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ALMOST ISOMETRIES OF BANACH SPACES AND MODULI OF PLANAR DOMAINS

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Let $C(X)$ and $C(Y)$ be the supremum normed Banach spaces of continuous complex valued functions on the compact Hausdorff spaces X and Y respectively. Let A and B be closed subspaces of $C(X)$ and $C(Y)$ respectively. A map from A to B will mean a continuous invertible linear map of A to B. The set of all such maps will be denoted by $L(A, B)$. For T in $L(A, B)$ define $c(T) = 1/(\| T \| \| T^{-1} \|)$. A generalization of the Banach-Stone theorem is proved which shows that there is a constant $d < 1$ such that if A and B satisfy certain additional technical restrictions and there is a T in $L(A, B)$ with $c(T) > d$ then X and Y are homeomorphic. Furthermore, T is, roughly, composition with this homeomorphism.

For S a connected subset of C bounded by a finite number of disjoint Jordan curves, denote by $A(S)$ the Banach space of functions in $C(S)$ which are analytic on the interior of S. For two such domains, S and S', set $d(S, S') = \inf\{-\log c(T); T$ a linear map of $A(S)$ onto $A(S')\}$. By analyzing maps T for which $c(T)$ is nearly one, it is shown that $d(\cdot, \cdot)$ is a metric on the space of moduli of such domain (considered as Riemann surfaces) and that this metric induces the classical moduli topology.

If T in $L(A, B)$ preserves norms, i.e., if $\| Tf \| = \| f \|$ for all f in A, T is called an isometry. If there is an isometry between A and B then A and B are called isometric. If T is of norm one then $c(T)$ is the largest constant such that $c(T) \| f \| \leqq \| Tf \| \leqq \| f \|$. It is immediate that $0 < c(T) \leqq 1$ for all T in $L(A, B)$ and that $T/\| T \|$ is an isometry if and only if $c(T) = 1$. Maps, T, for which $c(T)$ is nearly one will be called almost isometries.

Using this notation, the Banach-Stone theorem can be stated as follows.

THEOREM. *If there is a T in $L(C(X)), C(Y))$ with $c(T) = 1$ then X and Y are homeomorphic. Furthermore, any such T is of the form $Tf = g \cdot f \circ h$ where h is a homeomorphism from Y to X and g is a continuous function on Y of constant modulus.*

More recently, Cambern ([2], [3]) has extended this result to

THEOREM. *If there is a T in $L(C(X), C(Y))$ with $c(T) > 1/2$ then X and Y are homeomorphic.*

In § 2 we show that if A and B are closed subspaces (*not* necessarily subalgebras) of $C(X)$ and $C(Y)$ respectively which satisfy certain additional technical restrictions then the following is true.

THEOREM A. *There is a constant* $d < 1$ *so that if* A *and* B *are allowable subspaces of* $C(X)$ *and* $C(Y)$ *respectively and if there is a* T *in* $L(A, B)$ *with* $c(T) > d$, *then* X *and* Y *are homeomorphic. Furthermore, there is a function* $\varepsilon(x)$ *which decreases continuously to zero as* x *increases to one so that given such a* T *there is a homeomorphism* h *of* Y *to* X *and a function* g *of constant modulus so that for all* f *in* A, $\| Tf - g \cdot f \circ h \| \leqq \varepsilon(c(T)) \| f \|$.

If A and B are any two Banach spaces then the quantity $D(A, B) = \inf\{-\log c(T); T \text{ in } L(A, B)\}$ is a measure of how close A and B are to being isometric. Define $D(A, B) = \infty$ when $L(A, B)$ is empty. This function was first studied by Banach and Mazur ([1]) who observed that it is symmetric, positive semi-definite and satisfies the triangle inequality. Furthermore, if A and B are isometric then $D(A, B) = 0$.

This function, applied to certain algebras of analytic functions on planar Riemann surfaces, will be shown to define a metric on the moduli space of those surfaces. Specifically, we denote by $\mathscr{S}$ the set of conformal equivalence classes of Riemann surfaces realizable as connected subsets of the complex plane bounded by two or more (but a finite number of) disjoint Jordan curves. For $n \geqq 2$ denote by $\mathscr{S}_n$ the class of surfaces in $\mathscr{S}$ with n boundary contours. For any S in $\mathscr{S}$ we define $A(S)$ to be the subalgebra of $C(S)$ consisting of all functions in $C(S)$ which are analytic on the interior of S. It is known that this definition is independent of the particular realization of S. For S and S' in $\mathscr{S}$, define $d(S, S') = D(A(S), A(S'))$.

In § 3 we introduce a particular set of classical moduli for the sets $\mathscr{S}_n$.

In § 4 we introduce a set of conformal invariants for Riemann surface in $\mathscr{S}$ and study the relationship between these invariants and the function $d(\cdot, \cdot)$.

Sections 5 and 6 contain the major parts of the proof of the following.

THEOREM B. *For any integer* $n \geqq 2$, $d(\cdot, \cdot)$ *is a metric on the space* $\mathscr{S}_n$. *This metric induces the same topology as the classical moduli topology.*

A significant portion of the arguments use specific realizations of the Riemann surfaces in question as subsets of the complex plane. It is not clear to what extent these result can be extended to nonplanar surfaces or to surfaces of infinite connectivity. (Some results for nonplanar surfaces are presented in [7].)

These results are new, however, results very similar in spirit have been obtained by Nakai ([5]) using different methods. He shows that for a certain class of Riemann surfaces, the extent to which the Royden algebras of two surfaces are almost isometric is directly related to the minimal dilitation of quasiconformal maps between the two surfaces.

In addition to the notation previously introduced and that introduced in the individual sections we will use the following notation.

If a is an element of $C(X)$ and K is a subset of X, we set $\|a\|_K = \sup\{|f(x)|; x \text{ in } K\}$. We will use this notation even though $\|\cdot\|_K$ may not be a norm.

Let D be the unit disk of the complex plane. For z and z' in D we define $\delta(z, z') = |(z - z')/(1 - \bar{z}z')|$. If S is a Riemann surface which is conformally equivalent to D and w and w' are any two points of S, we set $\delta_S(w, w') = \delta(t(w), t(w'))$ where t is a conformal map of S onto D. It is well known that this last definition is independent of the choice of t.

2. Properties of almost isometries of Banach spaces. When working with an invertible linear map, T, between the Banach spaces we will often use the following notational convention. The spaces will be denoted by two capital Latin letters. Elements of the spaces will be denoted by the corresponding lower case letters. Finally, elements of the spaces will be individuated so that elements with corresponding individuation marks will be elements that correspond under T. For example, given T in $L(A, B)$, without further mention our convention guarantees that a, a', and a_3 are in A; b, b', and b'' are in B; and $T(a) = b$, $T(a_3) = b_3$, etc.

Given an element a in the Banach space $C(X)$ and a point x in X we will say that a *peaks at* x if a attains its maximum modulus at x, i.e., $|a(x)| = \|a\|$. We will say that a *peaks only at* x if x is the only point of X at which a peaks, i.e., $|a(x')| < \|a\|$ for all x' in X, $x' \neq x$. We will say that a sequence of functions $a_1, a_2, \cdots$ in $C(X)$ is a *fundamental sequence* at x if

(a) all of the a_i are of norm one and peak only at x, and

(b) for any y in $X, y \neq x$, the sequence of numbers $|a_1(y)|$, $|a_2(y)|, \cdots$ converges monotonically to zero.

It follows from this definition that if $a_1, a_2, \cdots$ is a fundamental sequence at x then the a_n converge uniformly to zero off every open neighborhood of x and that for any k between zero and one the set on which $|a_n(y)| > k$ shrinks to x as n becomes infinite. Let A be a closed subspace of $C(X)$. We will call A an *allowable subspace* if for every x in X, one can find in A a fundamental sequence at x.

We have not required that an allowable subspace be a subalgebra. It follows from the definition that a closed subalgebra of $C(X)$ is an allowable subspace of $C(X)$ if and only if for any x in X there is an a in A which peaks only at x (i.e., every point of X is a peak point for A). If S is a finite bordered Riemann surface and $A(S)$ is the algebra of functions continuous on the bordered surface and analytic on the interior of S, then $A(S)$, regarded as a subalgebra of $C(\partial S)$, satisfies this condition and hence is an allowable subspace of $C(\partial S)$.

For the remainder of this section we will be considering the following situation. X and Y will be compact Hausdorff spaces and A and B will be allowable subspaces of $C(X)$ and $C(Y)$ respectively. T will be an element of $L(A, B)$ of norm one. We set $c = c(T) = \|T^{-1}\|^{-1}$. Our goal will be to develop properties of T which follows from the assumption that c is sufficiently near one.

If $c = 1$ then some of the proofs in this section are not valid. However, in that case, the results remain valid and are simply known results and their direct corollaries.

2.1. We begin by showing that if c is large enough then, given x in X, there is a y in Y so that if a in A peaks at x then $b = Ta$ "almost" peaks at y.

THEOREM. *There is a $c_0 < 1$ such that if $c > c_0$ then given x_0 in X there is a unique y_0 in Y so that if a peaks at x_0, then $|b(y_0)| \geqq (2c-1)\|a\|$, and hence $|b(y_0)| \geqq (2c-1)\|b\|$.*

Proof. We begin with a lemma.

LEMMA. *Given a_1, a_2 in A, if $|a_1(x)| \geqq |a_2(x)|$ for all x in X and if b_2 peaks at y, then $|b_1(y)| \geqq c\|a_2\| - ((1-c)/c)\|a_1\|$.*

Proof of lemma. The result is immediate if b_1 peaks at y. Assume this is not the case. Given ε positive, pick U an open neighborhood of y so that $|b_1(w) - b_1(y)| < \varepsilon$ for all w in U. Pick b_3 so that $\|b_3\| = \|b_1\| - |b_1(y)|$, $\arg(b_3(y)) = \arg(b_2(y))$, b_3 peaks only at y, and $|b_3| < \varepsilon$ off U. Hence, for appropriate θ, since b_2 peaks at y,

$$\begin{aligned} c(\|b_2\| + \|b_3\|) = c(\|b_2 + b_3\|) &\leqq c(\|a_2 + a_3\|) \\ &\leqq c(\|e^{i\theta}a_1 + a_3\|) \leqq \|e^{i\theta}b_1 + b_3\| . \end{aligned}$$

But, by the construction of b_3,

$$\|e^{i\theta}b_1 + b_3\| \leqq \|b_1\| + \varepsilon$$

hence

$$c\|b_2\| \leqq \|b_1\| + \varepsilon - c\|b_3\| \leqq \|b_1\| + \varepsilon - c(\|b_1\| - |b_1(y)|) .$$

Since ε was arbitrary we have

$$|b_1(y)| \geqq \|b_2\| - \left(\frac{1-c}{c}\right)\|b_1\| \geqq c\|a_2\| - \left(\frac{1-c}{c}\right)\|a_1\| .$$

Proof of theorem. For any b in B we define $P_b = \{y \text{ in } Y: |b(y)| \geqq (4c-3)\|b\|\}$. Pick $a_1, \cdots, a_n, \cdots$ a fundamental sequence at x_0. We can assume $a_n(x_0) = 1$. Let $Q_\infty = \bigcap_{n=1}^{\infty} P_{b_n}$. Q_∞ is the required point y_0. We prove this by verifying a series of claims.

Claim A. Q_∞ is not empty. The P_{b_n} are closed subsets of the compact set Y. Hence it suffices to show that they have the finite intersection property. Given N, we must show that $Q_N = \bigcap_{n=1}^{N} P_{b_n}$ is not empty. Pick y in Y so that b_N peaks at y. Since a_N is of norm one, $|b_N(y)| \geqq c$. By construction y is in P_{b_N}. Since the a_n form a fundamental sequence, the previous lemma can be applied. For $k < N$, $|b_k(y)| \geqq c\|a_N\| - ((1-c)/c)\|a_k\| = c + 1 - 1/c$. Hence, if c is greater than 1/3 we have $|b_k(y)| \geqq 4c - 3$. So y is in P_{b_k}. Hence y is in Q_N, and thus $Q_\infty \neq \varnothing$.

Claim B. Q_∞ is a single point. Suppose Q_∞ contains two distinct points y' and y''. Choose a small positive ε. Choose b' and b'' from fundamental sequences at y' and y'' so that $\sup_\theta \|b' + e^{i\theta}b''\| \leqq 1 + \varepsilon$. Since b' peaks in Q_∞, for each n we can find a θ so that $4c - 2 \leqq \|b_n + e^{i\theta}b'\|$. Hence $4c - 2 \leqq \|a_n + e^{i\theta}a'\|$. Let x' be any point at which $|a_n(x') + e^{i\theta}a'(x')| \geqq 4c - 2$. We must have $|a'(x')| \geqq 4c - 3$ and $|a_n(x')| \geqq 4c - 2 - |a'(x')| \geqq 4c - 2 - 1/c$. Pick c so large that for some positive ε' we have

$$4c - 2 - 1/c \geqq \varepsilon' > 0 . \tag{2.1}$$

Hence a' takes a value of modulus greater than or equal to $4c - 3$ at a point x' where the modulus of a_n is greater than ε'. The a_n are a fundamental sequence at x_0. Letting n go to infinity, the set on which a_n is of modulus greater than ε' shrinks to the point x_0 and thus the points x' which were chosen depending on n converge to x_0. By continuity of a', $|a'(x_0)| \geqq 4c - 3$. Similarly, $|a''(x_0)| \geqq 4c - 3$. So, for some θ

$$\|a' + e^{i\theta}a''\| \geqq |a'(x_0)| + |a''(x_0)| \geqq 2(4c-3) = 8c - 6 .$$

So

$$8c - 6 \leqq \sup \|a' + e^{i\theta}a''\| \leqq \frac{1}{c}\sup\|b' + e^{i\theta}b''\| < \frac{1}{c}(1+\varepsilon) .$$

Since ε was arbitrary we must have $1/c \geqq 8c - 6$. This is impossible if $c > .89$. This contradiction shows that Q_∞ is a single point. Denote the point by y_0.

Claim C. If $|b_N(y)| > 2c - 1$ then $|b_k(y)| > 4c - 3$ for all k less than N. Assume $|b_N(w)| > 2c - 1$ and $|b_k(w)| < 4c - 3$ for some w and some k less than N. Pick b' peaking at w, $|b'(w)| = c$, $\arg b'(w) = \arg b_N(w)$, and modulus of b' less than some preassigned ε on P_{b_k}. (The assumption implies that w is not in P_{b_k}.) Hence $\| b_N + b' \| \geqq 2c$, and $\sup_\theta \| b_k + e^{i\theta} b' \| \leqq 4c - 2$. So, $2c \leqq \| a_N + a' \|$. Let x be a point at which $2c \leqq |a_N(x)| + |a'(x)|$. So

$$4c - 2 \geqq \sup_\theta \| b_k + e^{i\theta} b' \| \geqq c \sup_\theta \| a_k + e^{i\theta} a' \|$$
$$\geqq c(|a_k(x)| + |a'(x)|) .$$

The a_n are a fundamental sequence, hence $|a_N(x)| \leqq |a_k(x)|$. So, $4c - 2 \geqq c(|a_N(x)| + |a'(x)|) \geqq c \cdot 2c$ which is impossible if $c \neq 1$.

Claim D. Given an open set U in Y with y_0 in U, then for all but finitely many n, $\| b_n \|_{Y-U} < 2c - 1$. If not, then there are integers n_i increasing to infinity and points y_i in $Y - U$ with $|b_{n_i}(y_i)| \geqq 2c - 1$. Let y' be an accumulation point of the y_i. By Claim C, $|b_n(y_i)| \geqq 4c - 3$ for $n \leqq n_i$. Hence, for each n, by the continuity of b_n, $|b_n(y')| \geqq 4c - 3$. Thus y' is in $Q_\infty \subseteqq U$. But y' is not in U. This contradiction establishes the claim.

Claim E. If a' is in A and $\| a' \| = a'(x_0) = 1$ then $|b'(y_0)| \geqq 2c - 1$. If not, then there is an open set U in Y containing y_0 with $|b'| \leqq 2c - 1 - \varepsilon$ on U. Claim D shows that if we take n large enough we can insure that $|b_n(y)| < 2c - 1 - \varepsilon$ for y not in U. We know that $\| a_n + a' \| = 2$. Hence $2c \leqq \| b_n + b' \| \leqq \max (\| b_n + b' \|_U, \| b_n + b' \|_{Y-U}) \leqq 2c - \varepsilon$. This contradiction establishes Claim E.

Claim E combined with Claim B shows that the point y_0 is unique and independent of the original choices of the function a and the fundamental sequence. The theorem is proved.

DEFINITION. For any x_0 in X let $t(x_0)$ be the point y_0 in Y such that the previous theorem is satisfied.

During the proof a number of restrictions were placed on c. The most severe of these was (2.1). This inequality will be satisfied if $c > c_0$ where $c_0 = (3 + \sqrt{17})/8 = .8904$. There is no reason to assume that this is the best value for which the theorem is true. The results of Cambern referred to earlier are valid if $c > 1/2$. It may be that $c_0 = 1/2$ is also adequate for Theorem A.

COROLLARY 2.2. *If $c > c_0$ then, given x in X, there is a number*

$\theta(x)$ *such that if* $\|a\| = a(x) = 1$ *then* $\cos(\arg(b(t(x)) - \theta(x))) \geqq 2c - 1$.

Proof. Let $S = \{b(t(x)); b = Ta, \|a\| = a(x) = 1\}$. Let $\theta(x) = 1/2(\sup\{\arg(z); z \text{ in } S\} + \inf\{\arg(z); z \text{ in } S\})$. If z_1 and z_2 are in S then $|z_i| \leqq 1$, $i = 1, 2$ and by the previous theorem $|z_1 + z_2| \geqq 4c - 2$. Hence, by an elementary geometric argument, for any z in S, $\cos(\arg(z) - \theta(x)) \geqq 2c - 1$.

THEOREM 2.3. *If* $c > c_0$ *then* t *is a homeomorphism of* X *and* Y.

Proof. *t is continuous.* Since Y is compact it suffices to show that if x_n, $n = 1, 2, \cdots$ converge to x_0 and $y_n = t(x_n)$ converge to y', then $y' = y_0 = t(x_0)$. For $n = 0, 1, \cdots$ let $a_{n,1}, a_{n,2}, \cdots$ be a fundamental sequence at x_n. Choose b' of norm one peaking only at y'. Choose ε positive. By the previous theorem, for n sufficiently large and for all k, $\sup_\theta \|b_{n,k} + e^{i\theta}b'\| \geqq 2c - \varepsilon$. Hence, for the same n and k $\sup_\theta \|a_{n,k} + e^{i\theta}a'\| \geqq 2c - \varepsilon$. Fixing n sufficiently large and letting k become infinite we conclude $|a'(x_n)| \geqq 2c - 1 - \varepsilon$. Taking the limit as n becomes infinite and noting that ε was arbitrary we find $|a'(x_0)| \geqq 2c - 1$. Pick a of norm one, peaking only at x_0 and with $\arg a(x_0) = \arg a'(x_0)$. So, $\|a + a'\| \geqq 2c$. Hence $\|b + b'\| \geqq 2c^2$. Since b' could be any element of a fundamental sequence at y', this implies $|b(y')| \geqq 2c^2 - 1 \geqq 4c - 3$. Thus, for any a peaking only at x_0, y' is in $P_{T(a)} = \{y \in Y; |Ta(y)| \geqq 4c - 3\}$. y_0 is the intersection of all such $P_{T(a)}$. Thus $y' = y_0$. This argument also holds, mutatis mutandis, for convergence over nets.

t is one-to-one. Suppose $t(x') = t(x'')$. Let $a'_1, a'_2, \cdots$ be a fundamental sequence at x' and let $a''_1, a''_2, \cdots$ be a fundamental sequence a x''. Estimating $\sup\|a'_n + e^{i\theta}a''_n\|$ and $\sup\|b'_n + e^{i\theta}b''_n\|$ for large n shows $x' = x''$.

t is onto. If not, since $t(X)$ is closed, we can find U a nonempty open set in Y disjoint from $t(X)$. Pick y in U, b in B of norm one peaking only at y and $|b| < 2c - 1$ off U. Let x be a point of X at which a peaks. By Theorem 2.1 $|b(t(x))| \geqq (2c - 1)\|b\| = 2c - 1$. Thus $t(x)$ is in U: a contradiction.

COROLLARY. *If there is a* T *in* $L(A, B)$ *with* $c(T) \geqq c_0$ *then* X *and* Y *are homeomorphic.*

Using the homeomorphism t to identify X and Y, the function $\theta(\cdot)$ of Corollary 2.2 can be regarded as a function on X or on Y. For x in X, let $\theta(x)$ be the number produced in the proof of this corollary. For y in Y, set $\theta(y) = \theta(t^{-1}(x))$.

THEOREM 2.4. *If* $c > c_0$ *then for all* a *in* A, *for all* x *in* X, $\| b(t(x)) | - | a(x) \| \leqq 2(1-c)\| a \|$.

Proof. It suffices to consider the case $\| a \| = 1$, $a(x)$ real and positive. Let $y = t(x)$. Pick $a_1, a_2, \cdots$ a fundamental sequence for x with $a_i(x) = 1$ for all i. Set $r = 1 - a(x)$. Pick ε positive. Let x_n be the point at which $a + (1+\varepsilon)r\, a_n$ peaks. Let $y_n = t(x_n)$. By Theorem 2.1 $| b(y_n) + (1+\varepsilon)r\, b_n(y_n) | \geqq (2c-1)\| a + (1+\varepsilon)r\, a_n \| > 2c - 1$. Hence $| b(y_n) | \geqq 2c - 1 - (1+\varepsilon)r = 2c - 2 + | a(x) | - \varepsilon(1 - a(x))$. Letting n go to infinity, x_n approaches x and hence y_n approaches y so $| b(y) | \geqq 2c - 2 + | a(x) | - 2\varepsilon$. Since ε was arbitrary

$$| b(y) | \geqq 2c - 2 + | a(x) | \,. \tag{2.2}$$

Also,

$$\begin{aligned} 1 + | a(x) | &= \lim_{n\to\infty} \sup_{\theta} \| a + e^{i\theta} a_n \| \\ &\geqq \overline{\lim_{n\to\infty}} \sup_{\theta} \| b + e^{i\theta} b_n \| \geqq \overline{\lim}\, (| b(y) | + | b_n(y) |) \,. \end{aligned}$$

So

$$1 + | a(x) | \geqq | b(y) | + 2c - 1 \,. \tag{2.3}$$

The last inequality by Theorem 2.1. Inequalities (2.2) and (2.3) imply the desired conclusion.

THEOREM 2.5. *Let* $c > c_0$. *Given* K, $0 < K \leqq 1$, *and* $\varepsilon > 0$, *there is a* $d < 1$ *which depends only on* K *and* ε *such that if* $c > d$, *then given* a *of norm one and given* x *in* X *with* $| a(x) | \geqq K$ *and* $\arg(a(x)) = 0$, *then* $| \arg(b(t(x)) - \theta(x)) | < \varepsilon$.

Proof. We may assume $a(x)$ positive. Let $y = t(x)$. Choose $a' \neq a$ peaking only at x with $a'(x) = 1$. Set $a'' = (a - a(x)a')/(\| a - a(x)a' \|)$. a'' is a function of unit norm and $a''(x) = 0$. Applying the previous theorem to a'' at the point x we find $| b''(y) | \leqq 2(1-c)$. So $| b(y) - a(x)b'(y) | \leqq 2(1-c)\| a - a(x)a' \| \leqq 4(1-c)$. So $| b(y)/a(x) - b'(y) | \leqq 4(1-c)/a(x) \leqq 4(1-c)/K = \varepsilon_1$. Also, by applying Theorem 2.1 and Corollary 2.2 to a' at x we find $| b'(y) - \exp(i\theta(y)) | \leqq \varepsilon_2$ where ε_1 and ε_2 depend only on c and K and can be chosen to be arbitrarily small if c is close enough to one. Combining the last two inequalities we find $| b(y)/a(x) - \exp(i\theta(y)) | < \varepsilon_3$ where ε_3 can be made arbitrarily small if c is close enough to one. Since $| a(x) | < 1$ this implies $| b(y) - \exp(i\theta(y))a(x) | < \varepsilon_3$. But $a(x)$ is a positive real number greater than K. By elementary geometry the previous inequality implies that the quantity $| \arg(b(y)) - \theta(x) |$ can be made arbitrarily small by requiring that ε_3 be sufficiently small. This is accomplished by requiring that c be larger than some constant d; and the theorem is proved.

2.6. *Proof of Theorem* A. If suffices to prove the theorem for T of norm one. Let $g(y) = \exp(i\theta(y))$ and $h(y) = t^{-1}(y)$. Theorem 2.3 shows that h is a homeomorphism. The previous two theorems combine to show that g and h satisfy the requisite inequality.

2.7. If the allowable subspaces A and B are also algebras and if $T(1)$ is a real positive constant, then, for $c(T)$ large enough, T is almost an algebra isomorphism.

THEOREM A′. *There is a function $\varepsilon(x)$ which decreases continuously to zero as x increases to one and a constant $d < 1$ so that if A and B are algebras and $T(1)$ is a positive constant and $c(T) > d$ then there is an isometric algebra isomorphism R of $C(X)$ onto $C(Y)$ so that, as linear maps on A, $\| T - R \| \leqq \varepsilon(c(T))$.*

Proof. Define $Rf(y) = f(t^{-1}(y))$. The theorem follows from the following lemma.

LEMMA. *If 1 is in A then given ε positive there is a $d < 1$ which depends only on ε so that $c > d$ implies that for all a in A, for all x in X*

$$| b(t(x)) - \exp(i \arg(T1)(t(x)))a(x) | \leqq \varepsilon \| a \| .$$

Proof. We may assume $\| a \| = 1$. Note that Theorem 2.1 implies that $T1$ is bounded away from zero and hence has a well defined argument. Choose d large enough so that the desired result follows immediately from Theorem 2.4 whenever $| a(x) | < \varepsilon/3$. Now, by increasing d and applying Theorem 2.5 with $K = \varepsilon/3$ and Theorem 2.4 we can insure $| b(t(x)) - \exp(i\,\theta(x))a(x) | < \varepsilon/3$. Hence it suffices to show that $| \theta(x) - \arg(T(1)(t(x))) |$ can be made small for all x. This, however, follows from Corollary 2.2.

COROLLARY 2.7.1. *Given $\varepsilon > 0$ there is a $d < 1$ so that if $c > d$, A and B are algebras, and $T(1)$ is a positive constant, then for all a, a' in A, $\| (Ta)(Ta') - T(aa') \| \leqq \varepsilon \| a \| \| a' \|$.*

Proof. Let R be given by the previous theorem. $T = R + (T - R)$. R is multiplicative and $(T - R)$ is small. The result follows from a direct estimate using these facts.

COROLLARY 2.7.2. *Given $K > 0$, there is a $d < 1$ so that if $c >$*

d, A and B are algebras, and $T(1)$ is a positive constant, then, if a is invertible in A and $\|a\|\,\|a^{-1}\| \leqq K$, then b is invertible in B.

Proof. We may assume a is of norm 1. By applying the previous corollary to the functions a and a^{-1} with $\varepsilon = 1/(2K)$ we find $\|(Ta)(Ta^{-1}) - T1\| \leqq 1/2$. But $T1$ is a positive constant greater than c. By requiring that c be greater than one-half, we conclude $\|(Ta)(Ta^{-1}) - 1\| < 1$. By a standard Banach algebra result we conclude that $(Ta)(Ta^{-1})$, and hence also Ta, are invertible.

2.8. The following theorem allows us to use the results of 2.7 to study the implications of $D(A, B) = 0$ for function algebras.

THEOREM. *If A and B are algebras and $D(A, B) = 0$, then there is a sequence $T_1, T_2, \cdots$ of elements in $L(A, B)$ such that $T_n(1) = 1$ for all n and $\lim c(T_n) = 1$.*

Proof. Since $D(A, B) = 0$ there is a sequence $S_1, S_2, \cdots$ in $L(A, B)$ with $\lim c(S_n) = 1$. We can assume that $\|S_n\| = 1$ for all n. By removing a finite number of terms from the sequence we can assume that for all n, $c(S_n)$ is so large that all of the previous results hold. Let $a_n = S_n^{-1}(1)$. Applying the lemma of Theorem A′ three times to the map S_n and the functions 1, a_n, and a_n^2 produces three inequalities which combine to show $\|S_n(1)S_n(a_n^2) - (S_n(a_n))^2\| \leqq \varepsilon'$ where ε' can be made arbitrarily small if the ε of the lemma sufficiently small. Hence we conclude that $\|S_n(1)S_n(a_n^2) - 1\| < 1$. Hence $S_n(1)$ is invertible in B. Define T_n by $T_n(a) = (S_n(1))^{-1}S_n(a)$. It is clear that T_n is in $L(A, B)$ and that $T_n(1) = 1$. Theorem 2.1 allows us to conclude that $\|T_n\| \leqq 1/(2c(S_n) - 1)$. It is a direct estimate that $\|T_n^{-1}\| \leqq 1/c(S_n)$. The desired conclusion follows.

3. **Moduli.** We will use a specific set of moduli for surfaces in $\mathscr{S}$. For any S in $\mathscr{S}$ we will say that S is in *standard position* if it is realized as a subset of the complex plane bounded by n circles, $C_1, C_2, \cdots, C_n$ with C_1 the unit circle, C_2 concentric with and outside of the unit circle, and C_3 (if there is one) with center on the x axis. Let C_i have center $x_i + iy_i$ and radius r_i.

For S in standard position, we define the modulus of S, $m(S)$, to be the vector $(x_3, \cdots, x_n, y_4, \cdots, y_n, r_2, \cdots, r_n)$. When necessary, we denote the dependence of these quantities on S by writing $x_3(S)$, etc. For S in $\mathscr{S}_n$ with n greater than 2, $m(S)$ is a $(3n-6)$ – tuple.

It is known that, given S in $\mathscr{S}$, there is at least one and at most finitely many Riemann surfaces S' which are conformally equivalent to S and are in standard position. (E.g. [9] 424ff.)

We define the m-topology (moduli topology) on $\mathscr{S}$ as follows. A sequence $S_1, S_2, \cdots$ converges to S in the m-topology if and only if there is a sequence $S_1', S_2', \cdots$ such that S_i is conformally equivalent to S_i' for each i and the vectors $m(S_i')$ approach $m(S)$ in the Euclidean topology as i becomes infinite.

4. Conformal invariants and almost isometries.

In this section we introduce a set of conformal invariants for the Riemann surfaces in $\mathscr{S}$, develop some of the elementary properties of these invariants and relate these invariants to $d(\cdot, \cdot)$.

DEFINITIONS 4.1. For a function algebra A we define $e^A = \exp(A) = \{g \text{ in } A; g = e^h \text{ for some } h \text{ in } A\}$, $A^{-1} = \{g \text{ in } A; g^{-1} \text{ is in } A\}$. e^A and A^{-1} are commutative groups with respect to multiplication and e^A is a subgroup of A^{-1}. For a in A^{-1}, we will denote by (a) the element of the quotient group $A^{-1}/\exp(A)$ which contains a. For any a in A^{-1} we define $\rho((a)) = \inf\{\|g\|\,\|g^{-1}\|; g \text{ in } (a)\}$. Of course, we will often write $\rho(a)$ for $\rho((a))$.

For n larger than one, let S be a surface in $\mathscr{S}_n$. Pick a numbering of the boundary contours of S. For i and j between one and n, $i \neq j$, let τ_{ij} be the element of $A(S)$ which maps S conformally onto the surface $\tau_{ij}(S)$, a surface in standard position, so that the ith boundary contour of S is mapped to the unit circle and the jth to the circle concentric with the unit circle. The function $\rho(\cdot)$ is defined on $A(S)$. Set $\rho_{ij}(S) = \rho_{ij} = \rho((\tau_{ij}))$. The numbers ρ_{ij} are the conformal invariants we shall consider. Note that $\rho_{ij} = \rho_{ji}$, thus for S in $\mathscr{S}_n$ we have (at most) $n(n-1)/2$ distinct ρ_{ij}. If S and S' are conformally equivalent, then after some renumbering of the boundary contours of S', $\rho_{ij}(S) = \rho_{ij}(S')$ for all i and j. (Results relating to the converse of this observation are presented in [8].)

Given S in $\mathscr{S}$ and C a boundary contour of S and f in $A(S)$ we denote by $w(C, f)$ the winding number of the curve $f(C)$ about the origin. That is, $w(C, f)$ is the winding number of f on C.

4.2. Elementary properties. Let H be the free commutative group generated by the symbols $c_1, c_2, \cdots, c_n$. Let G be the subgroup of H which consists of those elements $\sum_{i=1}^n a_i c_i$ for which $\sum a_i = 0$. Let S be an element of $\mathscr{S}_n$ with boundary contours $C_1, C_2, \cdots, C_n$.

THEOREM. *The map k from $A(S)^{-1}/\exp(A(S))$ to G defined by $k((f)) = \sum_{i=1}^n w(C_i, f)c_i$ is a group isomorphism.*

NOTE. If A is a function algebra then $A^{-1}/\exp(A) \approx H^1(M(A), Z) =$ the first Čech cohomology group of $M(A)$ with integer coefficients

([6]). This theorem is just an explicit form of the isomorphism for the case of interest.

Proof. It is clear that k is well defined. $k((\tau_{ij})) = c_j - c_i$ and elements of the form $c_j - c_i$ generate G, hence k is onto. It remains to show that $k((f)) = 0$ implies that f is in $\exp(A(S))$, or equivalently, $\log(f)$ is in $A(S)$. f is invertible, hence by the monodromy theorem it suffices to show that $\operatorname{im}(\log(f))$ is single valued. Any smooth curve C in S is homologous to $\sum_{i=1}^{n} n_i C_i$ for some choice of n_i. By direct computation the change in $1/2\pi \operatorname{im}(\log(f))$ on traversing C is $\sum n_i w(C_i, f)$. Since $w(C_i, f) = 0$ for all i, this quantity is zero and the proof is complete.

4.3. The following reinterpretation of $\rho(\cdot)$ helps elucidate the elementary properties of $\rho(\cdot)$ and the relation between the invariants ρ_{ij} and the conformal structure of S. Let $C_R(\partial S)$ be the Banach space of real valued continuous functions on the boundary of S. Let $\operatorname{Re} A$ be the subspace of $C_R(\partial S)$ consisting of real parts of functions in A, and let $\overline{\operatorname{Re} A}$ be the closure in $C_R(\partial S)$ of $\operatorname{Re} A$. Let D be the Banach space $C_R(\partial S)/\overline{\operatorname{Re} A}$ and denote by $\| h \|_D$ the norm in D of the coset of the element h in $C_R(\partial S)$.

PROPOSITION. *For f in $A(S)^{-1}$,* $\log(\rho(f)) = 2\| \log |f| \|_D$.

Proof. The mapping which sends f to $\log|f|$ sends $A(S)^{-1}$ into $C_R(\partial S)$ and $\exp A(S)$ into $\operatorname{Re} A$. The equality follows directly from the definitions.

By Theorem 4.2 the mapping Φ of $A(S)^{-1}/\exp A(S)$ to L, the set of points in R^{n-1} with integer coordinates defined by $\Phi((f)) = (w(C_1, f), \cdots, w(C_{n-1}, f))$ is a group isomorphism. Let $\tilde{\rho}$ be the function ρ regarded via this isomorphism as a function on L, i.e., $\tilde{\rho}(\Phi((f))) = \log \rho((f))$.

PROPOSITION. *$\tilde{\rho}$ is the restriction to L of a norm on R^{n-1}.*

Proof. The mapping of L into D which sends l to $\log|\Phi^{-1}(l)|$ extends to a linear map R of R^{n-1} into D. Define $\tilde{\rho}$ on R^{n-1} by $\tilde{\rho}(l) = 2\| R(l) \|_D$. The proposition will be established if we show that R has kernel zero. To do this it suffices to show that $h = \sum_{i=1}^{n-1} a_i \log|r_{n,i}|$ in $\overline{\operatorname{Re} A}$ implies $a_i = 0$, $i = 1, \cdots, n-1$. The period of $^*h$, the harmonic conjugate of h about ρ_i, a curve interior to S and homotopic to C_i, is $2\pi a_i$. However h is in $\overline{\operatorname{Re} A}$. Hence by a standard approximation argument this period must be zero. Thus all the a_i are zero. The proposition is proved.

4.4. Continuity of the ρ_{ij}. We now show that the ρ_{ij} are continuous functions on $\mathscr{S}_n$ with respect to the topology induced by the metric $d(\cdot, \cdot)$.

THEOREM. *Given S in $\mathscr{S}_n$ and $\varepsilon > 0$ there is a δ so that if S' is in $\mathscr{S}_n$ and $d(S, S') < \delta$ then there is a renumbering of the boundary contours of S' after which $|\rho_{ij}(S) - \rho_{ij}(S')| \leqq \varepsilon$, $1 \leqq i, j \leqq n$.*

Proof. An argument similar to that in the proof of Theorem 2.8 shows that for δ small enough, $d(S, S') < \delta$ implies that there is a T in $L(A(S), A(S'))$ with $c(T) > 1 - \delta'$, and $T(1) = 1$, where δ' depends on δ and can be made arbitrarily small if δ is made sufficiently small. By choosing δ to be perhaps smaller still we can also insure that such a T determines a homeomorphism of ∂S and $\partial S'$ in the manner described in Theorems 2.1 and 2.2. Given S' with $d(S, S')$ less than this new δ, renumber the boundary components so as to be compatible with this homeomorphism. Denote the boundary contours of S and S' by $C_1, \cdots, C_n$ and $C'_1, \cdots, C'_n$ respectively.

Let $K = 3 \max\{\rho_{ij}(S); 1 \leqq i, j \leqq n\}$. Since winding numbers are integers, by choosing δ to be perhaps smaller still, Theorem 2.5 allows us to conclude that if f is in $A(S)^{-1}$ with $||f||\,||f^{-1}|| < K$ then $W(C_i, f) = W(C'_i, Tf)$ for $i = 1, \cdots, n$.

Given i and j between 1 and n, choose f in the coset of $A(S)^{-1}/\exp(A(S))$ which determines ρ_{ij} so that $||f|| \leqq ||T||^{-1}$ and $||f^{-1}|| \leqq \rho_{ij} + \varepsilon' < K$ for some small preassigned positive ε'. The remarks of the previous paragraph imply that Tf is in the coset of $A(S')^{-1}/\exp(A(S'))$ which determines $\rho_{ij}(S')$. Thus

$$\rho_{ij}(S') \leqq ||Tf||\,||(Tf)^{-1}|| \leqq ||(Tf)^{-1}|| \,.$$

But by Theorem 2.4

$$\begin{aligned} ||(Tf)^{-1}||^{-1} &= \inf\{|Tf(z)|; z \text{ in } \partial S'\} \\ &\geqq \inf\{|f(z)|; z \text{ in } \partial S\} + 2(c(T) - 1) \\ &\geqq ||f^{-1}||^{-1} + 2(c(T) - 1)\,. \end{aligned}$$

ε' was arbitrary so $\rho_{ij}(S') \leqq (\rho_{ij}(S)^{-1} + 2(c(T) - 1))^{-1}$. Hence $\rho_{ij}(S') \leqq \rho_{ij}(S) + 2(1 - c(T))\rho_{ij} \leqq \rho_{ij} + K(1 - c(T))$. Hence, if δ' is small enough, then the $\rho_{ij}(S')$ are bounded by the same K. In this case the role of S and S' in the previous argument can be interchanged and $\rho_{ij}(S) \leqq \rho_{ij}(S') + K(1 - c(T))$. The previous two inequalities imply the desired result.

5. Construction of elements of $L(A(S), A(S'))$. Let S be an element on $\mathscr{S}_n$ in standard position. The following Banach space direct sum decomposition of $A(S)$ will be called a *standard decomposi-*

tion.

$$A(S) = \boldsymbol{C} \oplus A_1 \oplus \cdots \oplus A_n$$

where $A_2 = \{f$ in $A(S)$, f is the restriction to S of a function analytic interior to C_2 and vanishing at $0\}$. For $i \neq 2$, $A_i = \{f$ in $A(S)$, f is the restriction to S of a function analytic exterior to C_i and vanishing at $\infty\}$.

Given f in $A(S)$, set $f_i^*(w) = 1/2\pi i \int_{c_i} f(z)(z - w)^{-1} dz$. Set $f_i = f_i^* - f_i^*(P_i)$ where $P_2 = 0$ and $P_i = \infty$ for $i \neq 2$. Since f_i is in A_i and $f = (\sum f_i^*(P_i)) + f_1 + \cdots + f_n$, the sum is all of A. Any element in $A_i \cap A_j$ for $i \neq j$ would be analytic on the Riemann sphere and vanish at ∞ and thus would be the zero function.

For f in $A(S)$, define $n(f) = |\sum f_i^*(P_i)| + \sum \|f_i\|$. $n(\cdot)$ is a norm on $A(S)$ and by the closed graph theorem the map from $A(S)$ into $A(S)$ normed by $n(\cdot)$ is bicontinuous. The projection $P(f) = f_i$ is clearly continuous with respect to this new norm, hence it is continuous with respect to the original norm and the sum is a Banach space direct sum.

For $n = 2$, the standard decomposition decomposes an element f in $A(S)$ into the sum of the terms of the Laurant series of F involving positive powers of the variable, those involving the negative powers of the variable, and the constant term.

THEOREM 5.1. *If S_i, $i = 0, 1, \cdots$ are in $\mathscr{S}_n$ and the S_i approach S_0 in m-topology as i becomes infinite, then* $\lim_{i\to\infty} d(S_i, S_0) = 0$.

Proof. We will show that for any compact subset W of $\mathscr{S}_n$ there is a continuous function $\lambda(t)$ which approaches zero as t goes to zero such that if S and S' are in W then $d(S, S') \leqq \lambda(\| m(S) - m(S') \|_\infty)$.

Let S and S' be any two elements of $\mathscr{S}_n$. We will construct a T in $L(A(S), A(S'))$. Let $A = A(S)$, $A' = A(S')$. Without loss of generality we can assume that S and S' are in standard position. Thus ∂S and $\partial S'$ are each a union of n disjoint circles, $C_1, \cdots, C_n$ and $C'_1, \cdots, C'_n$ respectively. Let $A = A_1 \oplus \cdots \oplus A_n \oplus \boldsymbol{C}$ and $A' = A'_1 \oplus \cdots \oplus A'_n \oplus \boldsymbol{C}$ be the standard decompositions of A and A' respectively. Let $t_1(z) = z$, and $t_2(z) = (r'_2/r_2)\, z$ (r'_2 and r_2 are the radii of C'_2 and C_2 respectively). For $k = 3, 4, \cdots, n$, let t_k be the Mobius transformation which takes the exterior of C_k to the exterior of C'_k fixes ∞ and moves C_k as little as possible, i.e., subject to the two previous conditions minimize $\sup\{|z - t_k(z)| : z \text{ in } C_k\}$.

For f in A we have $f = c + \sum f_i$ with c a constant and f_i in A_i. Let $(Tf)(w) = c + \sum f_i(t_i^{-1}(w))$. Note that for each i, T restricted to A_i is a subjective isometry of A_i onto A'_i and that the projection of

A onto A_i is continuous. Hence T is in $L(A, A')$.

For each k we have T_k mapping $A(S_k)$ to $A(S_0)$ constructed according to the prescription just given with $S = S_k$ and $S' = S_0$. It suffices to show $\overline{\lim}_{n\to\infty} \log \| T_n \| \| T_n^{-1} \| = 0$. Notice that T_n^{-1} is the map that we would have constructed according to the above prescription had we interchanged S and S' before starting. We will show that $\overline{\lim} \| T_n \| = 1$. Since $T_n(1) = 1$ we know that $\| T_n \| \geqq 1$ for all n. The same estimates with the requisite changes in subscripts would also show that $\overline{\lim} \| T_n^{-1} \| = 1$. We will not provide the details for this second claim.

We may limit consideration to those S_k for which $m(S_k)$ lies in W, some preassigned compact neighborhood of $m(S_0)$. This having been done, the constants R, K, and M of the next three lemmas may be chosen as universal constants, that is depending on W but independent of the choice of S in $\mathscr{S}_n$. (This uniformity follows from the fact that the constant R in the proof of Lemma 1 can be bounded away from 1 for all S with $m(S)$ in W.)

LEMMA 1. *There is a positive number k such that given j, $1 \leqq j \leqq n$, and given c in $\mathbf{C}$, f_i in A_i, $f = c + \sum f_i$ with $\|f_j\| = 1$ and $\| c + \sum f_i \|_{c_j} = \| c + \sum f_i \|$ (i.e., f peaks on C_j), then $\| c + \sum f_i \| \geqq k$.*

Proof of lemma. Assume $j \neq 2$ (the case $j = 2$ requires minor notational changes). Assume $\|f\| = \|f\|_{c_j} \leqq k$. Draw a circle Γ, "hyperbolically concentric" with C_j, i.e., if one maps ext (C_j) conformally to the unit disk and ∞ to 0, then the image of Γ will be concentric with the unit circle. We also require that the region between the two circles be contained in S. By Schwarz's lemma we have

$$\|f - f_j\|_\Gamma \leqq \|f\|_\Gamma + \|f_j\|_\Gamma \leqq k + 1/R .$$

The constant R which is greater than 1 is determined by the relative positions of Γ and c_j. Hence by the maximum modulus principle for $(\operatorname{int} \Gamma)$ applied to the function $f - f_j$ we have $\|f - f_j\|_{c_j} \leqq \|f - f_j\|_\Gamma$. So,

$$\begin{aligned} k \geqq \|f\| = \|f\|_{c_j} &\geqq \|f_j\|_{c_j} - \|f - f_j\|_{c_j} \\ &\geqq 1 - \|f - f_j\|_\Gamma \geqq 1 - k - 1/R . \end{aligned}$$

So $k \geqq (1 - 1/R)/2$. The same argument can be used for any j (with the obvious modifications if $j = 2$). Hence taking k to be the minimum of the finite number of k's that are produced by such arguments, the lemma is proved.

LEMMA 2. *There is a K such that if f_i is in A_i and if for some*

j, $\|c + \sum f_i\| = \|c + \sum f_i\|_{c_j} = 1$ *then* $\|f_j\| \leqq K$.

Proof of lemma. Let $f'_i = f_i/\|f_j\|$ for i equal one through n. Set $c' = c/\|f_j\|$. Apply Lemma 1 to $f' = c' + \sum f'_i$. So $1/\|f_j\| = \|f'\| \geqq k$. So $\|f_j\| \leqq 1/k = K$.

LEMMA 3. *There is an* M *such that, given* f *in* A, $\|f\| = 1$, *then* $\|f_i\| \leqq M$ *for all* i.

Proof. Let $f = c + \sum f_j$ with f_j is A_j. Assume $\|f\| = \|f\|_{c_i}$. By the previous lemma, $\|f_i\| \leqq K$. Let $g = (f - f_i)/\|f - f_i\|$. For some $j \neq i$, $\|g\| = \|g\|_{c_j}$. Denote the projection of g in A_j by g_j. By the previous lemma, $\|g_j\| \leqq K$. Hence $\|f_j\| = \|f - f_j\|\,\|g_j\| \leqq (1 + K)K$. Continuing in this manner (i.e., next $h = (g - g_j)/\|g - g_j\|$) gives a bound on all the f_k. Taking the greatest of these bounds, $\|f_j\| \leqq K(1 + K)^n$ for all j.

NOTE. This estimate (Lemma 3) also follows from the continuity of the projections of A onto each of the summands A_i. However, the constant produced by that observation depends on the particular surface rather than on the compact set W.

LEMMA 4. *There is a function* $M(i, j, m(S), m(S'))$ *such that if* f *is in* A_i *and* z_0 *is in* c_j *for* $i \neq j$, *then* $|f_i(z_0) - f_i(t_j^{-1}(z_0))| \leqq \|f_i\|$ $M(i, j, m(S), m(S'))$ *and* $M(i, j, m(S), m(S'))$ *approaches zero as* $\|m(S) - m(S')\|_\infty$ *approaches zero with* S *and* S' *restricted to lie in a preassigned compact subset* W *of* $\mathscr{S}_n$.

Proof of lemma. For simplicity we will assume that neither i nor j is equal to two. Let $D_i =$ (exterior of C_i) $\cup \{\infty\}$. Let $f'_i = f_i/\|f_i\|$. By Pick's lemma,

$$\delta(f'_i(z_0)), f'_i(t_i^{-1}(z_0)) \leqq \delta_{D_i}(z_0, t_i^{-1}(z_0)) .$$

Also

$$\frac{1}{2}|f'_i(z_0) - f'_i(t_i^{-1}(z_0))| \leqq \delta(f'_i(z_0)), f'_i(t_i^{-1}(z_0)) ,$$

hence

$$\frac{1}{2}|f_i(z_0) - f_i(t_i^{-1}(z_0))| \leqq \|f_i\|\delta_{D_i}(z, t_i^{-1}(z_0)) .$$

We set $M(i, j, m(S), m(S')) = 2 \sup [\delta_{D_i}(z, t_i^{-1}(z)) \colon z$ in $(c_j)]$. Since $M(i, j, m(S), m(S'))$ is continuous with respect to $m(S)$ and $m(S')$ the lemma is proved.

Proof of the theorem. Pick S_k and S_0. Let $S_k = S$, $S_0 = S'$. Let T be the map of A onto A' defined above. Pick f in A with $\|f\| = 1$. We will estimate $\|T\|$ by estimating $\|Tf\|$. We will assume that f peaks on C_i and will only estimate $\|Tf\|_{C_i'}$. Similar estimates apply to the other $\|Tf\|_{C_j'}$. Pick z_0' in C_i. Let $f = c + \sum_i f_i$ with c in C and f_i in A_i. Let $z_0 = t_i^{-1}(z_0')$. Thus z_0 is in C_i. We have $f(z_0) = c + \sum_i f_i(z_0)$. So

$$\begin{aligned}
|Tf(z_0') - f(z_0)| &= |\sum f_j(z_0) - \sum f_j(t_j^{-1}(z_0'))| \\
&= |\sum_{j\neq i}(f_j(z_0) - f_j(t_j^{-1}(z_0')))| \\
&\leqq \sum_{j\neq i} |f_j(z_0) - f_j(t_j^{-1}(z_0'))| \\
&\leqq \sum_{j\neq i} \|f_j\| M(i, j, m(S), m(S')) \text{ by Lemma 4} \\
&\leqq M'(m(S), m(S')) \sum_{j\neq i} \|f_j\| \text{ with } M' \text{ defined the obvious way} \\
&\leqq M'(m(S), m(S')) \sum M \text{ by Lemma 3} \\
&\leqq NM'(m(S), m(S')) \text{ for some constant } N\,.
\end{aligned}$$

So $|Tf(z_0')| \leqq |f(z_0)| + NM'(m(S), m(S')) \leqq 1 + NM'(m(S), m(S'))$. Once the compact set W in $\mathscr{S}_n$ has been fixed then the constant N in the previous inequality can be chosen uniformly and hence absorbed into the function $M'(\cdot, \cdot)$. z_0 was an arbitrary point on $\overline{\partial S'}$. Taking the supremum over all such z_0 we have

$$\|Tf\| \leqq |f| + M'(m(S), m(S'))\,.$$

Taking the supremum over all f in A of norm one we find

$$\|T\| \leqq 1 + M'(m(S), m(S'))\,.$$

Since M' has the required properties, the proof is complete.

6. Almost isometries and moduli of domains. In this section we prove the results necessary to complete the proof of Theorem B. The major remaining steps are the following two theorems.

THEOREM 6.1. *Given $n \geqq 2$, S and S' in $\mathscr{S}_n$, if $d(S, S') = 0$ then S and S' are conformally equivalent.*

Proof. Set $A = A(S)$, $A' = A(S')$. By Theorem 2.8 there is a sequence of maps T_i in $L(A, A')$ such that $c(T_i)$ approaches one and $T_i(1) = 1$. Without changing notation we normalize these T_i so that they are all of norm one. Hence, $T_i(1)$ will be a positive constant between $c(T_i)$ and one. By Theorem 2.2, for sufficiently large i, each T_i has associated with it a homeomorphism of the boundary of S with that of S'. This homeomorphism induces a renumbering of the boundary components of S'. Since there are only finitely many possible renumbering of the boundary components of S', we may pass to an infinite subsequence of the T_i's with the $c(T_i)$'s large enough to insure that the conclusions of Theorem 2.2 hold and with all of the associated boundary homeomorphisms inducing the same renumbering of the boundary components of S'. We will denote this new sequence by $\{T_i\}$ and will assume that the boundary contours have been renumbered so that for all j the induced homeomorphism carries the jth boundary component of S to the jth boundary component of S'.

The proof now consists of constructing an analytic map from S' into S, showing that the induced map between homology groups has kernel zero and concluding that the map is a conformal equivalence. Most of the previous results in this paper worked with the boundary points of the surfaces being considered. In this proof we will only show that int (S) and int (S') are conformally equivalent. The equivalence of the two as bordered surfaces then follows from standard results about the boundary behavior of conformal maps.

For convenience we will break the proof into a series of lemmas.

LEMMA 1. *There is a subsequence* $\{T_{n_i}\}$ *of the* T_i*'s such that for each* y *in the interior of* S' *there is a point* $x(y)$ *in* S *such that for all* f *in* A $\lim_{i\to\infty}(T_{n_i}f)(y) = f(x(y))$. *Furthermore, the mapping that sends* y *to* $x(y)$ *is analytic.*

Proof of lemma. Pick y in S'. All of the T_n are of norm one, hence all of the T_n^* are of norm one. Hence, by the weak-star compactness of the unit ball of A^*, the set $\{T_n^*(y)\}$ has a weak-star accumulation point $x(y)$.

We now perform two diagonalizations on the sequence $\{T_i\}$. Let $f_1, f_2, \cdots$ be a countable dense subset of A. (Since A is a direct sum of disk algebras with their constants identified, this is clearly possible.) We know that $\{T_n^*(y)(f_1)\}$ has the point $x(y)(f_1)$ as an accumulation point. By passing to a subsequence of the T_i's and renumbering we can insure that

$$\lim_{n\to\infty} T_n^*(y)(f) = x(y)(f) \tag{6.1}$$

for this particular y and for $f = f_1$. By passing to a further subsequence we can insure that (6.1) holds for this y and for f_2. Continuing in this manner and then replacing $\{T_i\}$ by the diagonal subsequence we insure that (6.1) holds for this y and for all f_i. Since the

f_i are dense we can conclude that (6.1) holds for this y and for all f in A.

Let $y_1, y_2, \cdots$ be a countable dense subset of the interior of S'. By another diagonalization, this time with respect to the y_i's, we obtain a subsequence of the T_i's such that (6.1) holds for all f in A and for y equal to any of the y_i. For the other y in S' we let $x(y)$ be any weak-star accumulation point of the sequence $\{T_n^*(y)\}$. In fact (6.1) now holds for all f in A and for all y in the interior of S'; for, by construction, $x(y)(f)$ is an accumulation point of the numbers $T_n^*(y)(f)$. Hence it suffices to show that this sequence has only one accumulation point. y is an interior point of S' and, for fixed f, the $T_n(f)$ are uniformly bounded on S'. Hence, in a small neighborhood of y, this set of functions is uniformly equicontinuous. But for a dense set of points, y_i, in this neighborhood, the sequences $\{T_n^*(y_i)(f)\}$ have unique accumulation points. Hence the sequence $\{T_n^*(y)(f)\}$ cannot have more than one accumulation point.

We now show that for any y, $x(y)$ is a nonzero multiplicative linear functional on A'. $x(y)(1) = \lim (T_n^*(y)(1)) = \lim T_n(1)(y) = 1$. Hence $x(y)$ is nonzero. It remains to show that for any f in A of norm one $x(y)(f^2) = (x(y)(f))^2$. We know that $\lim (T_n^*(y)(f))^2 = (x(y)(f))^2$ and that $\lim (T_n^*(y)(f^2)) = x(y)(f^2)$. We also know that $|(T_n^*(y)(f))^2 - T_n^*(y)(f^2)| \leqq \|(T_n f)^2 - T_n(f^2)\|$ and by Corollary 2.7.1 this last quantity must become arbitrarily small as n becomes infinite. Combining these observations show that $x(y)(f^2) = (x(y)(f))^2$. Since $x(y)$ is a multiplicative linear functional, it can be thought of as a point of S, the maximal ideal space of $A = A(S)$. We will denote this point by $x(y)$.

We now show that this mapping from y to $x(y)$ is analytic. Fix y in the interior of S'. Let h be the coordinate function on S. We know that the sequence of functions $\{T_n h(y)\}$ converges pointwise to the function $h(x(y)) = x(y)$. We also know that the sequence of functions $T_n h$ is a bounded sequence of analytic functions on S'. These two facts allow us to conclude that a subsequence of the $T_n h$ converge uniformly on some small neighborhood of y to the limit function x. Thus, at y, the function $x(y)$ is the uniform limit of the analytic functions $T_n h$, and hence is analytic. y was arbitrary so the lemma is proved.

NOTE. We have not ruled out the possibility that $x(y)$ is a constant function. Simple examples using the disk algebra show that this is, in fact, possible if we do not require S and S' each have more than one boundary component.

The presence of homology in S and S' prevents the map from being trivial. We have identified the integer cohomology groups

$H^1(S)$ and $H^1(S')$ with A^{-1}/e^A and $A'^{-1}/e^{A'}$ respectively. Since x maps S' to S there is an induced map x^* of $H^1(S)$ into $H^1(S')$.

LEMMA 2. Ker $x^* = 0$.

Proof. For $i = 1, \cdots, n-1$ let γ_i be a simple closed curve in the interior of S which is homotopic to the ith boundary contour of S and is similarly oriented. Define γ'_i in S' similarly. Suppose that f is in A^{-1} and (f) in A^{-1}/e^A is such that $x^*(f) = 0$. We must show that f is in e^A. It suffices to show that $W(\gamma_i, f) = 0$ for each i. Choose i. Since $x^*(f) = 0$, we know that $x^*(f)(\gamma'_i) = W(\gamma'_i, f\circ x) = 0$. Since γ'_i is a compact subset of the interior of S', $W(\gamma'_i, f\circ x) = \lim W(\gamma'_i, T_n f)$. Since the winding numbers are integers this implies that for all n sufficiently large $W(\gamma'_i, T_n f) = 0$. Let C_i and C'_i be the ith boundary contours of S and S' respectively. $W(\gamma'_i, T_n f) = W(C'_i, T_n f)$. Hence for all n sufficiently large, $W(C'_i, T_n f) = 0$. Hence, by the lemma of Theorem A′ $W(C_i, f) = 0$. But $W(\gamma_i, f) = W(C_i, f)$ so the proof is complete.

Proof of theorem. Let the mapping of y to $x(y)$ be the map of the interior of S' into S constructed in Lemma 1. Applying Lemma 1 again in the opposite direction gives a map of the interior of S into S' which sends x to $y(x)$. Let K be the map of the interior S into itself defined by $K(x)=x(y(x))$. The mapping of x into $y(x)$ is only defined on the interior of S. However, the previous lemmas show that the map of y to $x(y)$ is nonconstant and analytic. Hence the image of the interior of S' is contained in the interior of S. Thus K is well defined. K induces a map K^* of $H'(S)$ into itself and $K^* = y^*x^*$. By Lemma 2, Ker $x^* = 0$ and Ker $y^* = 0$. Hence Ker $K^* = 0$. Since $H^1(S)$ and $H_1(S)$, the integer homology group of S, are both free on $n-1$ generators, we may conclude that K_* mapping $H_1(S)$ to itself has trivial kernel. Landau and Osserman [4] have shown that if S is a finite planar domain and K is an analytic map of S into itself such that K_* has trivial kernel, then K is a conformal automorphism. Thus K, and similarly $H = y\circ x$ mapping S' to itself, are conformal automorphisms. Hence x is a conformal isomorphism of S and S' and the theorem is proved.

THEOREM 6.3. *If S_k is in $\mathscr{S}_n$, $k = 0, 1, 2, \cdots$ and* $\lim d(S_k, S_0) = 0$ *then $\{S_k\}_{k=0}^{\infty}$ lies in an m-topology compact subset of $\mathscr{S}_n$.*

Proof. Put S_0 in standard position. Let $S_0 - \varepsilon$ be the set of all points in S_0 of distance at least ε from ∂S_0. If ε is sufficiently small then $S_0 - \varepsilon$ and $S_0 - 2\varepsilon$ are in $\mathscr{S}_n$. By Theorem 2.8, for k sufficiently large we can find T_k in $L(A(S_k), A(S_0))$ with $T_k(1) = 1$ and $\lim c(T_k) = 1$. For N large and ε small, Theorem A′ holds and hence for all $n > N$, for all x in ∂S_0

$$|\, T_n(\tau_{12})(t_n(x)) - \tau_{12}(x)\,| < \varepsilon/2 \tag{6.2}$$

where t_n is the boundary homeomorphism induced by T_n in the manner described in Theorem 2.3. If α is in $S_0 - \varepsilon$ then, by (6.2), since winding numbers are integers $1 = W(\partial S_0, \tau_{12} - \alpha) = W(\partial S_n, T(\tau_{12}) - \alpha)$. Hence $T_n(\tau_{12})$ map a subset of S_n univalently onto $S_0 - \varepsilon$. Let h_n be the inverse of this map. Thus h_n maps $S_0 - \varepsilon$ into S_n and the components of $\boldsymbol{C} - S_n$ are contained in the components of $\boldsymbol{C} - h_n(S_0 - \varepsilon)$. Since $c(T_n)$ approach one, the functions h_n are uniformly bounded by some constant K. Hence, the set of numbers $\{r_2(S_k)\}$ is bounded for,if not, then the numbers $\rho_{12}(S_n) \geqq r_2(S_n)/K$ would not be bounded, an impossibility by Theorem 4.4.

For S in $\mathscr{S}_n$ and t a continuous function on S, define $B(t, S) = \inf\{|\, t(x) - t(y)\,|;\ x$ and y in different boundary components of $\partial S\}$. Let $B(S)$ be $B(t, S)$ with t the map which puts S in standard position.

Note that if some sequence of $r_i(S_n)$ approach zero as n becomes infinite and if the $B(S_n)$ are bounded away from zero then some of the $\rho_{ij}(S_n)$ must become arbitrarily large. Again, by Theorem 4.4, this is impossible.

The only way in which the moduli of the S_n can fail to lie in a compact subset are if the $r_2(S_n)$ are unbounded, or if some $r_i(S_n)$ become arbitrarily small, or if some $B(S_n)$ become arbitrarily small. We have already ruled out the first possibility and shown the second possibility cannot happen unless some $B(S_n)$ are arbitrarily small. We must now show that the $B(S_n)$ are bounded away from zero.

Suppose that, after passing to a subsequence, $B(S_n)$ approach zero. Since $B(h_n, S_0 - \varepsilon) \leqq B(S_n)$ we would have $B(h_n, S_0 - \varepsilon)$ approaching zero. Similarly, letting k_n be h_n restricted to $S_0 - 2\varepsilon$ we would have $B(k_n, S_0 - 2\varepsilon)$ approaching zero. The h_n are a uniformly bounded family of univalent functions on $S_0 - \varepsilon$; hence, after passing to a subsequence we can find h the uniform limit on $S_0 - 3\varepsilon/2$ of the h_n, h is constant or univalent. Since $W(|\,z\,| = 1 - 2\varepsilon; h_n) = 1$ for all n, h is not constant. Let k be h restricted to $S_0 - 2\varepsilon$. Since k is univalent on $S_0 - 2\varepsilon$, $B(k, S_0 - 2\varepsilon) > 0$. But $B(k, S_0 - 2\varepsilon) \leqq \varliminf B(k_n, S_0 - 2\varepsilon) = 0$. This contradiction completes the proof.

6.4. *Proof of Theorem* B. It suffices to show the following:

(1) $d(\cdot, \cdot)$ is a metric on $\mathscr{S}_n$ and

(2) Let $S_0, S_1, \cdots$ be elements on $\mathscr{S}_n$ then

(a) if the S_k approaches S_0 in the m-topology then $d(S_k, S_0)$ approaches zero.

(b) if $d(S_k, S_0)$ approaches zero then S_k approaches S_0 in the m-topology.

For 1 the only nontrivial fact is that $d(S, S') = 0$ implies S and S' are conformally equivalent. This is Theorem 6.1. 2(a) is Theorem 5.3. For 2(b) it suffices to show that the sequence S_k has an m-topology accumulation point, then by the triangle inequality for d and part 2(a), this point must be S_0. However, Theorem 6.2 guarantees that such an accumulation point exists.

Added in proof. A result very similar to Theorem 2.3 has been proved by B. Cengiz (Proc. Amer. Math. Soc., 40 (1973), 426–430).

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