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APPROXIMATION PROPERTIES OF VECTOR VALUED FUNCTIONS

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There are several analogues of the Weierstrass approximation theorem that characterize the uniform closure of a $C[X]$ submodule $\mathscr{M}$ of the space $C[X: E]$ of bounded continuous E-valued functions on a compact space X. In this paper, a strong form of such a theorem is obtained which is then applied to yield a characterization of all the functionals ϕ in the dual of $C[X: E]$ that are extreme among those of unit norm that vanish on an arbitrary chosen $\mathscr{M}$. Each is determined by a point $x_0 \in X$ and a unit functional L that is extreme in the annihilator of a closed subspace $M \subset E$.

The space $C[X]$ of bounded continuous real valued functions on a topological space X has been studied as an algebra, a lattice, and a normed linear space; each of these reflects an aspect of the structure of the reals. In the present paper, we are interested in the more general space $C[X: E]$ of bounded continuous functions from X to E, a fixed normed linear space, under the uniform topology defined by the norm $\|f\| = \sup |f(x)|$. Here, $|\ |$ denotes the norm in E.

The amount of structure in $C[X: E]$ depends upon that of E. In earlier work, Yood [21] assumed that E was itself a Banach algebra, and Kaplansky [7] chose E to be an algebra of operators; in both cases, $C[X: E]$ is then itself an algebra. Others have looked at $C[X: E]$ as a normed space and have sought Riesz-like representation theorems for its dual, or for its continuous linear transformations. (See Tucker [18] and the references therein.) In 1957–58, I obtained some theorems of the Weierstrass type for general E by regarding $C[X: E]$ as a module over $C[X]$, both in the uniform topology and in the strict topology, results later generalized by Todd [17] and Wells [20]. Other aspects of this have been studied by Singer [14, 15], by Nachbin and his students Prolla and Machado [1967], [1972], and by Prenter [12].

The main objective of the present paper is to identify all the extreme functionals in the unit ball of the dual of $C[X: E]$, and in the cross-section of the ball obtained by intersecting it with the annihilator of an arbitrary submodule $\mathscr{M}$ of $C[X: E]$. The former is known (Singer [15]) but the latter is new. The present result was initiated by a new proof of the simpler case by Strobele.

2. A special case. It is instructive to examine $C[X: E]$ when

$E = C[Y]$. Any function f in $C[X: E]$ gives rise to a continuous scalar valued function f^* on $X \times Y$ defined by $f^*(x, y) = f(x)(y)$. The map $f \to f^*$ is a linear isometry of $C[X: E]$ with a closed subspace of $C[X \times Y]$. The image does not have to be dense. Take $X = Y = (-\infty, \infty)$ and let $F(x, y) = e^{ixy}$. This is a bounded continuous function on $X \times Y$; however, F does not arise as the image of any $f \in C[X: E]$. This is because the only candidate to obey $f^* = F$ is $f(x) = e^{ix(\ )}$, and one finds $|f(x) - f(a)| = 2$ if $x \neq a$, $= 0$ if $x = a$, so that the function f is not continuous on X to $E = C[Y]$.

No such example can be constructed if Y is compact, and in this case, the mapping $f \to f^*$ maps $C[X: C[Y]]$ isometrically onto $C[X \times Y]$. For, let $F \in C[X \times Y]$ and define f on X to $C[Y]$ by $f(x)(y) = F(x, y)$. Take any $a \in X$ and $\delta > 0$ and let $\mathscr{O}$ be the open set in $X \times Y$ consisting of those (x, y) such that

$$|F(x, y) - F(a, y)| < \delta .$$

Clearly, $\mathscr{O}$ contains the set $\{a\} \times Y$. Since Y is compact, there is an open set $\mathscr{V}$ about a such that $\mathscr{V} \times Y \subset \mathscr{O}$, and therefore

$$|f(x) - f(a)| = \sup_y |F(x, y) - F(a, y)| < \delta$$

for all $x \in \mathscr{V}$, and f is continuous and $f^* = F$.

These arguments show that if $E = C[Y]$, then $C[X: E]$ can always be regarded as a closed subspace of a conventional function space $C[Z]$, and that in some cases it coincides with it. Is this typical? Given an arbitrary normed linear space E, we can first embed E as a closed subspace of $C[B^*]$, where B^* is the unit ball in the dual space E^* consisting of all functionals L on E with $\|L\| \leqq 1$, by representing any $u \in E$ as a function on B^* by the usual pairing $\langle u, L \rangle = L(u)$. $C[X: E]$ becomes a subspace of $C[X: C[B^*]]$, and thus in turn a subspace of $C[Z]$ where $Z = X \times B^*$. Explicitly, each function $f \in C[X: E]$ corresponds to a scalar function f^* on $X \times B^*$ defined by

$$f^*(x, L) = L(f(x)) .$$

While this permits one to reduce many questions dealing with $C[X: E]$ to more familiar ones dealing with a scalar function space $C[Z]$, (for example the determination of the dual space,) much of the essential structure of $C(X: E)$ is concealed in the embedding. We shall return to this viewpoint in § 6 when we complete the determination of the extreme points in the set of functionals that annihilate an arbitrary module $\mathscr{M}$.

3. The structure of submodules. Let $\mathscr{M}$ be a subspace of

$C[X: E]$ that is a module over the ring $C[X]$; we note in passing that there is little gained by replacing $C[X]$ by a function algebra $\mathfrak{A}$ that is dense in $C[X]$, but that the corresponding problem when $\mathfrak{A}$ is a proper closed (e.g., maximal) subalgebra of $C[X]$ are profound. Examples of modules are easily given. If M is a proper closed subspace of E and $x_0 \in X$, let $\mathscr{M}$ be the class of all functions $f \in C[X: E]$ such that $f(x_0) \in M$. Then, $\mathscr{M}$ is a proper closed submodule. Other examples can be produced by varying x_0 and M and taking the intersection of the resulting modules. The main results depend on the fact that this process produces all closed submodules of $C[X: E]$. For simplicity, we treat only the case where X is compact. It is convenient to introduce a special notation.

DEFINITION 1. For any module $\mathscr{M}$, let M_x be the closure in E of the set {all $f(x)$ for $f \in \mathscr{M}\} = \mathrm{cl}\,\mathscr{M}(x)$.

Thus, $x \rightarrow M_x$ is a mapping from X into the lattice of closed subspaces of E, associated with a particular $C[X]$ submodule $\mathscr{M}$ of $C[X: E]$. The first results is the Weierstrass theorem for modules.

THEOREM 1. *A function $g \in C[X: E]$ belongs to the uniform closure of a module $\mathscr{M}$ if and only if $g(x) \in M_x$ for every $x \in X$.*

COLOLLARY 1. *A module $\mathscr{M}$ is dense in $C[X: E]$ if and only if $\mathscr{M}(x)$ is dense in E for each $x \in X$.*

COLOLLARY 2. *The maximal proper submodules of $C[X: E]$ are those determined by the choice of a point $x_0 \in X$ and a nonvanishing continuous functional L in the dual of E, and characterized as the class of $f \in C[X: E]$ such that $Lf(x_0) = 0$.*

COLOLLARY 3. *Every closed submodule of $C[X: E]$ is the intersection of the maximal submodules that contain it.*

Each of these results is an immediate deduction from the following more general result which we call the *strong* Weierstrass theorem for modules.

THEOREM 2. *Let $g \in C[X: E]$ and let $\mathscr{M}$ be any module. Then,*

$$\inf_{f \in \mathscr{M}} \| g - f \| = \sup_{x \in X} \inf_{u \in M_x} | g(x) - u | .$$

Proof. The left side of this equation is the distance from g to $\mathscr{M}$. Let us denote this by ρ. The right side is $\lambda = \sup_{x \in X} \lambda(x)$ where $\lambda(x)$ is the distance in E from $g(x)$ to M_x. Since M_x is the closure of $\mathscr{M}(x)$, we may also write $\lambda(x) = \inf_{f \in \mathscr{M}} |g(x) - f(x)|$ from which it is immediately clear that $\rho \geqq \lambda$. To prove the reverse relation, let $\varepsilon > 0$. Then, for any $x_0 \in X$, there is an $f_0 \in \mathscr{M}$ such that $|g(x_0) - f(x_0)| < \lambda + \varepsilon$. Let $\mathscr{O}_0$ be the open set about x_0 such that $|g(x) - f_0(x)| < \lambda + \varepsilon$ for all $x \in \mathscr{O}_0$. As x_0 varies, the sets $\mathscr{O}_0$ cover the compact space X. We thus arrive at points x_i, open sets $\mathscr{O}_i$, and $f_i \in \mathscr{M}$ so that $X = \bigcup_1^n \mathscr{O}_i$ and $|g(x) - f_i(x)| < \lambda + \varepsilon$ for all $x \in \mathscr{O}_i$. Choose $\varphi_i \in C[X]$ so that φ_i vanishes off $\mathscr{O}_i$, $\varphi_i(x) \geqq 0$, and $\sum_1^n \varphi_i(x) = 1$ for all x. Set $f = \sum_1^n \varphi_i f_i \in \mathscr{M}$. Then,

$$\begin{aligned} |g(x) - f(x)| &\leqq |\sum \varphi_i(x) g(x) - \sum \varphi_i(x) f_i(x)| \\ &\leqq \sum |g(x) - f_i(x)| \varphi_i(x) \\ &\leqq \sum (\lambda + \varepsilon)\varphi_i(x) = \lambda + \varepsilon \,. \end{aligned}$$

Since this holds for all $x \in X$, we have $\|g - f\| \leqq \lambda + \varepsilon$ and we have produced a function in $\mathscr{M}$ that approximates g within $\lambda + \varepsilon$. Hence, $\rho \leqq \lambda + \varepsilon$ for every ε, and $\rho = \lambda$.

4. Duality and $\mathscr{M}^{\perp}$. If M is a subspace of a normed linear space E, and $M^{\perp}$ denotes the collection of all linear functionals $L \in E^*$ that vanish on M, then the following duality relations hold: (See Buck, [5])

$$\begin{aligned} \inf_{m \in M} |u - m| &= \max_{\substack{L \in M^{\perp} \\ \|L\| \leqq 1}} |L(u)| \\ \min_{L \in M^{\perp}} \|L - L_0\| &= \sup_{\substack{m \in M \\ |m| \leqq 1}} |L_0(m)| \end{aligned} \tag{1}$$

for any $u \in E$ and $L_0 \in E^*$.

If we apply these to the conclusion of Theorem 2, then the strong Weierstrass theorem can be restated as follows:

$$\max_{\substack{\Phi \in \mathscr{M}^{\perp} \\ \|\Phi\| \leqq 1}} |\Phi(g)| = \sup_{x \in X} \max_{\substack{L \in M_x^{\perp} \\ \|L\| \leqq 1}} |L(g(x)) \,.$$

(Because of the cumbersome notation, we will omit the norm restrictions in future formulae; it is to be understood that all functionals will have norm at most 1, unless otherwise specified.)

In the duality relations (1), sharpened forms can be obtained by maximizing merely over the set of extreme points of the appropriate convex sets. Thus, one may write

$$\sup_{\Phi \in e(\mathscr{M}^{\perp})} |\Phi(g)| = \sup_{\substack{x \in X \\ L \in e(M_x^{\perp})}} |L(g(x))|$$

where we have used $e(S)$ to denote the set of extreme points of a convex set S. This new formulation of the strong Weierstrass theorem suggests very strongly the conjecture that the extreme functionals Φ that annihilate $\mathscr{M}$ are precisely the special functionals ψ of the form $\psi(g) = L(g(x))$, where $x \in X$ and $L \in e(M_x^\perp)$. The remainder of the paper presents a proof of this.

THEOREM 3. *Every functional ψ, defined on $C[X: E]$ by the choice of a point $x_0 \in X$ and an extreme functional L in the convex set $B^* \cap M_{x_0}^\perp$, is extreme in the set of those of norm 1 that lie in $\mathscr{M}^\perp$.*

The key to the proof lies in the following characterization theorem for extreme functionals. (See Buck [4], Phelps [11].)

THEOREM 4. *Let E be a normed linear space and M a subspace of E. Then, a functional $L \in B^* \cap M^\perp$ is extreme in this set if and only if $E = M + (1/k)H_L - (1/k)H_L$ for $k = 1, 2, 3, \cdots$.*

Here, the set H_L is the unbounded convex neighborhood of the origin in E defined for a real space E by the inequality

$$u \in H_L \quad \text{if and only if} \quad |u| - L(u) \leqq 1$$

and for a complex space by

$$u \in H_L \quad \text{if and only if} \quad |u| - \operatorname{Re} L(u) \leqq 1 .$$

To prove Theorem 3, we will use the characterization theorem in both the direct and converse forms. For simplicity, we treat only the real case.

In order to prove that a functional ψ determined by the pair $x_0 \in X$, $L \in e(M_{x_0})$ is extreme in $\mathscr{M}^\perp$, it is sufficient to show that $C[X: E] = \mathscr{M} + H_\psi - H_\psi$. Given an arbitrary function $F \in C[X: E]$, we must show that $F = f + g - h$, where $f \in \mathscr{M}$ and g and h lie in H_ψ.

Let us write M for M_{x_0} = closure of $\mathscr{M}(x_0)$ in E. Since L is extreme in $M^\perp$, the characterization theorem shows that $E = M + (1/8)H_L - (1/8)H_L$. Since $F(x_0)$ is a point in E, we can choose $m \in M$, and a, b in $(1/8)H_L$ so that

$$F(x_0) = m + a - b .$$

The set H_L is unbounded, so we may choose $c \in (1/8)H_L$ such that $|c| > 2\|F\| + |a|$. H_L is convex so that $a_0 = a + c$ and $b_0 = b + c$ both lie in $(1/4)H_L$, and

$$F(x_0) = m + a_0 - b_0$$

while $|a_0| \geqq |c| - |a| > ||F||$.

Since $m \in M$, the closure of $\mathscr{M}(x_0)$, we can choose a function $f \in \mathscr{M}$ such that $f(x_0) = m_0$, and $|m - m_0| < 1/8$. Choose an open set $\mathscr{O}$ about x_0 such that $|f(x) - f(x_0)| < 1/4$ and $|F(x) - F(x_0)| < 1/4$ for all $x \in \mathscr{O}$, and then a real valued function $\varphi \in C[X]$ such that $\varphi(x) = 0$ off $\mathscr{O}$, while for all x, $0 \leqq \varphi(x) \leqq 1 = \varphi(x_0)$.

Define three functions in $C[X: E]$ by

$$\begin{aligned} f_0(x) &= \varphi(x)f(x) \\ g(x) &= F(x) + (b_0 - f(x))\varphi(x) \\ h(x) &= b_0\varphi(x) . \end{aligned}$$

Since $\mathscr{M}$ is a $C[X]$ module, $f_0 \in \mathscr{M}$. Clearly, $F = f_0 + g - h$. All that remains, in the proof of Theorem 3, is to show that g and h lie in H_ψ.

For h, this is immediate; $||h|| = ||b_0|$, and $\psi(h) = L(h(x_0)) = L(b_0)$, so that

$$||h|| - \psi(h) = |b_0| - L(b_0) \leqq \frac{1}{4} < 1 ,$$

where we have used the fact that b_0 is in $(1/4)H_L$.

If x lies outside $\mathscr{O}$, then

$$|g(x)| = |F(x)| \leqq ||F|| < |a_0| .$$

If x lies in $\mathscr{O}$, then

$$\begin{aligned} |g(x)| &= |F(x) - F(x_0) + F(x_0) + (b_0 - f(x))\varphi(x)| \\ &\leqq |F(x) - F(x_0)| + |F(x_0) + (b_0 - f(x_0))\varphi(x)| \\ &\quad + |(f(x_0) - f(x))\varphi(x)| \\ &\leqq \frac{1}{4} + |F(x_0) + (b_0 - m)\varphi(x) + (m - m_0)\varphi(x)| + \frac{1}{4} \\ &\leqq \frac{1}{2} + |F(x_0) + (a_0 - F(x_0))\varphi(x)| + |m - m_0| \\ &\leqq \frac{5}{8} + |F(x_0)|(1 - \varphi(x)) + |a_0|\varphi(x) \\ &\leqq \frac{5}{8} + |a_0|(1 - \varphi(x)) + |a_0|\varphi(x) \\ &\leqq \frac{5}{8} + |a_0| . \end{aligned}$$

Accordingly, we have $||g|| \leqq |a_0| + 5/8$. Now, $\psi(g) = L(g(x_0)) = L(F(x_0) + b_0 - m_0) = L(a_0) + L(m - m_0)$ and

$$\|g\| - \psi(g) \leqq |a_0| - L(a_0) - L(m - m_0) + \frac{5}{8}$$
$$\leqq \frac{1}{4} + \frac{1}{8} + \frac{5}{8} = 1$$

and $g \in H_\psi$. This completes the proof of Theorem 3.

When the module $\mathscr{M}$ consists merely of 0, this theorem identifies extreme functionals in the unit ball of the dual space of $C[X: E]$; here the result is not new. (See Singer [15] and Brosowski and Deutsch). [2]. The present result was initiated by a proof of this special case by Strobele which was based upon the use of the Characterization Theorem (Theorem 4), and which was simplified by suggestions from Overdeck [16], [10].

5. **The converse.** To complete our results, we prove that every extreme functional in $\mathscr{M}^\perp$ is among the functions ψ discussed in Theorem 3. The method depends upon the embedding of $C[X: E]$ in $C[Z]$ with $Z = X \times B^*$ that was discussed in § 2. We show that any functional Φ that is extreme in $\mathscr{M}^\perp$ has an extension to $C[Z]$ that is extreme in a certain convex set associated with $\mathscr{M}$, and then that it can be identified as one of the functionals of the form ψ.

First, recall that any function $f \in C[X: E]$ corresponds to a function f^* in $C[Z]$ defined by $f^*(x, L) = L(f(x))$. This is an isometric embedding of $C[X: E]$ as a subspace $\mathscr{C}$ of $C[Z]$. The image functions f^* are quite special, among all functions $F \in C[Z]$. For example, a general function F may have arbitrary values at two distinct points (x, L) of Z, but if $F = f^*$, then for example, $F(x, (1/2)L_1 + (1/2)L_2) = (1/2)F(x, L_1) + (1/2)F(x, L_2)$. Also, $F(x, 0) = 0$ for all x. Thus $\mathscr{C}$ is a very special subspace.

If $\mathscr{M}$ is an arbitrary closed submodule of $C[X: E]$, then under the canonical embedding, it is mapped onto a subspace of the special subspace $\mathscr{C}$. Our first result characterizes this image, which for convenience we continue to denote by $\mathscr{M}$.

DEFINITION 2. Given $\mathscr{M}$, let Λ be the subset of $Z = X \times B^*$ consisting of all (x, L) such that $x \in X$ and $L \in M_x^\perp$.

It is easily seen that Λ is compact.

DEFINITION 3. Given any subset $G \subset Z$, let $\mathscr{N}(G)$ be the space of functions $F \in C[Z]$ that vanish on G.

THEOREM 5. $\mathscr{M} = \mathscr{C} \cap \mathscr{N}(\Lambda)$. *Explicitly, a function f^* arises from a function $f \in M$ if and only if f^* vanishes on Λ.*

Proof. Suppose that $f \in \mathscr{M}$. Then, for any $(x, L) \in \Lambda$, $f^*(x, L) = 0$, since $f(x) \in M_x$ and $L \in M_x^{\perp}$. Thus, $f^* \in \mathscr{N}(\Lambda)$. Conversely, let $f \in C[X: E]$ and suppose that $f^* \in \mathscr{N}(\Lambda)$. If there were a point $x \in X$ such that $f(x) \notin M_x$, then there would be a functional $L \in B^*$ such that L vanishes on M_x but $Lf(x) \neq 0$. But, this would imply that $f^*(x, L) \neq 0$ for some $(x, L) \in \Lambda$. Thus, we must have $f(x) \in M_x$ for all x. By the Weierstrass theorem for modules (Theorem 1), $f \in \mathscr{M}$.

The next result is the key to proving the converse of Theorem 3.

THEOREM 6. *Any functional ϕ on $\mathscr{C}$ that vanishes on $\mathscr{M}$ has a norm-preserving extension to a functional ψ on $C[Z]$ that vanishes on $\mathscr{N}(\Lambda)$.*

Proof. Any $p = (x, L)$ in $Z = X \times B^*$ defines a point functional ψ_p on $C[Z]$ with $\psi_p(F) = F(x, L)$. If (x, L) is restricted to the set Λ, then the functionals ψ_p belong to $\mathscr{N}(\Lambda)^{\perp}$, and in fact are precisely the extreme functionals among those of norm 1. Their convex hull is therefore exactly $\mathscr{N}(\Lambda)^{\perp}$. Hence, for any $F_0 \in C[Z]$, by the duality relation,

$$\sup_{p \in \Lambda} |\psi_p(F_0)| = \inf_{F \in \mathscr{N}(\Lambda)} \|F_0 - F\| .$$

If this is applied to the choice $F_0 = g^*$ for any $g \in C[X: E]$, we have

$$\sup_{\substack{x \in X \\ L \in M_x^{\perp}}} |Lg(x)| = \inf_{F \in \mathscr{N}(\Lambda)} \|g^* - F\| .$$

However, by the strong Weierstrass theorem (Theorem 2), this in turn is equivalent to

$$\inf_{f \in \mathscr{M}} \|g - f\| = \inf_{F \in \mathscr{N}(\Lambda)} \|g^* - F\|$$

(interesting in itself), as well as the result we need,

$$(2) \qquad \sup_{\Phi \in \mathscr{M}^{\perp}} |\Phi(g)| = \inf_{F \in \mathscr{N}(\Lambda)} \|g^* - F\| .$$

Let Φ be any functional on $\mathscr{C}$ that vanishes on $\mathscr{M}$. We may assume that Φ is of norm 1. Then, (2) shows that

$$|\Phi(g^*)| \leqq \|g^* + F\|$$

for every $g^* \in \mathscr{C}$ and $F \in \mathscr{N}(\Lambda)$. This therefore permits us to extend Φ from $\mathscr{C}$ to $\mathscr{C} + \mathscr{N}$ by defining $\Phi_0(g^* + F) = \Phi(g^*)$, without increasing the norm. Note that the extension now vanishes on $\mathscr{N}$. We now extend Φ_0 to a functional ψ on $C[Z]$ in the usual way, obtaining the desired result.

The next result is an elementary observation about extensions of functionals.

LEMMA. *Let E be a subspace of the normed linear space F, and let M and N be subspaces of F with $M = E \cap N$. Suppose that every functional Φ on E that is zero on M has a norm preserving extension to a functional ψ on F that vanishes on N. Then, every functional Φ that is extremal in $M^{\perp}$ has an extension ψ that is extremal in $N^{\perp}$.*

Proof. Given $\Phi \in E^*$, extremal among those of norm 1 that vanish on M, let S be the set of all $\psi \in F^*$ that extend Φ and vanish on N. Choose ψ_0 to be extremal in S. Then, it is easily seen that ψ_0 is extremal in $N^{\perp}$.

In passing, we note that the spaces E, F, M, N have this extension property if and only if

$$\inf_{m \in M} |u - m| = \inf_{n \in N} |u - n|$$

for every $u \in E$; this is something like a generalized orthogonality relationship for the subspaces E and N in the space F. [1] The proof of this is essentially that of Theorem 6.

We are now ready to prove the converse of Theorem 3.

THEOREM 7. *Every functional Φ on $C[X{:}\,E]$ that is extremal in the set of those of norm 1 that vanish on a closed submodule $\mathscr{M}$ is given by $\Phi = \psi_{(x_0, L_0)}$ where $x_0 \in X$ and $L_0 \in e(M_{x_0}^{\perp})$.*

Proof. Applying Theorem 6 and the lemma, Φ can be extended to a functional ψ that is extreme in $\mathscr{N}(\Lambda)^{\perp}$. Hence, $\psi = \psi_p$ for some $(x_0, L_0) = p \in \Lambda$. We must now prove that L_0 is extreme in $M_{x_0}^{\perp}$. Suppose that $L_0 = (L_1 + L_2)/2$ where $\| L_i \| \leqq 1$ and $L_i = 0$ on M_{x_0}. Because of the special nature of functions $g^* \in \mathscr{C}$, $\psi_p(g^*) = L_0 g(x_0) = (1/2)\{L_1 g(x_0) + L_2 g(x_0)\}$, so that $\psi_p = (1/2)(\psi_1 + \psi_2)$, where $\psi_i \in \mathscr{N}(\Lambda)^{\perp}$. Since ψ_p was extreme, $\psi_1 = \psi_2$, from which we find $L_1 = L_2$, and L_0 is extreme in $M_{x_0}^{\perp}$.

REFERENCES

1. G. Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. J., **1** (1935), 169-172.
2. B. Brosowski and F. Deutsch, *On some geometric properties of sums*, J. Approx. Theory, **10** (1974), 245-267.
3. R. C. Buck, *Bounded functions on a locally compact space*, Mich. Math. J., **5** (1958), 95-104.
4. ———, *A complete characterization for extreme functionals*, Bull. Amer. Math. Soc., **65** (1959), 130-133.
5. ———, *Applications of duality in approximation theory*, in "Approximation of

Functions", Garabedian (ed.) Elsevier, Amsterdam, (1965), 27-42.
6. M. Jerison, *The space of bounded maps into a Banach space*, Ann. of Math., **52** (1950), 309-327.
7. I. Kaplansky, *The structure of certain operator algebras*, Trans. Amer. Math. Soc., **70** (1951), 219-225.
8. J. T. Marti, *The approximation of continuous nonlinear operators in normed spaces by polynomial operators*, J. Appl. Math. Phy., **22** (1971), 991-996.
9. L. Nachbin, *Elements of Approximation Theory*, Van Nostrand, 1967.
10. J. Overdeck, private communication, (1972).
11. R. R. Phelps, *Extreme points of polar convex sets*, Proc. Amer. Math. Soc., **12** (1961), 291-296.
12. P. M. Prenter, *A Weierstrass theorem for real separable Hilbert space*, J. Approx. Theory, **3** (1970), 341-351.
13. J. B. Prolla, *Bishop's generalized Stone-Weierstrass theorem for weighted spaces*, to appear.
14. I. Singer and C. Foias, *Some remarks on the representation of linear operators in spaces of vector valued continuous functions*, Rev. Math. Pures Appl., **5** (1960), 729-752.
15. I. Singer, *Best Approximation in Normed Vector Spaces*, (Roum.), Bucharest, 1967. (Springer-Verlag, 1970).
16. W. Strobele, *The representation of the extreme functionals on* $C_0(T, X)$, J. Approx. Theory, **10** (1974), 64-68.
17. C. Todd, *Stone-Weierstrass theorems for the strict topology*, Proc. Amer. Math. Soc., **16** (1965), 654-659.
18. D. H. Tucker, *A representation theorem for a continuous linear transformation on a space of continuous functions*, Proc. Amer. Math. Soc., **16** (1965), 946-953.
19. D. J. Uherka, *General Stieltjes integrals, and a strong representation theorem for continuous linear maps on a function space*, Math. Ann., **182** (1969), 60-66.
20. J. Wells, *Bounded continuous vector valued functions on a locally compact space*, Mich. Math. J., **12** (1965), 119-126.
21. B. Yood, *Banach algebras of bounded functions*, Duke Math. J., **16** (1949), 151-163.

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