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A NOTE ON THE CONTINUITY OF BEST POLYNOMIAL APPROXIMATIONS

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An example is given to show that while best uniform polynomial approximation in the complex plane is continuous, it is not in geneal uniformly continuous.

For a continuous complex valued function f defined on E, a compact set in the plane and $n \in \{0, 1, 2, \cdots\}$, let $p_n(f, E)$ denote the polynomial of degree n of best uniform approximation to f on E and let $\|f\|_E$ denote the uniform norm of f on E. In [2] it was shown that for any such f and E there exists for each n and each β, $0 < \beta < 1/2$, a constant $M(n, \beta) > 0$ such that

$$\|p_n(f, E) - q_n\|_E \leqq M(n, \beta)[\|f - q_n\|_E - \|f - p_n(f, E)\|_E]^\beta ,$$

where q_n is any polynomial of degree n. If we in fact let $M(n, \beta)$ denote the least such constant then we ask if the sequence $\{M(n, \beta)\}_{n=0}^{\infty}$ is bounded. The purpose of this note is to show that in general it is not.

Let $f(z) = 1/z$ and $E = U = \{|z| = 1\}$. Then $p_n(f, U) \equiv 0$ for $n = 0, 1, 2, \cdots$. Now for each $k = 1, 2, \cdots$ and each β, $0 < \beta < 1/2$, let $\Omega_{k,\beta}$ denote a simply connected Jordan region containing the origin such that

1. $\Omega_{k,\beta} \subset \{|z| < k^\beta\}$
2. $k^\beta \in \bar{\Omega}_{k,\beta}$
3. $\Omega_{k,\beta} \cap \{|z| > k(\beta - 1)/\beta\} \cap \{\mathrm{Re}\, z \leqq 0\} = \Phi$.

The region $\Omega_{k,\beta}$ can in fact be chosen to be a displaced ellipse. Futhermore, there exists a conformal map $g_{k,\beta}$ of the unit disc $\{|z| < 1\}$ onto $\Omega_{k,\beta}$ such that $g_{k,\beta}\,(0) = 0$. Furthermore $g_{k,\beta}$ will be continuous in $\{|z| \leqq 1\}$ and map U onto the boundary of $\Omega_{k,\beta}$. As a consequence of these definitions we have that

$$\|1 - g_{k,\beta}(z)/k\|_U \leqq 1 + (1/k)^{1/\beta}$$

and

$$\|g_{k,\beta}(z)/k\|_U = k^\beta/k \ .$$

Now there exists [1, p. 98] a polynomial p such that

$$\|g_{k,\beta}(z)/z - p(z)\|_U < k^{(\beta-1)/\beta} \ .$$

Thus

$$||1/z - p(z)/k||_U \leqq 1 + 2(1/k)^{1/\beta}$$

and

$$||p(z)/k||_U \geqq \frac{k^\beta - k^{(\beta-1)/\beta}}{k} = \frac{k^\beta - k^{(\beta-1)/\beta}}{2^\beta}\left(\frac{2^\beta}{k}\right).$$

Consequently, we see that for this particular choice of f and E, and for each $0 < \beta < 1/2$,

$$\lim_{n\to\infty} M(n, \beta) \geqq \lim_{k\to\infty}\left(\frac{k^\beta - k^{(\beta-1)/\beta}}{2^\beta}\right) = \infty .$$

REFERENCES

1. Gunter Meinardus, *Approximation of Functions: Theory and Numerical Methods*, Springer-Verlag, (1967).

2. S. J. Poreda, *On the continuity of best polynomial approximations*, Proc. Amer. Math. Soc., November, 1972.

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