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HYPERSPACES OF GRAPHS ARE HILBERT CUBES

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The authors prove that 2^Γ is a Hilbert cube where Γ is any nondegenerate, finite, connected graph and 2^Γ is the space of nonvoid closed subsets of Γ metrized with the Hausdorff metric. This extends their result that 2^I is a Hilbert cube. They also prove corresponding theorems for local dendrons D as well as for the space of subcontinua $C(D)$ of D.

1. Introduction. In [9] the authors outlined their proof that 2^I, the space of nonvoid, closed subsets of $I = [0, 1]$ metrized with the Hausdorff metric, is a Hilbert cube Q and announced the main results concerning graphs in this paper. Here we give the complete proof, assuming that 2^I is a Hilbert cube, that 2^Γ is a Hilbert cube for any finite, connected graph Γ. We also prove that if D is any local dendron, then 2^D is a Hilbert cube and prove some results about the space of subcontinua $C(D)$ of a local dendron D that extend the results of [13].

In [10] the authors give a complete proof that 2^I is a Hilbert cube. This settled a conjecture raised by Wojdyslawski [16] in 1938 where he also asked if 2^X is a Hilbert cube for any nondegenerate Peano space X. The first author and D. W. Curtis have announced the proof of this latter conjecture in [5] as well as the theorem that says that $C(X)$ is always a Q-factor for any Peano space X, and $C(X)$ is a Hilbert cube iff X is a nondegenerate Peano space that contains no free arcs. These results are strongly dependent upon the results of this paper. The complete proofs of the 2^X and $C(X)$ results appear in [6].

This paper assumes the 2^I result and not the techniques of the proof. The proofs given here use some of the fundamental results of infinite-dimensional topology, but if the reader takes these results, listed in § 2, as axioms, then no previous knowledge of infinite-dimensional topology is necessary for understanding this paper.

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2. Definitions and infinite-dimensional topology background. If X is a compact metric space, then the Hausdorff metric D on 2^X can be defined by

$$D(A, B) = \inf \{\varepsilon > 0 \colon A \subset U(B, \varepsilon) \text{ and } B \subset U(A, \varepsilon)\}$$

where $U(C, \varepsilon)$ is the open ε-neighborhood of $C \subset X$. If V is a subset of X, then 2^X_V is the subspace of 2^X consisting of all members of 2^X that contain V, and likewise for $C_V(X)$.

Let Q denote the countable infinite product of I with itself and define a *Hilbert cube* as any space homeomorphic ($\approx$) to Q. A space X is a *Q-factor* if $X \times Q \approx Q$. A *Q-manifold* is a separable metric space such that each point has an open neighborhood homeomorphic to an open subset of Q.

A *map* is a continuous function. If X and Y are homeomorphic compact metric spaces, then a map $f: X \to Y$ is a *near-homeomorphism* if for each $\varepsilon > 0$ there exists a homeomorphism $h: X \to Y$ such that $d(f, h) < \varepsilon$. We say that $f: X \to Y$ *stabilizes* to a near-homeomorphism if $f \times id: X \times Q \to Y \times Q$ is a near-homeomorphism. By a *graph* we will mean a 1-dimensional polyhedron with a specific triangulation.

R. D. Anderson's notion of Z-set [1] is extensively used in this paper and is one of the fundamental concepts in infinite-dimensional topology. There have been various definitions of Z-sets in the literature [1], [2], [4], and [7]. The following is the most convenient formulation for this paper.

DEFINITION 2.1. A closed subset A of a Q-factor X is a *Z-set* in X if for each $\varepsilon > 0$ there exists a map $f: X \to X \backslash A$ such that $d(f, id) < \varepsilon$.

We list below two well-known properties of Z-sets, the proofs of which are very easy. All spaces below are Q-factors.

2.2. Z-set Properties.

(a) *If A is a Z-set in X, then $A \times Y$ is a Z-set in $X \times Y$.*

(b) *Any finite union of Z-sets is a Z-set.*

One of the important theorems in infinite-dimensional topology is the following theorem of Anderson. See [11] and [14] for generalizations.

2.3. First Sum Theorem [1]. *If A, B, and $A \cap B$ are Hilbert cubes (Q-factors) and $A \cap B$ is a Z-set in A and in B, then $A \cup B$ is a Hilbert cube (Q-factor).*

If X and Y are disjoint spaces, A a closed subset of X, and $f: A \to Y$ a map, then the *adjunction space* of f, denoted $X \bigcup_f Y$, is $(X \cup Y)/R$, where R is the equivalence relation on $X \cup Y$ generated by $aRf(a)$ for each $a \in A$. We say X is *attached* to Y by f. If $g: X \to Y$ is a map, then the *mapping cylinder* of g, denoted M_g, is the adjunction space $(X \times I) \bigcup_{g'} Y$ where $g': X \times \{0\} \to Y$ is defined by $g'(x, 0) = g(x)$. The following is one of the basic theorems in the theory of Q-factors.

2.4. Mapping Cylinder Theorem [11] and [14]. *Let X and Y be Q-factors and let $g: X \to Y$ be a map of X into Y, then the mapping cylinder of g, M_g, is also a Q-factor. Furthermore, if $c: M_g \to Y$ is the map defined by $c([x, t]) = g(x)$, then c stabilizes to a near-homeomorphism.*

An important corollary of this is the following.

2.5. The Attaching Theorem [10]. *Let X and Y be Q-factors and let A be a closed subset of X that is a Z-set in X. If $f: A \to Y$ is any map, then the adjunction space $X \bigcup_f Y$ is also a Q-factor.*

A *relative homeomorphism* $f: (X, A) \to (Y, B)$ is a map of the pairs where $f | X \backslash A: X \backslash A \to Y \backslash B$ is a homeomorphism. The next remark is just a convenient alternative way of viewing adjunction spaces and will not be proved. Let all spaces below be compact metric.

REMARK 2.6. If $f: (X, A) \to (Y, B)$ is a relative homeomorphism, then Y is homeomorphic to the adjunction space $X \bigcup_g B$ where $g = f | A$.

The main tool of this paper is the following theorem.

2.7. Compactification Theorem [13]. *Let A be a closed subset of the space X where*

(1) *X is a Q-factor,*
(2) *A is a Q-factor,*
(3) *A is a Z-set in X, and*
(4) *$X \backslash A$ is a Q-manifold.*

Then X is a Hilbert cube.

The above theorem gives us conditions as to when the Q-manifold $X \backslash A$ can be compactified to be a Hilbert cube. We list the parts of the hypothesis because in practice the verification of each part will often be a separate result. To prove that 2^Γ is a Hilbert cube we will use the Compactification Theorem where $X = 2^\Gamma$ and $A = C_w(\Gamma)$ for some vertex $w \in \Gamma$. In § 3 we will prove that 2^Γ is a Q-factor and in § 4 we will prove that 2^Γ and $C_w(\Gamma)$ satisfy the other three conditions.

3. 2^{Γ} is a Q-factor. All of our results will be for the more general case 2^{Γ}_V where V is any set of vertices (possibly empty) of a finite, connected graph. Note that if V is empty, then $2^{\Gamma}_V = 2^{\Gamma}$. We first prove two lemmas.

Let Γ be a finite, connected, acyclic graph and let V be any subset (possibly empty) of the vertices of Γ. Let w be a vertex of Γ which separates it and let $\Gamma_1, \cdots, \Gamma_n$ be the closure of the components of $\Gamma \backslash \{w\}$, denoting by V_i the set $V \cap \Gamma_i$, $i = 1, \cdots, n$. Suppose that $w \notin V$ and let $W = V \cup \{w\}$ and for each i, let $W_i = V_i \cup \{w\}$. Let $X_i = \bigcup_{j=1}^{i} (2^{\Gamma_j}_{W_j} \times \prod_{k \neq j, k=1}^{i} 2^{\Gamma_k}_{V_k})$.

LEMMA 3.1. *X_n is a Q-factor if the $2^{\Gamma_j}_{V_j}$ and $2^{\Gamma_j}_{W_j}$ are Q-factors.*

Proof. For $i < n$, $X_{i+1} = 2^{\Gamma_{i+1}}_{V_{i+1}} \times X_i \cup 2^{\Gamma_{i+1}}_{W_{i+1}} \times \prod_{j=1}^{i} 2^{\Gamma_j}_{V_j}$, and $2^{\Gamma_{i+1}}_{V_{i+1}} \times X_i \cap 2^{\Gamma_{i+1}}_{W_{i+1}} \times \prod_{j=1}^{i} 2^{\Gamma_j}_{V_j} = 2^{\Gamma_{i+1}}_{W_{i+1}} \times X_i$. Since Γ is acyclic, w is a free vertex of each Γ_i and thus by a direct verification of the definition of a Z-set, each $2^{\Gamma_i}_{W_i}$ is a Z-set in $2^{\Gamma_i}_{V_i}$ and by 2.2(b), X_i is a Z-set in $\prod_{j=1}^{i} 2^{\Gamma_j}_{V_j}$. Thus, by 2.2(a), $2^{\Gamma_{i+1}}_{W_{i+1}} \times X_i$ is a Z-set in $2^{\Gamma_{i+1}}_{V_{i+1}} \times X_i$ and in $2^{\Gamma_{i+1}}_{W_{i+1}} \times \prod_{j=1}^{i} 2^{\Gamma_j}_{V_j}$. Note that a finite product of Q-factors is a Q-factor. Hence, by the First Sum Theorem, X_{i+1} is a Q-factor if X_i is one and since $X_1 = 2^{\Gamma_1}_{W_1}$ is a Q-factor by hypothesis, then X_n is a Q-factor by induction and the proof is complete.

Let Y_n be the set of all members of 2^{Γ}_V which meet each Γ_i.

LEMMA 3.2. *Y_n is a Q-factor if 2^{Γ}_W and the $2^{\Gamma_j}_{V_j}$ and $2^{\Gamma_j}_{W_j}$ are Q-factors.*

Proof. If $F: \prod_{i=1}^{n} 2^{\Gamma_i}_{V_i} \to 2^{\Gamma}_V$ is defined by $F(A_1, \cdots, A_n) = A_1 \cup \cdots \cup A_n$, then $F: (\prod_{i=1}^{n} 2^{\Gamma_i}_{V_i}, X_n) \to (Y_n, 2^{\Gamma}_W)$ is a relative homeomorphism and hence Y_n is homeomorphic to the adjunction space $\prod_{i=1}^{n} 2^{\Gamma_i}_{V_i} \bigcup_f 2^{\Gamma}_W$ where $f = F | X_n$. Since each of $\prod_{i=1}^{n} 2^{\Gamma_i}_{V_i}$, X_n, and 2^{Γ}_W is a Q-factor and since X_n is a Z-set in $\prod_{i=1}^{n} 2^{\Gamma_i}_{V_i}$, then Y_n is a Q-factor by the Attaching Theorem.

PROPOSITION 3.3. *If Γ is a finite, connected, acyclic graph and V is any subset (possibly empty) of the vertices of Γ, then 2^{Γ}_V is a Q-factor.*

Proof. (By induction on the number of edges in Γ.) If Γ is degenerate (no edges), this is clear, and if Γ has only one edge, this is shown in [10]. Now suppose that Γ has more than one edge and that the proposition is true for graphs with fewer edges than Γ. Adopt the notation of this section but allow w to belong to V. If

$w \in V$, then the mapping $\prod_{i=1}^{n} 2^{\Gamma_i}_{V_i} \to 2^{\Gamma}_{V}$ given by $(A_1, \cdots, A_n) \to A_1 \cup \cdots \cup A_n$ is a homeomorphism and since each of the $2^{\Gamma_i}_{V_i}$ is a Q-factor by the inductive hypothesis, 2^{Γ}_{V} is also a Q-factor.

If $w \notin V$, then by the above we have that 2^{Γ}_{W} is a Q-factor and hence by Lemma 3.2, Y_n is a Q-factor. For $k = 1, \cdots, n-1$, let Y_k be the subset of 2^{Γ}_{V} composed of those members which meet at least k of the Γ_i's. If $Y_{k+1} \neq 2^{\Gamma}_{V}$, let $\sigma_1, \cdots, \sigma_p$ be the subsets of $\{1, \cdots, n\}$ with exactly k members which contain $\{i: 1 \leqq i \leqq n, V_i \neq \emptyset\}$, and let

$$X_{\sigma_j} = \bigcup_{i \in \sigma_j} (2^{\Gamma_i}_{W_i} \times \prod_{m \in \sigma_j \setminus \{i\}} 2^{\Gamma_m}_{V_m}) .$$

Then exactly as in the proof of Lemma 3.1, each X_{σ_j} is a Q-factor and a Z-set in $\prod_{i \in \sigma_j} 2^{\Gamma_i}_{V_i}$. For $i = 1, \cdots, p$, let $Y_{k,i}$ be the subset of 2^{Γ}_{V}, composed of those members that are contained in $\bigcup_{j \in \sigma_i} \Gamma_j$ and which meet each Γ_j, $j \in \sigma_i$; let $Y^i_k = (\bigcup_{j=1}^{i} Y_{k,j}) \cup Y_{k+1}$, and let Y^0_k denote Y_{k+1}. Then $Y_k = Y^p_k$ and $f_{k,i}: (\prod_{j \in \sigma_i} 2^{\Gamma_j}_{V_j}, X_{\sigma_i}) \to (Y^i_k, Y^{i-1}_k)$ defined by $f_{k,i}(A_1, \cdots, A_k) = A_1 \cup \cdots \cup A_k$ is a relative homeomorphism and hence $Y^i_k \approx \prod_{j \in \sigma_i} 2^{\Gamma_j}_{V_j} \bigcup_g Y^{i-1}_k$, where $g = f_{k,i} \mid X_{\sigma_i}$. Thus, by induction we have that $Y_k = Y^p_k$ is a Q-factor if $Y_{k+1} = Y^0_k$ is one. Thus, since Y_n is a Q-factor we have by induction that $Y_1 = 2^{\Gamma}_{V}$ is a Q-factor.

THEOREM 3.4. *If Γ is a finite, connected graph and V is any subset (possibly empty) of the vertices of Γ, then 2^{Γ}_{V} is a Q-factor.*

Proof. As this is a topological result, new vertices may be introduced in Γ at will and therefore, one may assume without loss of generality that for some connected, acyclic graph Γ_0 and some collection $v_1, w_1, \cdots, v_n, w_n$ of free vertices of Γ_0, that $\Gamma = \Gamma_0/R$ where R is the equivalence relation on Γ_0 generated by $v_i R w_i$ for $i = 1, \cdots, n$. For $1 \leqq k \leqq n$, let R_k be the equivalence relation on Γ_0 generated by $v_i R w_i$ for $i = 1, \cdots, k$, and let $\Gamma_k = \Gamma_0/R_k$. Since $R_{k-1} \subset R_k$, we have a natural map $\varphi_k: \Gamma_{k-1} \to \Gamma_k$ induced by the identity map on Γ_0.

The theorem is true for Γ_0 by Proposition 3.3. Suppose the theorem is true for Γ_{k-1}, let X be any subset of the vertices of Γ_k and let $X' = \varphi_k^{-1}(X)$. Let $f_k: 2^{\Gamma_{k-1}}_{X'} \to 2^{\Gamma_k}_{X}$ be the map induced by φ_k and observe that f_k carries $2^{\Gamma_{k-1}}_{X' \cup \{v_k, w_k\}}$ homeomorphically onto $2^{\Gamma_k}_{X \cup \varphi_k(\{v_k, w_k\})}$. Thus, if $\varphi_k(\{v_k, w_k\}) \in X$, then $2^{\Gamma_k}_{X}$ is a Q-factor. If $\varphi_k(\{v_k, w_k\}) \notin X$, let $Y_1 = X' \cup \{v_k\}$, $Y_2 = X' \cup \{w_k\}$, and $Y_3 = X' \cup \{v_k, w_k\}$. Then $2^{\Gamma_{k-1}}_{X'}$, $2^{\Gamma_{k-1}}_{Y_i}$, $i = 1, 2, 3$, and $2^{\Gamma_k}_{\varphi_k(Y_3)}$ are Q-factors and $2^{\Gamma_{k-1}}_{Y_3} = 2^{\Gamma_{k-1}}_{Y_1} \cap 2^{\Gamma_{k-1}}_{Y_2}$. Moreover, since v_k and w_k are free vertices, $2^{\Gamma_{k-1}}_{Y_3}$ is a Z-set in each of them and thus by the First Sum Theorem $2^{\Gamma_{k-1}}_{Y_1} \cup 2^{\Gamma_{k-1}}_{Y_2}$ is a Q-factor. Also, since each of $2^{\Gamma_{k-1}}_{Y_i}$, $i = 1, 2$, is a Z-set in $2^{\Gamma_{k-1}}_{X'}$, their union is also a Z-set by 2.2(b). Moreover, $f_k\colon (2_{X'}^{\Gamma_{k-1}}, 2_{Y_1}^{\Gamma_{k-1}} \cup 2_{Y_2}^{\Gamma_{k-1}}) \to (2_X^{\Gamma_k}, 2_{\varphi_k(Y_3)}^{\Gamma_k})$ is a relative homeomorphism and hence $2_X^{\Gamma_k} \approx 2_{X'}^{\Gamma_{k-1}} \bigcup_{g_k} 2_{\varphi_k(Y_3)}^{\Gamma_k}$ where $g_k = f_k \mid 2_{Y_1}^{\Gamma_{k-1}} \cup 2_{Y_2}^{\Gamma_{k-1}}$, and thus by the Attaching Theorem $2_X^{\Gamma_k}$ is a Q-factor and the theorem follows.

4. 2^Γ is a Hilbert cube.

In this section we verify the last three conditions of the Compactification Theorem.

LEMMA 4.1. *If Γ is a finite, connected graph and V is any set of vertices (possibly empty) of Γ, then $C_V(\Gamma)$ is a Q-factor.*

Proof. First we show that $C_V(\Gamma)$ is contractible. Let Γ be endowed with a convex metric, i.e., one for which there always exists a point half way between any two given points. Then the function $F\colon C_V(\Gamma) \times I \to C_V(\Gamma)$ defined by $F(A, t)$ is equal to the closed $t\delta$-neighborhood of A in Γ, where δ is the diameter of Γ, is a contraction of $C_V(\Gamma)$ to the point $\Gamma \in C_V(\Gamma)$.

Next, in [8], R. Duda proves that $C(\Gamma)$ is a polyhedron and since it is contractible we have by [11] that $C(\Gamma)$ is a Q-factor. If $V \neq \emptyset$, then $C_V(\Gamma)$ is geometrically easier to classify than $C(\Gamma)$ and although it was not specifically dealt with in [8], it is a subpolyhedron of $C(\Gamma)$, and since it is contractible, it is a Q-factor. For a considerably more general result see [6].

LEMMA 4.2. *If Γ is a finite, connected, nondegenerate graph, w is a vertex of Γ, and V is a collection (possibly empty) of vertices of Γ, then $C_{V \cup \{w\}}(\Gamma)$ is a Z-set in 2_V^Γ.*

Proof. We will first prove the result for the case that $w \in V$ by constructing for each $\varepsilon > 0$ a map $f\colon 2_V^\Gamma \to 2_V^\Gamma \backslash C_V(\Gamma)$ that is within ε of the identity. Let w_i, $i = 1, \cdots, n$, be the vertices of Γ which are joined to w by edges $E_i = [w, w_i]$ and assume, for the metric on Γ, that each E_i is isometric with $[0, 1]$ so that for each $0 < \varepsilon \leqq 1$ the open ε-ball about w, $U(w, \varepsilon)$, is precisely the set $\{(1 - t)w + tw_i\colon 0 \leqq t < \varepsilon, i = 1, \cdots, n\}$. Let $V(w, \varepsilon)$ be the closure in Γ of $U(w, \varepsilon)$ and let $\operatorname{Bd} U(w, \varepsilon) = V(w, \varepsilon) \backslash U(w, \varepsilon)$. For a fixed $0 < \varepsilon < 1$, and for $A \in 2_V^\Gamma$, let

$$f(A) = [A \backslash U(w, \varepsilon/2)] \cup \{w\} \cup \operatorname{Bd} U(w, \varepsilon/2) .$$

It is clear that $[A \backslash U(w, \varepsilon/2)] \cup \{w\} \in 2_V^\Gamma \backslash C_V(\Gamma)$ but this assignment of A would not be continuous basically for the reason that one may have two points $x \in U(w, \varepsilon/2)$ and $y \notin U(w, \varepsilon/2)$ that are very close together. Including the set $\operatorname{Bd} U(w, \varepsilon/2)$ in the image under f of A

establishes the continuity of f, which is within ε of the identity map because in 2^{Γ}_{V} the distance between $\{w\}$ and Bd $U(w, \varepsilon/2)$ is $\varepsilon/2 < \varepsilon$. Thus, since f is continuous and the image of f misses $C_V(\Gamma)$, $C_V(\Gamma)$ is a Z-set in 2^{Γ}_{V}.

We will now modify these techniques to prove the theorem in the case $w \notin V$: Let $W = V \cup \{w\}$. If the above map f were defined on 2^{Γ}_{V} it would not be within ε of the identity, as is seen by comparing $f(A)$ and A for sets A with no points close to w. Since our main technique of mapping 2^{Γ}_{V} off $C_W(\Gamma)$ is to delete an open set about w, we will phase out this process so that we will be deleting open sets about w only from those members of 2^{Γ}_{V} that contain points close to w.

For $0 \leqq a \leqq 1$ we denote the point $(1-a)w + aw_i \in [w, w_i]$ simply by $[a]_i$. For $A \in 2^{\Gamma}_{V}$, let $a_i \in [0, 1]$ be the number such that $[a_i]_i$ is the point of $A \cap E_i$ nearest to w, if $A \cap E_i \neq \emptyset$. If $0 \leqq a_i \leqq \varepsilon$, let $a'_i = \max\{0, 2a_i - \varepsilon\}$ observing that if $0 \leqq a_i \leqq \varepsilon/2$, then $a'_i = 0$; and if $a_i = \varepsilon$, then $a'_i = a_i$. For $A \in 2^{\Gamma}_{V}$, let

$$f(A) = \begin{cases} A \cup \{[a'_i]_i : 1 \leqq i \leqq n,\ 0 \leqq a_i \leqq \varepsilon\}\,, & \text{if } \delta \geqq \varepsilon/2 \\ A \cup \{[(2\delta/\varepsilon)a'_i + (1 - 2\delta/\varepsilon)a_i]_i : 1 \leqq i \leqq n,\ 0 \leqq a_i \leqq \varepsilon\}\,, & \\ & \text{if } 0 \leqq \delta \leqq \varepsilon/2 \end{cases}$$

where $\delta = \delta(A) = D(A, 2^{\Gamma}_{W})$, which in this case is the minimum distance between points of A and w. Then f is a well-defined function since it is uniquely defined for elements $A \in 2^{\Gamma}_{V}$, where $\delta = \varepsilon/2$. Also, f is phased back to the identity at $\delta = 0$, that is, if $\delta(A) = 0$, then $f(A) = A$; and this establishes the continuity of f. Also observe that if $\delta(A) = \varepsilon/2$, then $w \in f(A)$ and if $\delta(A) \geqq \varepsilon$, then $f(A) = A$. Let $\alpha(A) = \max\{0, \varepsilon/2 - \delta(A)\}$ and define g on $f(2^{\Gamma}_{V})$ by

$$gf(A) = \begin{cases} [f(A)\backslash U(w, \alpha(A))] \cup \operatorname{Bd} U(w, \alpha(A)) & \text{if } \delta(A) < \varepsilon/2 \\ f(A) & \text{if } \delta(A) \geqq \varepsilon/2\,. \end{cases}$$

The continuity of g follows since α is continuous and since for $A \in 2^{\Gamma}_{V}$ where $\delta(A)$ is less than $\varepsilon/2$ but close to $\varepsilon/2$, then Bd $U(w, \alpha(A))$ is close to $\{w\}$, and hence $gf(A)$ is close to $f(A)$. Furthermore, the composition $gf\colon 2^{\Gamma}_{V} \to 2^{\Gamma}_{V}$ is within ε of the identity and $gf(2^{\Gamma}_{V}) \cap C_W(\Gamma) = \emptyset$ and thus, $C_W(\Gamma)$ is a Z-set in 2^{Γ}_{V}.

The next lemma will be the inductive step for the main theorem of this section. Let $L_1, \cdots, L_m$ be a finite collection of finite, connected graphs, let W be a collection of vertices from $\bigcup_{i=1}^{m} L_i$ where W contains at least one vertex of each L_i, and let $K = (\bigcup_{i=1}^{m} L_i)/W$ be the quotient space obtained by taking the disjoint union of the L_i and identifying all the vertices in W. Let $p\colon \bigcup_{i=1}^{m} L_i \to K$ be the quotient map and let $w = p(W)$.

LEMMA 4.3. *If each $2_{V_i}^{L_i}$ is a Hilbert cube for each collection V_i (possibly empty) of vertices of L_i, then 2_V^K is a Hilbert cube for each set of vertices V (possibly empty) of K.*

Proof. To apply the Compactification Theorem, we have that 2_V^K is a Q-factor by 3.4, $C_W(K)$ is a Q-factor by 4.1 where $W = V \cup \{w\}$, and $C_W(K)$ is a Z-set in 2_V^K, by 4.2. It remains to be shown that $2_V^K \backslash C_W(K)$ is a Q-manifold.

If $A \in 2_V^K \backslash C_W(K)$, then A has a component missing w. If A is connected, then it has an open neighborhood U in 2_V^K homeomorphic to an open set of $2_{V_i}^{L_i}$, for some i and some collection V_i of vertices of L_i. Since $2_{V_i}^{L_i}$ is by hypothesis a Hilbert cube, U is homeomorphic to an open subset of the Hilbert cube. If A is not connected, then it has a separation into two disjoint closed nonempty subsets A_1 and A_2 such that $A = A_1 \cup A_2$. Assuming that $w \notin A_2$, let U_1 and U_2 be disjoint open sets of K containing A_1 and A_2, respectively. Now, for some $i_1, \cdots, i_k$, $1 \leqq k \leqq m$, A_2 has an open neighborhood W_2 in $2_{A_2 \cap V}^K$ consisting of sets lying entirely within U_2, which is homeomorphic to a product $U_{21} \times U_{22} \times \cdots \times U_{2k}$ of open sets of the Hilbert cubes $2_{V_j}^{L_j}$, $j = i_1, \cdots, i_k$ where $V_j = L_j \cap p^{-1}(A_2 \cap V)$. On the other hand, the set $W_1 = \{B \in 2_{V'}^K \colon B \subset U_1\}$, where $V' = V \cap A_1$, is an open neighborhood of A_1 in $2_{V'}^K$ which is by 3.4 a Q-factor. Now $U = \{B \cup C \colon B \in W_1, C \in W_2\}$ is an open neighborhood of A in 2_V^K which is homeomorphic to $W_1 \times W_2$ and hence, to an open subset of the Hilbert cube $2_{V'}^K \times \Pi\{2_{V_j}^{L_j} \colon j = i_1, \cdots, i_k\}$. Therefore, $2_V^K \backslash C_W(K)$ is a Q-manifold and the proof is complete.

THEOREM 4.4. *If Γ is a nondegenerate, finite, connected graph and V is any set (possibly empty) of vertices of Γ, then 2_V^Γ is a Hilbert cube.*

Proof. Let $\mathscr{G}$ be the class of all nondegenerate, finite, connected graphs. For each $K \in \mathscr{G}$, let $V(K)$ be the number of vertices of K, $E(K)$ the number of edges of K, and $R(K) = E(K) - V(K) + 1$. ($R(K)$ is the rank of the first homology group $H_1(K)$; it is also $E(K) - E(L)$ for each maximal acyclic subgraph L of K.) Let $\mathscr{G}_i$ be the class of all members K of $\mathscr{G}$ for which $R(K) = i$, and let $\mathscr{G}_{ij}$ be the subclass of $\mathscr{G}_i$ composed of all members K of $\mathscr{G}_i$ with $E(K) = j$.

The theorem holds for $\mathscr{G}_{01}$, being the main results of [9] and [10]. Specifically, 2^I, 2_0^I, 2_1^I, and 2_{01}^I are all Hilbert cubes. Now fix $(i, j) \neq (0, 1)$ and suppose that the theorem holds for each $\mathscr{G}_{i'j'}$ with $i' < i$ or $i' = i$ and $j' < j$.

Let $K \in \mathscr{G}_{ij}$ and let V be a set of vertices (possibly empty) of K and let w be a vertex of K which is not a free vertex of K. Construct a new complex K' by "splitting" K at w. That is, let $v_1, \cdots, v_n$ be the vertices of K which are joined to w by edges $[w, v_i]$ of K and let $w_1, \cdots, w_n$ be abstract vertices not in K. Then $K' = (K \backslash \bigcup_{i=1}^n [w, v_i)) \cup \bigcup_{i=1}^n [w_i, v_i]$ and K' has as vertices all vertices of K except w together with $w_1, \cdots, w_n$ and has as edges all edges of K which do not contain w together with the new edges $[w_i, v_i]$, $i = 1, \cdots, n$. Now, if w separates K, each component L of K' has $E(L) < E(K)$ and $R(L) \leqq R(K)$, while if w does not separate K, then $K' \in \mathscr{G}$ and $R(K') < R(K)$. Thus, by the induction hypothesis, each component of K' satisfies the theorem and hence by Lemma 4.3, 2_V^K is a Hilbert cube and thus by induction the theorem is proved.

5. 2^D and $C(D)$ for local dendrons D. In this section we generalize the theorems to each *dendron*, that is, a Peano space which contains no simple closed curve, and to each *local dendron*, that is, a Peano space such that each point has a closed neighborhood which is a dendron. In particular, each dendron is a local dendron. We can express (see [13]) each dendron D as the limit of an inverse sequence (T_n, r_n), $\lim (T_n, r_n)$, where T_1 is an arc and for each $n \geqq 1$, T_{n+1} is the union of T_n and an arc $[a_n, b_n]$ where $T_n \cap [a_n, b_n] = \{a_n\}$, and where $r_n: T_{n+1} \to T_n$ is the retraction which collapses $[a_n, b_n]$ to a_n. The inverse sequence (T_n, r_n) induces the inverse sequence $(2^{T_n}, r_n^*)$ where $r_n^*: 2^{T_{n+1}} \to 2^{T_n}$ is defined by $r_n^*(A) = r_n(A)$. Then 2^D is homeomorphic to $\lim (2^{T_n}, r_n^*)$.

The corresponding inverse limit representation for local dendrons is the same except that T_1 is allowed to be a finite, connected graph. We argue this as follows. For a local dendron D there exists an $\varepsilon > 0$ such that each closed connected subset of D with diameter less than ε is a dendron. Cover D with a finite collection of closed connected neighborhoods $\{D_i\}$ with diameter less than $\varepsilon/2$. The pairwise intersections of the D_i are connected. In each nonempty intersection of elements of the $\{D_i\}$ pick a point and then in each D_i construct a tree connecting each of the selected points contained in that D_i. Then the union of these trees will be a finite connected graph, a candidate for T_1 in the above inverse limit presentation. Now we can add the remaining stickers to the trees in the prescribed manner to obtain the local dendron D as the $\lim (T_n, r_n)$. Such an inverse limit for a local dendron D will be called a *standard* inverse limit representation for D. Also, for a given finite subset V of D we can easily construct T_1 to contain V by including it in the set of points picked in the intersections of the D_i. We will need the next result.

THEOREM 5.1. Morton Brown [3]. *Let* $S = \lim(X_n, f_n)$, *where the* X_n *are all homeomorphic to a given compact metric space* X *and each* f_n *is a near-homeomorphism. Then* S *is homeomorphic to* X.

LEMMA 5.2. *If* $f: Q \to Q$ *is a map that stabilizes to a near-homeomorphism, then* f *is a near-homeomorphism.*

Proof. Define $\alpha_n: Q \times Q \to Q$ by $\alpha_n((x_1, x_2, \cdots), (y_1, y_2, \cdots)) = (x_1, \cdots, x_n, y_1, x_{n+1}, y_2, x_{n+2}, y_3, \cdots)$. Then each α_n is a homeomorphism and hence each $\alpha_n \circ (f \times id) \circ \alpha_n^{-1}$ is a near-homeomorphism since $f \times id$ is one by assumption. Furthermore, $d(f, \alpha_n \circ (f \times id) \circ \alpha_n^{-1}) \to 0$ as $n \to \infty$ and hence f is a uniform limit of near-homeomorphisms and thus is a near-homeomorphism.

THEOREM 5.3. *If* D *is a nondegenerate local dendron and* V *is any finite subset (possibly empty) of* D, *then* 2_V^D *is a Hilbert cube.*

Proof. We follow the proof of [Theorem 2, 13] which states a corresponding result for $C(D)$. Choose a standard inverse limit representation for D where $V \subset T_1$. Let $r'_n: 2^{T_{n+1}}_{V \cup \{b_n\}} \to 2_V^{T_n}$ be the restriction of the map r_n^*, let $M_{r'_n}$ be the mapping cylinder of r'_n, and let $c_n: M_{r'_n} \to 2_V^{T_n}$ be the natural projection defined by $c_n([A, t]) = r'_n(A)$. Since $2^{T_{n+1}}_{V \cup \{b_n\}}$ and $2_V^{T_n}$ are Q-factors by 3.4, it follows by the Mapping Cylinder Theorem that c_n stabilizes to a near-homeomorphism. We will show below that $M_{r'_n}$ is homeomorphic to $2_V^{T_{n+1}}$ in such a way that the projection map c_n is topologically equivalent to r_n^*. Thus, since each of $2_V^{T_n}$ and $2_V^{T_{n+1}}$ is a Hilbert cube, we have by 5.2 that c_n is a near-homeomorphism and hence so is r_n^*. The proof that $2_V^D \approx Q$ will then be complete by 5.1 since 2_V^D is homeomorphic to an inverse limit of Hilbert cubes $2_V^{T_n}$ where the bonding maps are near-homeomorphisms. We now verify the above stated fact about $M_{r'_n}$. Define $g_n: 2_V^{T_{n+1}} \to M_{r'_n}$ as follows where we parametrize $[a_n, b_n]$ to be order isomorphic with $[0, 1]$ and let $\sup(A \cap [a_n, b_n]) = d$ if it exists. Let

$$g_n(A) = \begin{cases} [A], & \text{if } A \cap (a_n, b_n] = \varnothing \\ [(A \cap T_n) \cup (1/d(A \cap [a_n, b_n]), d)], & \text{if } A \cap (a_n, b_n] \neq \varnothing\,. \end{cases}$$

Then g_n is a homeomorphism so that the following diagram is

$$\begin{tikzcd} 2_V^{T_{n+1}} \arrow[rr, "g_n"] \arrow[dr, "r_n^*"'] & & M_{r'_n} \arrow[dl, "c_n"] \\ & 2_V^{T_n} & \end{tikzcd}$$

commutative and this completes the proof.

In [13], it is proved that the subcontinua $C(D)$ of a dendron D form a Q-factor which is a Hilbert cube if and only if the branch points of D are dense. We will extend this result to local dendrons D and to spaces $C_V(D)$ where V is a finite subset of D.

LEMMA 5.4. *For each local dendron D and each finite subset V (possibly empty) of D, $C_V(D)$ is a Q-factor.*

Proof. Choose a standard inverse limit representation, $\lim(T_n, r_n)$, for D where $V \subset T_1$. Then $C_V(D) \approx \lim(C_V(T_n), r_n^*)$. As in the proof of Theorem 5.3 the space $C_V(T_{n+1})$ is naturally homeomorphic to the mapping cylinder $M_{r_n'}$ where $r_n': C_{V \cup \{b_n\}}(T_{n+1}) \to C_V(T_n)$ is the restriction of r_n^*. Furthermore, the map r_n^* is topologically equivalent to the natural projection $c_n: M_{r_n'} \to C_V(T_n)$ which stabilizes to a near-homeomorphism. Since each space $C_V(T_n)$ is a Q-factor by Lemma 4.1 and since each bounding map r_n^* stabilizes to a near-homeomorphism, then $C_V(D) \approx \lim(C_V(T_n), r_n^*)$ is a Q-factor and the proof is complete.

To prove that $C_V(D)$ is a Hilbert cube if the branch points of D are dense, we will need Lemmas 4.1 and 5.4 together with the next two lemmas to satisfy the hypothesis of the Compactification Theorem where $X = C_V(D)$ and $A = C_V(T_1)$.

LEMMA 5.5. *Let D be a local dendron with a dense set of branch points, let V be a finite subset (possibly empty) of D, and let $\lim(T_n, r_n)$ be a standard inverse limit representation for D where $V \subset T_1$. Then $C_V(T_1)$ is a Z-set in $C_V(D)$.*

Proof. A local dendron admits a convex metric. Using a convex metric on D, for sufficiently small $\varepsilon > 0$, the map f on $C_V(D)$ defined by setting $f(A)$ equal to the closed ε-neighborhood of A in D is a map from $C_V(D)$ into itself where $d(f, id) < \varepsilon$. Since the branch points of D are dense, we also have that $f: C_V(D) \to C_V(D) \backslash C_V(T_1)$ and hence $C_V(T_1)$ is a Z-set in $C_V(D)$.

LEMMA 5.6. *If D, V, and $\lim(T_n, r_n)$ are as above, then $C_V(D) \backslash C_V(T_1)$ is a Q-manifold.*

Proof. Let $A \in C_V(D) \backslash C_V(T_1)$. It is sufficient, since $C_V(D) \backslash C_V(T_1)$ is open in $C_V(D)$, to show that A has an open neighborhood in $C_V(D)$ that is homeomorphic to an open subset of the Hilbert cube. If $A \cap T_1$ is either empty or a single point, then V is either empty or is a single point and there exists an open set U in D containing A and a dendron D_1 such that $A \subset U \subset D_1 \subset D$. If W is the set of all elements of $C_V(D)$ contained in U, then W is an open neighborhood of A in $C_V(D)$ and is an open subset of $C_V(D_1)$ which is a Hilbert cube by an obvious modification of West's proof [13] that $C(D_1)$ is a Hilbert cube.

If $A \cap T_1$ is nondegenerate, let E be the closure of some component of $D \backslash T_1$ that contains some points of A and let F be the closure of $D \backslash E$. Then E is a dendron and F is a local dendron containing T_1 and each has a dense set of branch points and $E \cap F$ is one point, say q. Then $C_q(E)$ is a Hilbert cube by modifying West's argument and $C_W(F)$, where $W = V \cup \{q\}$, is a Q-factor by Lemma 5.4 and hence $C_q(E) \times C_W(F)$ is a Hilbert cube. The map $\alpha: C_q(E) \times C_W(F) \rightarrow C_V(D)$ defined by $\alpha(A, B) = A \cup B$ is an embedding into $C_V(D)$ where the image of α is a closed neighborhood (not a small one) of A and thus $C_V(D) \backslash C_V(T_1)$ is a Q-manifold.

THEOREM 5.7. *If D is a local dendron and V is a finite subset (possibly empty) of D, then $C_V(D)$ is a Q-factor, and furthermore if the branch points of D are dense, then $C_V(D)$ is a Hilbert cube.*

Proof. The first part of the theorem is Lemma 5.4 and the second part follows from applying Lemmas 4.1 and 5.4–5.6 to the Compactification Theorem and observing that D admits a standard inverse limit representation $\lim(T_n, r_n)$ where $V \subset T_1$.

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