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THE CENTER OF A SIMPLE ALGEBRA

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The following theorem is proved: *Let A be a finite-dimensional simple K-algebra, K a field. If E is an extension of K and if M is an absolutely irreducible left $A \otimes_K E$-module with character $\chi: A \otimes_K E \rightarrow E$, then $\chi(A)$ is a subfield of E which is K-isomorphic to the center of A.*

The purpose of this note is to give a short demonstration of the above theorem, proved first by Brauer [1] and later by Fein [3, 4] in case K is a perfect field (or, more generally, when A is a separable K-algebra). We make no assumptions on separability.

All algebras are assumed to be finite-dimensional, and all modules are unital left modules. Let B be a K-algebra, K a field, and let M be a B-module. We say M is *absolutely irreducible* if $M \otimes_K E$ is an irreducible $B \otimes_K E$-module for all extensions E of K; an extension E of K is said to be a *splitting field* for B if every irreducible $B \otimes_K E$-module is absolutely irreducible (cf. [2, p. 202]). We will always identify B with its natural image in $B \otimes_K E$.

LEMMA. *Let A be a central simple L-algebra, L a field, and let F be a Galois extension of L. If N is an irreducible $A \otimes_L F$-module with character $\chi: A \otimes_L F \rightarrow F$ such that $\chi \neq 0$, then $\chi(A) = L$.*

Proof. For each $\sigma \in G(F/L)$, the Galois group of F over L, define an L-automorphism (still denoted by σ) of $A \otimes_L F$ by $\sigma(\sum_i a_i \otimes f_i) = \sum_i a_i \otimes \sigma(f_i)$. Each such L-automorphism of $A \otimes_L F$ gives rise to an irreducible $A \otimes_L F$-module σN: The additive group of $\sigma N = \{\sigma n: n \in N\}$ is the same as that of N, but the module structure on σN is defined by $(\sigma x)(\sigma n) = \sigma(xn)$ for all $x \in A \otimes_L F$ and $n \in N$. One checks that the character of σN is $\sigma \chi \sigma^{-1}$. Since $A \otimes_L F$ is simple [2, (68.1)], $\sigma N \cong N$, and so $\sigma \chi \sigma^{-1} = \chi$. This says that for each $a \in A$, $\sigma \chi(a) = \chi(a)$ for all $\sigma \in G(F/L)$; hence $\chi(A) \subseteqq L$. Since $\chi(A)$ is a nonzero L-subspace of L, it follows that $\chi(A) = L$, as desired.

LEMMA. *Let A be a central simple L-algebra, L a field, and let E be an extension of L. If M is an absolutely irreducible $A \otimes_L E$-module with character $\zeta: A \otimes_L E \rightarrow E$, then $\zeta(A) = L$.*

Proof. It is well known that there is a Galois extension F of L which is a splitting field for A. Let N be an irreducible $A \otimes_L F$-module with character $\chi: A \otimes_L F \rightarrow F$. Then N is absolutely irreducible, and $\chi \neq 0$. By the above lemma, $\chi(A) = L$.

Let W be a compositum of E and F. Now $(A \otimes_L E) \otimes_E W$ and $(A \otimes_L F) \otimes_F W$ are both isomorphic to $A \otimes_L W$, and $M \otimes_E W$ and $N \otimes_F W$ are irreducible $A \otimes_L W$-modules with characters ζ and χ, respectively, on A. Since $A \otimes_L W$ is simple, $M \otimes_E W \cong N \otimes_F W$, so $\zeta = \chi$ on A. It follows that $\zeta(A) = \chi(A) = L$, as desired.

Observe that the restriction ζ_A of ζ to A is the reduced trace of A into its center L.

THEOREM. *Let A be a simple K-algebra with center L. Let E be an extension of K, and let M be an absolutely irreducible $A \otimes_K E$-module with character $\chi: A \otimes_K E \to E$. Then $\chi(A)$ is a K-subfield of E, and $\chi(A) \cong L$ as K-algebras.*

Proof. Since L is contained in the center of $A \otimes_K E$, L is K-isomorphic to a subfield of $\mathrm{End}_{A \otimes E}(M) \cong E$, and we regard this as an identification [2, (29.13)]. It follows that M can be made into an $A \otimes_L E$-module, and that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dd, equal] & A \underset{L}{\otimes} E \arrow[dr, "\alpha"] & & \\ & & \mathrm{End}_E(M) \arrow[r, "T"] & E \\ A \arrow[r] & A \underset{K}{\otimes} E \arrow[uu] \arrow[ur, "\beta"'] & & \end{tikzcd}$$

commutes, where T is the trace map, and where α and β are the E-algebra homomorphisms afforded by the module structures on M. Since M is an absolutely irreducible $A \otimes_K E$-module, it follows that β is an epimorphism, and so α is also an epimorphism. Thus M is an absolutely irreducible $A \otimes_L E$-module, with character $T\alpha: A \otimes_L E \to E$. By the previous lemma, $T\alpha(A) = L$. Now $\alpha(A) = \beta(A)$, so $\chi(A) = T\beta(A) = T\alpha(A) = L$, as desired.

With a little extra effort, it is possible to generalize this result to orders. In particular, let R be a Krull domain with quotient field K, and let A be a simple K-algebra. An R-order Λ in A is a unital R-subalgebra of A which spans A over K, and each element of Λ is integral over R. Let E be an extension of K, and let M be an absolutely irreducible $A \otimes_K E$-module with character $\chi: A \otimes_K E \to E$. If Λ is an R-order in A which is separable over its center, then one can prove that $\chi(\Lambda)$ is R-isomorphic to the center of Λ.

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