sigma \in I(R)$, then by the discussion on page 34 of [1], $\sigma = \inf\{\mu_{p_{i_1}}, \cdots, \mu_{p_{i_j}}\}$, which, by Corollary 1.17, has Property (T). Since 0 and ∞ both have Property (T), all $\sigma \in$

$I(R)$ have Property (T). Conversely suppose that every $\sigma \in I(R)$ has Property (T) and let $P \neq \mathscr{M}$ be a prime ideal of R. Then the ring R/P is a local noetherian integral domain. Furthermore, it is easy to check that: $K(R) \rightarrow K(R/P)$, induced by $R \rightarrow R/P$, is onto. Then using Theorem 1.1 (iv) one can show that every $\rho \in I(R/P)$ has Property (T), and so by Theorem 1.15, $\mathscr{M}/P$ is the only nonzero prime ideal of R/P. Hence P is a minimal prime ideal of R, and the proof is complete.

Finally we give an example to show that the noetherian hypothesis is essential in the preceding theorems. Let X be the following semigroup: as a set, $X = \{x^\alpha \mid \alpha$ is a positive real number$\}$, and $x^\alpha \cdot x^\beta = x^{\alpha+\beta}$. If K is a field, let $S = K[X]$, the semigroup algebra of X on K. Then $\mathscr{M}$, the ideal generated by all the x^α (i.e., the ideal consisting of elements with no constant term), is a maximal ideal of S. Let $R = S_{\mathscr{M}}$. It is routine to check that the ideals of R are linearly ordered and that $\mathscr{M}R$ and 0 are the only prime ideals of R. Furthemore $(\mathscr{M}R)^2 = \mathscr{M}R$. Thus $\{\mathscr{M}R, R\}$ defines some $\sigma \in I(R)$. But since $\mathscr{M}R$ is not finitely generated σ cannot have Property (T). Thus R is an integral domain for which every nonzero prime ideal is maximal, yet not every idempotent kernel functor has Property (T).

The author wishes to express his gratitude to Paul Eakin for his generous expenditure of time and information.

REFERENCES

1. O. Goldman, *Rings and modules of quotients*, J. Algebra, **13** (1969), 10–47.
2. I. Kaplansky, *Commutative Algebra*, Allyn & Bacon, Inc., Boston, 1970.
3. J. Lambek, *Torsion Theories, Additive Semantics, and Rings of Quotients*, Springer Lecture Notes No. 177, 1971.
4. M. Nagata, *Local Rings*, Interscience, New York, 1962.
5. D. G. Northcott, *Ideal Theory*, Cambridge University Press, 1953.

Received August 14, 1972 and in revised form July 17, 1973.

UNIVERSITY OF KENTUCKY
AND
PAHLAVI UNIVERSITY, SHIRAZ, IRAN