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ON SYMMETRY OF SOME BANACH ALGEBRAS

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A Banach *-algebra is called symmetric, if the spectra of elements of the form a^*a contain only nonnegative real numbers. Symmetric Banach *-algebras have a series of important properties, especially with respect to their representation theories. Here it is proved that tensoring with finite dimensional matrix rings preserves symmetry. As an application it is shown that the category of locally compact groups with symmetric L^1-algebras is closed under finite extensions.

In recent years there was a growing interest in the problem of symmetry of involutive Banach algebras. In particular very substantial progress has been made towards a solution of the problem of characterizations of such locally compact groups G for which the group algebra $L^1(G)$ is symmetric. The most striking results in this direction are due to J. Jenkins, who first proved that the discrete "$ax + b$"-group has a nonsymmetric algebra [3] and that the same holds for noncompact semisimple Lie groups [4] (independently this was also proved—but not published—by R. Takahashi). On the other hand, Hulanicki proved symmetry for discrete nilpotent groups (Studia Math. 35) and for class finite groups (Pacific J. Math. 18). Moreover, in Studia Math. 48 I obtained the same results for connected nilpotent Lie groups of class 2.

In [1] D. W. Bailey states a theorem (Theorem 2, p. 417) that a semi-direct product extension of a locally compact group G with a finite group F has a symmetric group algebra, if G has. As (implicitely) in [2] and as in the present note this is reduced to the preservation of symmetry under tensoring with matrix algebras over $\boldsymbol{C}$. His reduction of the $n \times n$-case to the 2×2-case is the same as ours, but his proof of the 2×2-case (Lemma 2, p. 415) contains a definitely false inequality for the spectral radius and thus is wrong.

Let Γ be a locally compact group, H a closed normal subgroup and let $G = \Gamma/H$ be the quotient group. Assume that we have a measurable cross section from G into Γ. Then in [6] and [7] it was shown that $L^1(\Gamma)$ is isomorphic with a generalized L^1-algebra $L^1(G, L^1(H); T, P)$. Thus our result on groups will be a consequence of a more general one: Let G be a finite group, $\mathscr{A}$ a Banach *-algebra on which G acts and let P be a factor system of G with values in the unitary multipliers of $\mathscr{A}$ (see [6], [7]). Then the generalized L^1-algebra $L^1(G, \mathscr{A}; T, P)$ is symmetric, if $\mathscr{A}$ is symmetric. For commutative $\mathscr{A}$ and generalized L^1-algebras in the sense of [7] this has been proved by Glaser [2], using determinants. Our theorem will be a consequence of

THEOREM 1. *Let $\mathscr{A}$ be an involutive Banach algebra and let $\mathscr{A}^{[n]}$ be the algebra of $n \times n$-matrices $x = (x_{ij})$ over $\mathscr{A}$, $x_{ij} \in \mathscr{A}$, $i, j = 1, \cdots, n$. If $\mathscr{A}$ is symmetric, then $\mathscr{A}^{[n]}$ is symmetric.*

Proof. Assume at first that $\mathscr{A}$ contains an identity $\mathbf{1}$ and that $n = 2$. Write $\mathscr{M} = \mathscr{A}^{[2]}$. We have to prove that for every $a = (a_{ij}) \in \mathscr{M}$ the left ideal

$$\mathscr{L} = \mathscr{M}(\mathbf{1} + a^*a)$$

equals $\mathscr{M}$. (Here of course $\mathbf{1}$ is the unitmatrix and $a^* = (a'_{ij})$ with $a'_{ij} = a^*_{ji}$). More generally consider the left ideal

$$\mathscr{J} = \mathscr{M}(b^*b + a^*a)$$

where $b = (b_{ij})$ with $b_{21} = 0$, $b_{11} = b_{22} = \mathbf{1}$. The set $\mathscr{J}_{lk}$ of all $u \in \mathscr{A}$ for which there is an $x = (x_{ij}) \in \mathscr{J}$ with $u = x_{lk}$ plainly is a left ideal in $\mathscr{A}$. Since left multiplication with (e_{ij}), $e_{11} = e_{22} = 0$, $e_{12} = e_{21} = \mathbf{1}$, interchanges rows, we have $\mathscr{J}_{1k} = \mathscr{J}_{2k} = \mathscr{J}_k$ for $k = 1, 2$. Now $b^*b + a^*a = (z_{ij}) \in \mathscr{J}$. Hence $z_{11} = \mathbf{1} + a^*_{11}a_{11} + a^*_{21}a_{21} \in \mathscr{J}_1$ and $z_{22} = \mathbf{1} + b^*_{12}b_{12} + a^*_{22}a_{22} + a^*_{12}a_{12} \in \mathscr{J}_2$. Clearly no nontrivial positive functional can vanish on z_{ii} and hence on $\mathscr{J}_i$. This implies $\mathscr{J}_i = \mathscr{A}$ for $i = 1, 2$ (see [8], (4.7.14)). Now let $x = (x_{ij}) \in \mathscr{J}$ with $x_{11} = \mathbf{1}$ and let $y = (y_{ij})$ with $y_{12} = y_{22} = 0$. Then yx is in $\mathscr{J}$ and has (y_{11}, y_{21}) as its first column. Similarly one shows that $\mathscr{J}$ contains elements with given second column.

Now consider again $\mathscr{L}$ and let $x \in \mathscr{L}$ with $x_{11} = \mathbf{1}$. Multiplying x from the left by $f = (f_{ij})$, $f_{11} = \mathbf{1}$, $f_{ij} = 0$ otherwise, we may assume that $x_{21} = x_{22} = 0$. Define $b = (b_{ij})$ by $b_{11} = b_{22} = \mathbf{1}$, $b_{12} = -x_{12}$ and $b_{21} = 0$, $c = ab$. We have $\mathscr{M}(b^*b + c^*c) = \mathscr{M}b^*(\mathbf{1} + a^*a)\ b = \mathscr{L}b$. It follows that $\mathscr{J} = \mathscr{M}(b^*b + c^*c)$ contains $xb = f$. From this one derives easily that every matrix with second column equal to zero is contained in $\mathscr{J}$. But we know already that arbitrary second columns occur in elements from $\mathscr{J}$. This implies that $\mathscr{J}$ contains all matrices, i.e., $\mathscr{J} = \mathscr{M}$. Since b is regular, also $\mathscr{L} = \mathscr{M}$.

2. Now let n be arbitrary. By induction it follows that all $\mathscr{A}^{[2^l]}$ are symmetric. For $n \leqq 2^l$, $\mathscr{A}^{[n]}$ can be considered as subalgebra of $\mathscr{A}^{[2^l]}$, thus [8], (4.7.7) $\mathscr{A}^{[n]}$ is symmetric.

3. If $\mathscr{A}$ has no unit, the algebra $\tilde{\mathscr{A}} = (1) \oplus \mathscr{A}$ is symmetric

[8], (4.7.9) and contains $\mathscr{A}$. It follows that $\tilde{\mathscr{A}}^{[n]}$ is symmetric and contains $\mathscr{A}^{[n]}$, which in turn is symmetric. Thus Theorem 1 is proved.

Now let G be a locally compact group which acts on the involutive Banach algebra $\mathscr{A}$. This means that there exists a mapping T of G into the group Aut $(\mathscr{A})$ of isometric *-automorphisms of $\mathscr{A}$, such that $T_x T_y T_{xy}^{-1}$ is contained in the adjoint algebra $\mathscr{A}^b$ of $\mathscr{A}$(= algebra of "double centralizers") [7]. If P is a unitary 2-cocycle one can form the generalized L^1-algebra $L^1(G, \mathscr{A}; T, P)$, see [6].

THEOREM 2. *If G is finite and $\mathscr{A}$ is symmetric, then $L^1(G, \mathscr{A}; T, P)$ is symmetric.*

Proof. Let n be the order of G. The matrix algebra $\mathscr{A}^{[n]}$ acts naturally on the n-fold direct sum $n\mathscr{A} = \mathscr{A} \oplus \cdots \oplus \mathscr{A}$. On the other hand, identifying $\mathscr{L} = L^1(G, A; T, P)$ and $n\mathscr{A}$ as Banach spaces convolution also defines an action of $\mathscr{L}$ on $n\mathscr{A}$. The formula for the convolution product in $\mathscr{L}$ shows, that for $f \in \mathscr{L}$ the convolution operator on $n\mathscr{A}$ corresponds to the matrix

$$M(f) = (P_{yx^{-1},x} T_{x^{-1}} f(yx^{-1}))_{x,y} .$$

Thus $f \rightarrow M(f)$ is an isomorphism of $\mathscr{L}$ into $\mathscr{A}^{[n]}$. Also $M(f^*) = M(f)^*$ holds. It suffices to prove this only for functions of the form $f(x) = \delta_{x,z} a$ with fixed $a \in \mathscr{A}$ and $z \in G$. For such a function the claim is equivalent with the identity

$$(P_{z,y} T_{y^{-1}} a)^* = P_{z^{-1},zy} T_{(zy)^{-1}} P^0_{z,z^{-1}} T_z a^*$$

which can be proved by using the definitions and relations of [6], esp. (1.2), (1.3) and the definition of $T_{x,y}$ on p. 595.

It follows that $\mathscr{L}$ can be considered as a subalgebra of $\mathscr{A}^{[n]}$. Hence by Theorem 1 and [8], (4.7.7) symmetry of $\mathscr{A}$ implies that of $\mathscr{L}$.

THEOREM 3. *Let G be a locally compact group and let H be a closed subgroup of finite index. Then $L^1(G)$ is symmetric if and only if $L^1(H)$ is symmetric.*

Proof. Since H is open in G, $L^1(H)$ is a subalgebra of $L^1(G)$ and consequently is symmetric if $L^1(G)$ is.

Now let $L^1(H)$ be symmetric. The intersection H_0 of the finite number of conjugate subgroups xHx^{-1} of H in G is closed, normal and also of finite index in G. Moreover, $L^1(H_0) = \mathscr{A}$ as we have seen is symmetric. But $L^1(G) \cong L^1(G/H_0, \mathscr{A}; T, P)$ ([6], Satz 5). Therefore, Theorem 3 follows now from Theorem 2.

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