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CHARACTERISTIC IDEALS IN GROUP ALGEBRAS

I. SINHA

If $\mathfrak{F}G$ is the group-algebra of a group G over a field $\mathfrak{F}$, and $\mathfrak{A}$ is any subgroup of the automorphism group of the $\mathfrak{F}$-algebra $\mathfrak{F}G$, then an ideal I of $\mathfrak{F}G$, is called $\mathfrak{A}$-characteristic if $I^\alpha \subseteqq I$, $\forall^\alpha \in A$. If A is the whole automorphism group itself, then we merely say that I is characteristic. Then D.S. Passman has proved the following result:

"Let $H \trianglelefteq G$ such that G/H is $\mathfrak{F}$-complete. Then for each characteristic ideal I of $\mathfrak{F}G$, $I = (I \cap \mathfrak{F}H)\mathfrak{F}G$."

The main concern in this paper is to consider the converse of this result.

2. Some preliminaries. For a given ideal $I \trianglelefteq \mathfrak{F}G$, let $\mathscr{R}(I)$ be the set of all $H \leqq G$ such that $I = (I \cap \mathfrak{F}H)\mathfrak{F}G$. Let $C(I)$ be the set of all H in G such that if for some right $\mathfrak{F}H$-module $\mathfrak{M}$, $I \cap \mathfrak{F}H \subseteqq \operatorname{Ann}\mathfrak{M}$, then $I \subseteqq \operatorname{Ann}\mathfrak{M}^G$, the induced $\mathfrak{F}G$-module. We first of all have:

THEOREM 1. (i) *For any* $I \trianglelefteq \mathfrak{F}G$, $C(I) \subseteqq \mathscr{R}(I)$.
(ii) *If* $H \trianglelefteq G$, *then* $H \in \mathscr{R}(I)$ *if and only if* $H \in C(I)$.

Proof. (i) Let $I \cap \mathfrak{F}H \subseteqq \operatorname{Ann}\mathfrak{M}$ imply that $I \subseteqq \operatorname{Ann}\mathfrak{M}^G$. Let $\sum p_i x_i \in I$ with $p_i \in \mathfrak{F}H$, where $G = \cup Hx_i$ is a coset-decomposition. We have $(\sum \mathfrak{M} \otimes x_i)(\sum p_i x_i) = 0$ if $I \cap \mathfrak{F}H \subseteqq \operatorname{Ann}\mathfrak{M}$. In particular $(m \otimes I)(\sum p_i x_i) = 0$, $\forall m \in \mathfrak{M}$, i.e., $\sum m p_i \otimes x_i = 0$, $\forall m \in \mathfrak{M}$. So $\mathfrak{M}\cdot p_i = 0$ for each i. Thus $p_i \in \operatorname{Ann}\mathfrak{M}$. Since $\mathfrak{M}$ is arbitrary with the property that $I \cap \mathfrak{F}H \subseteqq \operatorname{Ann}\mathfrak{M}$, so we may take $\mathfrak{M} = \mathfrak{F}H/I \cap \mathfrak{F}H$, and conclude that each $p_i \in \operatorname{Ann}\mathfrak{M} = I \cap \mathfrak{F}H$. Thus $\sum p_i x_i \in (I \cap \mathfrak{F}H)\mathfrak{F}G$.

(ii) Suppose $I = \mathfrak{F}G(I \cap \mathfrak{F}H)$ and $I \cap \mathfrak{F}H \subseteqq \operatorname{Ann}\mathfrak{M}$, for some $\mathfrak{F}H$-module $\mathfrak{M}$. Note that $H \trianglelefteq G$ implies that $\mathfrak{F}G(I \cap \mathfrak{F}H) = (I \cap \mathfrak{F}H)\mathfrak{F}G$. Let $a = \sum x_i p_i \in I$ where $p_i \in I \cap \mathfrak{F}H$. So $a\mathfrak{M}^G = (\sum x_i p_i)(\sum x_j \otimes \mathfrak{M}) = \sum x_i x_j \otimes p_i^{x_j}\mathfrak{M} = 0$ since $p_i^{x_j} \in I \cap \mathfrak{F}H \subseteqq \operatorname{Ann}\mathfrak{M}$. Thus $a\ \mathfrak{M}^G = 0$ and $I \subseteqq \operatorname{Ann}\mathfrak{M}^G$.

Theorem 17.4 of [1] then gives us:

COROLLARY 1. *Let* $H \trianglelefteq G$ *such that* G/H *is* $\mathfrak{F}$*-complete. Then* $H \in C(I)$ *for every characteristic ideal* I *of* $\mathfrak{F}G$.

Also Theorem 17. 7 of [1] implies:

COROLLARY 2. *If* $H \trianglelefteq G \ni G/H$ *is abelian and has no elements of order* $p = \mathrm{Char.}\ \mathscr{F}$, *then* $H \in C(J(G))$, *where* J *denotes the*

Jacobson-radical of $\mathfrak{F}G$.

3. **Main result.** We will prove:

THEOREM 2. *For* $I = [\mathfrak{F}G, \mathfrak{F}G]$, *the commutator ideal and for* $J = J(G)$, *if* $H \leqq G$ *such that* $H \in \mathscr{R}(I)$ *and* $H \in \mathscr{R}(J)$ *then* $H \trianglelefteq G$, G/H *is abelian with no elements of order* p. *In particular,* $\mathfrak{F}(G/H)$ *is semi-simple.*

Further, if $\mathfrak{F}$ *is algebraically closed then* G/H *is* $\mathfrak{F}$*-complete.*

We observe that the last two statements in the theorem follows from 17.8 and 17.1 (i) respectively of [1]. The rest of the theorem will be proved by a series of results proved below.

LEMMA 1. *Let* $H \leqq G$, $I \trianglelefteqq \mathfrak{F}G$ *and* $H \in \mathscr{R}(I)$. *Then* $H \supseteqq \mathfrak{A}^{-1}(I) = \{g \in G \mid g - 1 \in I\}$.

Proof. Let $G = \cup Hx_i$ be a coset-decomposition, and $g \in \mathfrak{A}^{-1}(I)$ such that $g \notin H$. Then $g = hx_i$ for some i, where $x_i \neq 1$, and $h \in H$; and $hx_i - 1 \in (I \cap \mathfrak{F}H)\mathfrak{F}G = \sum (I \cap \mathfrak{F}H)x_i$. Since $\{x_i\}$ are linearly independent over $\mathfrak{F}H$, $h \in I \cap \mathfrak{F}H$, and $x_i \neq 1$, so $g \in I$ which implies that $1 \in I$, a contradiction.

LEMMA 2. *If* $I = [\mathfrak{F}G, \mathfrak{F}G]$, *and* $H \in \mathscr{R}(I)$ *then* $H \trianglelefteq G$ *and* G/H *is abelian.*

Proof. Observe that I is a proper ideal in $\mathfrak{F}G$, since $\mathfrak{A}(I) = 0$. Also by Lemma 1, $H \supseteqq \mathfrak{A}^{-1}(I)$. Since $(ghg^{-1}h^{-1} - 1)hg = gh - hg \in I$, for all $g, h \in G$, so $(ghg^{-1}h^{-1} - 1) \in I$. Hence $ghg^{-1}h^{-1} \in \mathfrak{A}^{-1}(I) \subseteqq H$; i.e., G', the commutator-subgroup is in H. Hence $H \trianglelefteq G$ and G/H is abelian.

Now let H satisfy the hypothesis of Lemma 2. Then we have:

LEMMA 3. *Let* $I = J(G)$ *and* $H \in \mathscr{R}(I)$. *Then* $\mathfrak{F}(G/H)$ *is semi-simple and* G/H *has no elements of order* $p =$ Char. $\mathfrak{F}$.

Proof. $J(G) = (J(G) \cap \mathfrak{F}H)\mathfrak{F}G \subseteqq J(H)\cdot\mathfrak{F}G$ by 16.9 of [1]. Now $\mathfrak{F}H[\mathfrak{A}_H(H) \cong \mathfrak{F}$ where $\mathfrak{A}_H(H)$ is the ideal of $\mathfrak{F}H$, generated by $\{h - 1 \mid h \in H\}$. So $\mathfrak{A}_H(H) \supseteqq J(H)$. Hence $\mathfrak{A}_H(H)\mathfrak{F}G = \mathfrak{A}_G(H) \supseteqq J(H)\cdot\mathfrak{F}G \supseteqq J(G)$, where $\mathfrak{A}_G(H)$ is the ideal in $\mathfrak{F}G$, generated by $\{h - 1 \mid h \in H\}$. Now $\mathfrak{A}_G(H)$ is the kernel of the natural map of $\mathfrak{F}G$ onto $\mathfrak{F}(G/H)$; {see for example proof of Theorem 1 in [2]}. Thus $\mathfrak{F}(G/H) \cong \mathfrak{F}G/\mathfrak{A}_G(H)$ is semi-simple. Since G/H is abelian by Lemma

2, so it is clear that it has no elements of order p, as $\mathfrak{F}(G/H)$ is semi-simple.

This also completes the proof of Theorem 2.

REFERENCES

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MICHIGAN STATE UNIVERSITY