GROUPS OF MATRICES ACTING ON
DISTRIBUTION SPACES

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Let E be a locally convex space of temperate distributions on the n-dimensional Euclidean space R^n, and G a closed subgroup of $Gl(n, R)$, the general linear group over R^n. An attempt is made to identify those distributions which can be approximated in E by linear combinations of distributions of the form $u(Ax + b)$, where u is a fixed element of E, A varies over G, and b varies over R^n. A cancellation theorem is proved; this then allows the support of the Fourier transform of any annihilator of the set of distributions of the form $u(Ax + b)$ to be localized. This in turn is used to obtain approximation results.

1. **Notation.** Throughout, R^n denotes n-dimensional Euclidean space. The character group of R^n is again R^n, the identification being made in such a way that multiplicative factors in the Fourier inversion formula are eliminated. The Haar measure on R^n is denoted by dx.

We denote by $C_c^\infty(U)$ the space of indefinitely differentiable functions on R^n which have compact support inside the open set U in R^n. $S(R^n)$ is the space of rapidly decreasing indefinitely differentiable functions on R^n. The space of all Schwartz distributions on R^n is designated by $D'(R^n)$, and its subspace consisting of temperate distributions is denoted by $S'(R^n)$.

$Gl(n, R)$ is the general linear group over R^n. The determinant of an element A in $Gl(n, R)$ is written $\det A$, and A' denotes the adjoint matrix of A.

Now, consider a fixed element A in $Gl(n, R)$. Then it is easy to see that the function

$$x \longrightarrow \phi(Ax) \quad (x \in R^n)$$

belongs to $C_c^\infty(R^n)$ whenever ϕ does. We write ϕ^A for this function. This definition is extended to include all distributions by making use of the adjoint of the map which carries ϕ onto ϕ^A. More precisely, if u is a distribution, then we define u^A to be the unique distribution which satisfies

$$\langle \phi, u^A \rangle = |\det A^{-1}| \langle \phi^{A^{-1}}, u \rangle \qquad (\phi \in C_c^\infty(R^n)) .$$

The translate of a distribution u by an element b in R^n is defined in the usual fashion, and denoted by u_b. We write u_b^A for $(u_b)^A$.

These notational conventions entail that when u is a function, then u_b^A is the function defined by

$$u_b^A(x) = u(Ax + b) \quad (x \in R^n) \,.$$

Let G be a closed subgroup of $\mathrm{Gl}\,(n, R)$. We write G' for its adjoint group. G will always be equipped with the topology induced by $\mathrm{Gl}\,(n, R)$. With this topology, G is a Lie group. We write dA for left Haar measure on G. If V is an open set in G, then $C_c^\infty(V)$ denotes the set of all smooth functions on G having compact support in V.

Finally, again consider a closed subgroup G of $\mathrm{Gl}\,(n, R)$. If E is a locally convex space of temperate distributions which is invariant under all mappings of the form $u \to u^A$, where A is a fixed element of G, we say that E is G-invariant. Similarly, if E is invariant under all maps of the form $u \to u_b$, where b belongs to R^n, then we say that E is translation invariant. Now, suppose that E is both translation invariant and G-invariant, and that u is an element of E. We denote by $T_c[u]$ the closed linear subspace of E spanned by the set of all distributions of the form u_b^A, with A in G and b in R^n.

2. **Preliminaries.** We shall be concerned with the problem of characterizing $T_c[u]$; that is, given two distributions u and w in E, when can we say that w belongs to $T_c[u]$? Without some restrictions on E, the problem is a rather intractable one. However, quite mild assumptions about E enable us to obtain a great deal of information about $T_c[u]$. These assumptions are mild in the sense that they allow us to include most of the classical distribution spaces in our results. In essence, the assumptions are that the topology on E is barrelled, and that the action of the topological dual of E can be expressed as a "convolution". We examine this situation below.

DEFINITION 2.1. (cf. [1]). A locally convex space of temperate distributions is *admissible* if
(i) $S(R^n)$ is a dense subspace of E,
(ii) The inclusion maps $S(R^n) \to E \to S'(R^n)$ are continuous.

The topological dual space of an admissible space can be identified with a subspace of $S'(R^n)$ which contains $S(R^n)$. We shall always make this identification in such a way that the relations

$$\langle \phi, v \rangle = \phi * v(0) \qquad (\phi \in S(R^n),\ v \in E')$$
$$\langle u, \phi \rangle = u * \phi(0) \qquad (u \in E,\ \phi \in S(R^n))$$

hold.

DEFINITION 2.2. Let G be a closed subgroup of $\mathrm{Gl}(n, R)$. If E is a G-invariant and translation invariant admissible space, we say that E is a *G-space* if it has the following properties.

(i) For each u in E, the map $x \rightarrow u_x$ is continuous from R^n into E for the weak topology on E.

(ii) For each x in R^n, the map $u \rightarrow u_x$ is weakly continuous from E into E.

(iii) For each u in E, the map $A \rightarrow u^A$ is continuous from G into E for the weak topology on E.

(iv) For each A in G, the map $u \rightarrow u^A$ is weakly continuous from E into E.

(v) For each u in E and each v in E', the continuous (by (i)) function $x \rightarrow \langle u_x, v \rangle$ is a temperate distribution on R^n.

We shall find it convenient to denote by $u \bar{*} v$ the function in (v) above.

The next result lists some of the basic properties of G-spaces. These properties will be used repeatedly and without further reference in what follows.

PROPOSITION 2.3. *Let E be a G-space, where G is a closed subgroup of $\mathrm{Gl}(n, R)$. Then*

(a) *E' is translation invariant, and for each b in R^n*

$$u \bar{*} v_b = u_b \bar{*} v = (u \bar{*} v)_b \qquad (u \in E,\ v \in E'),$$

(b) *E' is G-invariant, and for each A in G*

$$u \bar{*} v^A = |\det A^{-1}| (u^{A^{-1}} \bar{*} v)^A \qquad (u \in E,\ v \in E').$$

If E is barrelled, then we have also

(c) *E' is a module over $S(R^n)$ with respect to convolution, and for each $\phi \in S(R^n)$*

$$u \bar{*} (v * \phi) = (u \bar{*} v) * \phi \qquad (u \in E,\ v \in E').$$

Properties (a) and (b) reflect the fact that the maps $u \rightarrow u_b$ and $u \rightarrow u^A$ of E into E, being weakly continuous, have adjoints which are weakly continuous from E' into E'. Property (c) derives from the fact that, if E is barrelled, the map $u \rightarrow u \bar{*} v$ is continuous from E into $S'(R^n)$, for each fixed v in E'. This may be verified by making use of the Closed Graph theorem; see [2].

Most of the classical spaces of temperate distributions are $\mathrm{Gl}(n, R)$-spaces, and hence G-spaces for every closed subgroup G of $\mathrm{Gl}(n, R)$. The following result identifies a large class of G-spaces. However, since we shall not need this result in what follows, we omit its proof.

PROPOSITION 2.4. *Let G be a closed subgroup of $\mathrm{Gl}\,(n, R)$ and E a G-invariant and translation invariant admissible space. Suppose that E is barrelled, B_r-complete and a module over $S(R^n)$ with respect to convolution. Then E is a G-space if and only if the following conditions hold.*

   (i) *For each u in E, the map $x \to u_x$ of R^n into E is bounded on compact subsets of R^n.*

   (ii) *For each u in E, the map $A \to u^A$ of G into E is bounded on compact subsets of G.*

3. **The function $q\nabla F$.** In this section, we construct a certain function, which will serve as the basic tool in proving our approximation results. Before we carry out this construction, it is convenient to introduce some additional notation.

Suppose that H is a closed subgroup of $\mathrm{Gl}\,(n, R)$, of Lie group dimension $m \geqq n$. Let $\theta\colon (t_1, \cdots, t_m) \to \theta(t_1, \cdots, t_m)$ be a diffeomorphism of a neighborhood of the origin in R^n onto a neighborhood of the identity in H, and such that $\theta(0, \cdots, 0)$ is the identity matrix. We write $\theta_{jk}(t_1, \cdots, t_m)$ for the (j, k)th matrix element of $\theta(t_1, \cdots, t_m)$. Let 1 be the set of all subsets of $\{1, \cdots, m\}$ which have cardinality n. Suppose that $i = \{i_1, \cdots, i_n\}$ is an element of I (we adopt this notation because we shall be using I as an index set). We define a polynomial P_i on R^n by setting $P_i(x) = P(x_1, \cdots, x_n)$ to be the determinant which has, as its (j, k)th entry, the expression

$$x_1\frac{\partial \theta_{j1}}{\partial t_{i_k}} + x_2\frac{\partial \theta_{j2}}{\partial t_{i_k}} + \cdots + x_n\frac{\partial \theta_{jn}}{\partial t_{i_k}}$$

evaluated at $t_1 = t_2 = \cdots = t_m = 0$. Then each P_i is a homogeneous polynomial of degree n. We write $Z(H)$ for the set of common zeros of these polynomials; that is,

$$Z(H) = \bigcap_{i \in I} \{x\colon P_i(x) = 0\}\;.$$

We can now turn our attention to the promised construction. Let H be a closed subgroup of $\mathrm{Gl}\,(n, R)$. Consider a distribution q in $D'(R^n)$ and a function F in $C_c^\infty(H)$. If ϕ is in $C_c^\infty(R^n)$, then the mapping

$$A \longrightarrow \langle q^A, \phi \rangle \qquad (A \in G)$$

defines a continuous function on H. It then follows that we can define a linear functional

$$\phi \longrightarrow \int_H \langle q^A, \phi \rangle F(A)dA$$

on $C_c^\infty(R^n)$. We claim that this functional is continuous, and hence a distribution on R^n. For if (ϕ_i) converges to ϕ in $C_c^\infty(R^n)$, then it does so uniformly on equicontinuous subsets of $D'(R^n)$. Now, the set $\{q^A : A \in \operatorname{supp} F\}$ is the image in $D'(R^n)$ of the compact subset $\operatorname{supp} F$ of H under the map A to q^A. Since this map is continuous for the weak topology on $D'(R^n)$, $\{q^A : A \in \operatorname{supp} F\}$ is weakly compact as a subset of $D'(R^n)$. It is thus an equicontinuous subset of $D'(R^n)$ (because $C_c^\infty(R^n)$ is barrelled) and so

$$\lim_i \langle q^A, \phi_i \rangle = \langle q^A, \phi \rangle$$

uniformly for A in $\operatorname{supp} F$. Form this, it follows that

$$\lim_i \int_H \langle q^A, \phi_i \rangle F(A) dA = \int_H \langle q^A, \phi \rangle F(A) dA$$

which proves our assertion.

In view of the above, we may now define a distribution $q \nabla F$ on R^n by setting

$$\langle \phi, q \nabla F \rangle = \int_H \langle q^A, \phi \rangle F(A) dA \qquad (\phi \in C_c^\infty(R^n)) .$$

Using a similar argument, with $S(R^n)$ in place of $C_c^\infty(R^n)$, it can be shown that $q \nabla F$ is a temperate distribution whenever q is.

We have the following result.

LEMMA 3.1. *Suppose that H has dimension $m \geq n$. Then $q \nabla F$ is an indefinitely differentiable function on $R^n \backslash Z(H)$ for every q in $D'(R^n)$ and F in $C_c^\infty(H)$. Moreover, for each multi-index α*

$$D^\alpha(q \nabla F) = \sum_{\|\beta\| \leq |\alpha|} (D^\beta q) \nabla (G_\beta^\alpha F)$$

where each G_β^α is a smooth function on H depending only on α and β.

Proof. The proof proceeds in several stages.

First, we make the observation that if f is a continuous function on R^n, then for each ϕ in $C_c^\infty(R^n)$, and each F in $C_c^\infty(H)$

$$\begin{aligned} \langle \phi, f \nabla F \rangle &= \int_H \langle f^A, \phi \rangle F(A) dA \\ &= \int_H \left\{ \int_{R^n} f(Ax) \phi(-x) dx \right\} F(A) dA \\ &= \int_{R^n} \phi(-x) \left\{ \int_H f(Ax) F(A) dA \right\} dx \end{aligned}$$

from which it follows that $f \nabla F$ is the function

(3.1) $$f \nabla F(x) = \int_H f(Ax)F(A)dA \ .$$

It is easy to see that this function is continuous on R^n.

We now show that if x is not in $Z(H)$ then there exists a neighborhood U_x of x in R^n and a neighborhood V_x of the identity in H such that, for every continuous function f on R^n and every F in $C_c^\infty(V_x)$, $f \nabla F$ is indefinitely differentiable in U_x. To do this, choose a diffeomorphism θ of a neighborhood of the origin R^m onto a neighborhood W of the identity in H, as in the opening paragraphs of this section. If W is chosen to be sufficiently small, then there exists an indefinitely differentiable function h on R^m such that

$$\int_H F(A)dA = \int_{R^m} F(\theta(t))h(t)dt$$

for all F in $C_c^\infty(W)$; see [3]. In particular, (3.1) becomes

(3.2) $$f \nabla F(x) = \int_{R^m} f(\theta(t)x)F(\theta(t))h(t)dt$$

for every continuous f on R^n and every F in $C_c^\infty(W)$. Now, in view of the choice of x, there is an i in I such that $P_i(x) \neq 0$. We may, without loss of generality, assume that $i = \{1, 2, \cdots, n\}$. Consider now the map

$$(t_1, \cdots, t_m, y_1, \cdots, y_n) \longrightarrow (s_1, \cdots, s_m, z_1, \cdots, z_n)$$

of R^{m+n} into R^{m+n} which is defined by

$$\begin{aligned}
s_k &= \theta_{k_1}(t_1, \cdots, t_m)y_1 + \cdots + \theta_{k_n}(t_1, \cdots, t_m)y_n \quad &(1 \leq k \leq n) \\
s_k &= t_k &(n+1 \leq k \leq m) \\
z_k &= y_k &(1 \leq k \leq n) \ .
\end{aligned}$$

It is easy to verify that the Jacobian of this transformation, evaluated at $t_1 = \cdots = t_m = 0$, $y_1 = x_1, \cdots, y_n = x_n$, is just $P_i(x)$, which is nonzero. Thus, there exists a neighborhood U_x of x in R^n and a neighborhood V_x of the identity in H, such that the Jacobian of the transformation (3.3) does not vanish on $U_x \times \theta^{-1}(V_x)$. The Inverse Function theorem now tells us that the transformation, when restricted to $U_x \times \theta^{-1}(V_x)$, has an inverse

$$(s_1, \cdots, s_m, z_1, \cdots, z_n) \longrightarrow (t_1, \cdots, t_m, y_1, \cdots, y_n)$$

which clearly has the form

$$\begin{aligned}
t_k &= g_k(s_1, \cdots, s_m, z_1, \cdots, z_n) \quad &(1 \leq k \leq m) \\
y_k &= z_k &(1 \leq k \leq n)
\end{aligned}$$

where $g_1, \cdots, g_m$, are indefinitely differentiable functions on R^{m+n}. We write this briefly as

(3.4)
$$t = g(s, z)$$
$$y = z \,.$$

We assume that V_x is contained in W. If this is the case, then, in view of (3.2),

(3.5) $$\int_{R^n} \phi(y) f \nabla F(y) dy = \int_{R^{n+m}} \phi(y) f(\theta(t) y) F(\theta(t)) h(t) dt dy$$

for all ϕ in $C_c^\infty(U_x)$, F in $C_c^\infty(V_x)$, and each continuous function f on R^n. If we now make the change of variables defined by (3.4) in the integral on the right of (3.5), we find that

(3.6)
$$\int_{R^n} \phi(y) f \nabla F(y) dy = \int_{R^{n+m}} \phi(z) f'(s) F''(s, z) h'(s, z) J(s, z) ds dz$$
$$= \int_{R^{n+m}} \phi(y) f'(s) F''(s, y) h'(s, y) J(s, y) ds dy$$

where the functions f', F'', and h' are defined by

$$f'(s) = f(s_1, \cdots, s_n)$$
$$F''(s, z) = F(\theta(g(s, z)))$$
$$h'(s, z) = h(g(s, z))$$

and J is the Jacobian of the transformation (3.4). From (3.6), we deduce that

$$f \nabla F(y) = \int_{R^m} f'(s) F''(s, y) h'(s, y) J(s, y) ds$$

for all y in U_x. Since F'', h', and J are indefinitely differentiable on R^{m+n}, the indefinite differentiability of $f \nabla F$ in U_x is now evident.

We now go a step further, and show that $f \nabla F$ is indefinitely differentiable everywhere in $R^n \backslash Z(H)$ for all F in $C_c^\infty(H)$ and all continuous functions f on R^n. First, we make the observation that

(3.6) $$P_i(x) = \det A^{-1} \cdot P_i(Ax)$$

for all x in R^n, A in H, and i in I. To see this, assume, without loss of generality, that $i = \{1, 2, \cdots, n\}$. Temporarily denote by T the map (3.3) of R^{m+n} into itself. Then T is composed of the maps

(3.7) $$(t, y) \longrightarrow (t, A^{-1}y)$$

and

(3.8) $$(t, y) \longrightarrow T(t, Ay) \,.$$

Hence, the Jacobian of T is the product of the Jacobians of the maps
(3.7) and (3.8). Evaluating these Jacobians at $t = 0, y = x$, we get
(3.6). It now follows that if x is in $R^n \backslash Z(H)$ then the same is true
for Ax, for all A in H. Thus, in the notation of the preceding
paragraph, we may choose, for each A in H, a neighborhood U_{Ax} of
Ax and a neighborhood V_{Ax} of the identity in H, such that $f \nabla G$
is indefinitely differentiable in U_{Ax} whenever G is in $C_c^\infty(V_{Ax})$ and f
is a continuous function on R^n. Now let F be an arbitrary element
of $C_c^\infty(H)$. Then the collection of open sets $\{V_{Ax}A : A \in \operatorname{supp} F\}$ covers
the compact set $\operatorname{supp} F$ in H. (Here, $V_{Ax}A$ denotes the set $\{BA : B \in V_{Ax}\}$.) Thus, there exists a finite number of elements $A_1, \cdots, A_r$ in
H such that

$$\operatorname{supp} F \subset V_{A_1 x}A_1 \cup \cdots \cup V_{A_r x}A_r .$$

Now choose functions $F_1, \cdots, F_r$ in $C_c^\infty(H)$ such that

$$F_1 + \cdots + F_r = 1$$

on $\operatorname{supp} F$ and

$$\operatorname{supp} F_k \subset V_{A_k x}A_k$$

for $k = 1, \cdots, r$. Then we clearly have

$$(3.9) \qquad f \nabla F = f \nabla (FF_1) + \cdots + f \nabla (FF_r) .$$

We claim that each of the functions $f \nabla (FF_k)$ is indefinitely diffe-
rentiable on some neighborhood U of x. To prove this, write

$$U = A_1^{-1}(U_{A_1 x}) \cap \cdots \cap A_r^{-1}(U_{A_r x})$$

and define (for $k = 1, \cdots, r$) the functions G_k by

$$G_k(A) = F(AA_k)F_k(AA_k) \qquad\qquad (A \in H) .$$

Then G_k belongs to $C_c^\infty(V_{A_k x})$, and so $f \nabla G_k$ is indefinitely differentiable
in $U_{A_k x}$. But, if $\varDelta$ denotes the modular function of H, then for each
ϕ in $C_c^\infty(R^n)$

$$\int_{R^n} f \nabla (FF_k)(x)\phi(-x)dx = \int_H \langle f^A, \phi \rangle F(A)F_k(A)dA$$

$$= \varDelta(A_k^{-1}) \int_H \langle f^{AA_k}, \phi \rangle F(AA_k)F_k(AA_k)dA$$

$$= \varDelta(A_k^{-1})|\det A_k^{-1}| \int_H \langle f^A, \phi^{A_k^{-1}} \rangle G_k(A)dA$$

$$= \varDelta(A_k^{-1})|\det A_k^{-1}| \int_{R^n} f \nabla G_k(x)\phi^{A_k^{-1}}(-x)dx$$

$$= \varDelta(A_k^{-1}) \int_{R^n} f \nabla G_k(A_k x)\phi(-x)dx .$$

Thus, we conclude that

$$f \nabla (FF_k)(x) = \varDelta(A_k^{-1}) f \nabla G_k(A_k x)$$

for all x in R^n. Since $f \nabla G_k$ is indefinitely differentiable in $U_{A_k x}$, it follows that $f \nabla(FF_k)$ is indefinitely differentiable in $A_k^{-1}(U_{A_k x})$, and hence in U. Thus $f \nabla F$ is also indefinitely differentiable in U.

Now consider an arbitrary distribution q in $D'(R^n)$. Again, let x be an arbitrary point in $R^n \backslash Z(H)$. We first notice that for each multi-index α, and each ϕ in $C_c^\infty(R^n)$, we clearly have

$$D^\alpha(\phi^A) = \sum_{|\beta| \leq |\alpha|} G_\beta^\alpha(A)(D^\beta \phi)^A$$

where the G_β^α are C^∞ functions on H depending only on α and β. It follows easily that for each distribution q on R^n,

(3.10) $$D^\alpha(q^A) = \sum_{|\beta| \leq |\alpha|} G_\beta^\alpha(A)(D^\beta q)^A$$

and

(3.11) $$(D^\alpha q)^A = \sum_{|\beta| \leq |\alpha|} \check{G}_\beta^\alpha(A) D^\beta(q^A)$$

where we have written

$$\check{G}_\beta^\alpha(A) = G_\beta^\alpha(A^{-1}) \qquad (A \in H) .$$

Now choose a neighborhood W of x which has compact closure. Then, inside W, q can be expressed as

$$q = \sum_\alpha D^\alpha f_\alpha$$

where the f_α are continuous functions on R^n, and the sum is finite. Using (3.11), we see that if F is in $C_c^\infty(H)$ and ϕ is in $C_c^\infty(R^n)$, then

$$\langle \phi, (D^\alpha f_\alpha) \nabla F \rangle = \int_H \langle (D^\alpha f_\alpha)^A, \phi \rangle F(A) dA$$

$$= \sum_\beta \int_H \langle D^\beta(f_\alpha^A), \phi \rangle \check{G}_\beta^\alpha(A) F(A) dA$$

(3.11) $$= \sum_\beta \int_H \langle f_\alpha^A, D^\beta \phi \rangle \check{G}_\beta^\alpha(A) F(A) dA$$

$$= \sum_\beta \langle D^\beta \phi, f_\alpha \nabla(\check{G}_\beta^\alpha F) \rangle$$

$$= \sum_\beta \langle \phi, D^\beta[f_\alpha \nabla(\check{G}_\beta^\alpha F)] \rangle .$$

We infer that

$$(D^\alpha f_\alpha) \nabla F = \sum_\beta D^\beta[f_\alpha \nabla(\check{G}_\beta^\alpha F)]$$

as distributions on R^n. But, in view of what we have already shown, each of the functions $f_\alpha \nabla (G^\alpha_\beta F)$ is indefinitely differentiable on some neighborhood U of x. The same is therefore true of the distributions $(D^\alpha f_\alpha) \nabla F$. Now, it is evident that

$$q \nabla F = \sum_\alpha (D^\alpha f_\alpha) \nabla F$$

in W. Since we may obviously assume that U is contained in W, it now follows that the distribution $q \nabla F$ is an indefinitely differentiable function in the neighborhood U of x. This completes the proof of the first assertion in Lemma 3.1.

To prove the second assertion, we refer to (3.10) and carry out a computation similar to (3.12).

LEMMA 3.2. *Suppose that* H *has dimension* $m \geq n$. *Let* q *be a distribution on* R^n *and* x *a point in* $\operatorname{supp} q$ *which is not in* $Z(H)$. *Then there is a function* F *in* $C^\infty_c(H)$ *such that* $q \nabla F(x) \neq 0$.

Proof. Write S for the set of all points z in $R^n \backslash Z(H)$ such that, for some F in $C^\infty_c(H)$, $q \nabla F(z) \neq 0$.

We first observe that x is in the closure of S. For suppose that there were a neighborhood U of x which contained no points of S. This would entail that each function $q \nabla F$ vanishes identically in U. It would then follow that, for each ϕ in $C^\infty_c(U)$ and each F in $C^\infty_c(H)$,

$$\int_G \langle q^A, \phi \rangle F(A) dA = \int_{R^n} q \nabla F(y) \phi(-y) dy$$
$$= 0 .$$

This would lead to the conclusion that the function $A \to \langle q^A, \phi \rangle$ is identically zero on H, for every ϕ in $C^\infty_c(U)$. In particular, we would have

$$\langle q, \phi \rangle = 0 \qquad (\phi \in C^\infty_c(U))$$

contradicting the fact that x is in $\operatorname{supp} q$.

We know that $P_i(x) \neq 0$ for some i, and, as usual, we assume that $i = \{1, 2, \cdots, n\}$. Consider the map

$$(t_1, \cdots, t_m) \longrightarrow (y_1, \cdots, y_m)$$

of R^m into itself defined by

$$y_k = \theta_{k1}(t_1, \cdots, t_m)x_1 + \cdots + \theta_{kn}(t_1, \cdots, t_m)x_n \quad (1 \leq k \leq n)$$
$$y_k = t_k \qquad\qquad\qquad\qquad\qquad\qquad (n + 1 \leq k \leq m) .$$

The Jacobian of this transformation is $P_i(x)$, and is therefore nonzero at $t_1 = \cdots = t_m = 0$. Hence this map carries a neighborhood W of

the origin in R^m onto a neighborhood of the point $(x_1, \cdots, x_n, 0, \cdots, 0)$. It follows that the set $\{\theta(t)x: t \in W\}$ is a neighborhood of x in R^n and, consequently, so is the set $\{Bx: B \in H\}$. Since x is in the closure of S, there must be a point z in S which has the form $z = Bx$, for some B in H. Choose a function G in $C_c^\infty(H)$ such that $q\nabla G(z) \neq 0$, and define the function F by

$$F(A) = G(AB^{-1}) \qquad\qquad (A \in H).$$

Then F is in $C_c^\infty(H)$. Moreover, using an argument similar to one used in the course of the proof of Lemma 3.1, it may be shown that

$$q\nabla F(y) = q\nabla G(By) \qquad\qquad (y \in R^n).$$

It follows that

$$q\nabla F(x) = q\nabla G(z) \neq 0$$

as we wished to show.

4. **A cancellation theorem.** In this section, we prove our main theorem. Before we do so, however, it is necessary to make some brief comments about notation. We shall want to make use of the Schwartz-Fourier transform. Now, if q is a temperate distribution and A is in $\mathrm{Gl}(n, R)$, then it is easy to verify that

(4.1) $$(q^A)^\wedge = |\det A^{-1}|\,\hat{q}^{A'^{-1}}.$$

It appears, then, that we shall be dealing with both a closed subgroup G of $\mathrm{Gl}(n, R)$ and its adjoint group G'. It is actually to the group G' that we shall apply the theory developed in the preceding section. As far as the operation ∇ is concerned, we shall use it in both the case when we are considering the group G and in the case when we are considering the group G'. Thus, if q is a distribution on R^n and F is in $C_c^\infty(G)$, we shall write $q \nabla F$ for the distribution defined in §3 (with G playing the role of H); but also, if q is a distribution and F'' is in $C_c^\infty(G')$, we shall write $q\nabla F''$ for the distribution arising when we replace H by G' in the definitions of §3. The context makes evident the group (G or G') with respect to which the operation ∇ is being defined.

We begin with a lemma. This will then yield the main theorem as an easy corollary.

LEMMA 4.1. *Let G be a closed subgroup of $\mathrm{Gl}(n, R)$ of dimension $m \geq n$. Let E be a barrelled G-space. Then for all u in E, v in E', F'' in $C_c^\infty(G')$, and ϕ in $C_c^\infty(R^n)$ such that*

$$\operatorname{supp} \phi \subset R^n \backslash Z(G')$$

we have the identity

$$\langle \phi, \hat{u} \nabla F' \cdot \hat{v} \rangle = \int_{G'} \langle u^{A^{-1}}, v * \hat{\phi} \rangle F'(A') dA' .$$

Proof. Recall that if q is a temperate distribution, then so is $q \nabla F'$. Thus if ϕ is in $C_c^\infty(R^n)$, then

$$
\begin{aligned}
[q \nabla F' \cdot \phi]^\wedge(x) &= (q \nabla F')^\wedge * \hat{\phi}(-x) \\
&= \langle \hat{\phi}_x, (q \nabla F')^\wedge \rangle \\
&= \langle (\hat{\phi}_x)^\wedge, q \nabla F' \rangle \\
&= \int_{G'} \langle q^{A'}, (\hat{\phi}_x)^\wedge \rangle F'(A') dA' \\
&= \int_{G'} \langle (q^{A'})^\wedge, \hat{\phi}_x \rangle F'(A') dA' \\
&= \int_{G'} |\det A^{-1}| \langle q^{A^{-1}}, \phi_x \rangle F'(A') dA' \\
&= \int_{G'} |\det A^{-1}| q^{A^{-1}} * \phi(x) F'(A') dA' .
\end{aligned}
$$

(4.2)

Now consider a temperate distribution v, a function F' in $C_c^\infty(G')$, and functions ψ in $S(R^n)$ and ϕ in $C_c^\infty(R^n)$. Then, if (η_i) is a net extracted from $S(R^n)$ which converges to v in $S'(R^n)$, we have, by virtue of (4.2),

$$
\begin{aligned}
\langle \phi, \hat{\psi} \nabla F' \cdot \hat{v} \rangle &= \lim_i \langle \hat{\psi} \nabla F' \cdot \phi, \hat{\eta}_i \rangle \\
&= \lim_i \int_{R^n} [\hat{\psi} \nabla F' \cdot \phi]^\wedge(x) \eta_i(-x) dx \\
&= \lim_i \int_{R^n} \int_{G'} |\det A^{-1}| \psi^{A^{-1}} * \hat{\phi}(x) F'(A') dA' \eta_i(-x) dx \\
&= \lim_i \int_{G'} |\det A^{-1}| \psi^{A^{-1}} * \hat{\phi} * \eta_i(0) F'(A') dA' .
\end{aligned}
$$

Now, the set

$$\{\psi^{A^{-1}} * \hat{\phi} : A' \in \operatorname{supp} F'\}$$

is a bounded subset of $S(R^n)$. Hence

$$\lim_i \psi^{A^{-1}} * \hat{\phi} * \eta_i(0) = \psi^{A^{-1}} * \hat{\phi} * v(0)$$

uniformly for A' in $\operatorname{supp} F'$. Thus (4.3) becomes

(4.4) $$\langle \phi, \hat{\psi} \nabla F' \cdot \hat{v} \rangle = \int_{G'} |\det A^{-1}| \psi^{A^{-1}} * \hat{\phi} * v(0) F'(A') dA' .$$

Now consider a fixed ϕ in $C_c^\infty(R^n)$ with

$$\operatorname{supp} \phi \subset R^n \backslash Z(G')$$

and a fixed F' in $C_c^\infty(G')$. Then, for each u in E, $\hat{u} \nabla F'$ is indefinitely differentiable on a neighborhood of $\operatorname{supp} \phi$ (by Lemma 3.1). Thus $\hat{u} \nabla F' \cdot \phi$ is in $C_c^\infty(R^n)$ and so $[u \nabla F' \cdot \phi]^\wedge$ is in $S(R^n)$. We claim that the map

$$(4.5) \qquad\qquad u \longrightarrow [\hat{u} \nabla F' \cdot \phi]^\wedge$$

is continuous from E into $S(R^n)$. Since E is barrelled and $S(R^n)$ is fully complete, it suffices to show that the graph of this map is closed. Thus, assume that (u_i) converges to u in E and that

$$\lim_i [\hat{u}_i \nabla F' \cdot \phi]^\wedge = \eta$$

in $S(R^n)$. Then, for each λ in $C_c^\infty(R^n)$, we have, in view of (4.2),

$$
\begin{aligned}
\eta * \lambda(0) &= \lim_i [\hat{u}_i \nabla F' \cdot \phi]^\wedge * \lambda(0) \\
&= \lim_i \int_{R^n} [\hat{u}_i \nabla F' \cdot \phi]^\wedge(x) \lambda(-x) dx \\
&= \lim_i \int_{R^n} \int_{G'} |\det A^{-1}| u_i^{A^{-1}} * \hat{\phi}(x) F'(A') \lambda(-x) dA' dx \\
&= \lim_i \int_{G'} |\det A^{-1}| u_i^{A^{-1}} * \hat{\phi} * \lambda(0) F'(A') dA' \\
&= \lim_i \int_{G'} \langle u_i, (\hat{\phi} * \lambda)^A \rangle F'(A') dA' \ .
\end{aligned}
$$

Now, since $\operatorname{supp} F'$ is compact, the set $\{A : A' \in \operatorname{Supp} F'\}$ is a compact subset of G. Hence, since E is a G-space, the set

$$\{(\hat{\phi} * \lambda)^A : A' \in \operatorname{supp} F'\}$$

is a weakly compact, and hence equicontinuous (E being barrelled) subset of E'. It follows that

$$\lim_i \langle u_i, (\hat{\phi} * \lambda)^A \rangle = \langle u, (\hat{\phi} * \lambda)^A \rangle$$

uniformly for A' in $\operatorname{supp} F'$. Using this in (4.6), we get

$$
\begin{aligned}
\eta * \lambda(0) &= \int_{G'} \langle u, (\hat{\phi} * \lambda)^A \rangle F'(A') dA' \\
&= \int_{G'} |\det A^{-1}| \langle u^{A^{-1}}, \hat{\phi} * \lambda \rangle F'(A') dA' \\
&= \int_{G'} |\det A^{-1}| u^{A^{-1}} * \hat{\phi} * \lambda(0) F'(A') dA' \\
&= \int_{R^n} \int_{G'} |\det A^{-1}| u A^{-1} * \hat{\phi}(x) F'(A') dA' \lambda(-x) dx \\
&= \widehat{\hat{u} \nabla F' \cdot \phi} * \lambda(0) \ .
\end{aligned}
$$

We conclude that η coincides with $[\hat{u}\nabla F''\cdot\phi]^\wedge$ and therefore the graph of the map (4.5) is indeed closed.

Now consider again a function ϕ in $C_c^\infty(R^n)$ with

$$\text{supp }\phi \subset R^n\backslash Z(G')\ .$$

Let (ψ_i) be a net extracted from $S(R^n)$ which converges to u in E. Using the continuity of the map (4.5) and relation (4.4), we find that

$$
\begin{aligned}
\langle\phi,\ \hat{u}\nabla F''\cdot\hat{v}\rangle &= \langle\hat{u}\nabla F''\cdot\phi,\ \hat{v}\rangle \\
&= \lim_i \langle\hat{\psi}_i\nabla F''\cdot\phi,\ \hat{v}\rangle \\
&= \lim_i \int_{G'} |\det A^{-1}|\,\psi_i^{A^{-1}}*\hat{\phi}*v(0)F''(A')dA' \\
&= \lim_i \int_{G'} |\det A^{-1}|\langle\psi_i^{A^{-1}},\ \hat{\phi}*v\rangle F''(A')dA' \\
&= \int_{G'} |\det A^{-1}|\langle u^{A^{-1}},\ \hat{\phi}*v\rangle F''(A')dA'
\end{aligned}
$$

the last equality following from a now familiar argument. This completes the proof of our lemma.

THEOREM 4.2. *Let* G *be a closed subgroup of* $\text{Gl}\,(n, R)$ *of dimension* $m \geqq n$. *Suppose that* E *is a barrelled* G-*space. Let* u *in* E *and* v *in* E' *be such that*

$$(4.7)\qquad\qquad\qquad \langle u_b^A, v\rangle = 0$$

for all A *in* G *and* b *in* R^n. *Then*

$$\text{supp }\hat{v} \cap \left\{\bigcup_{A\in G} A'(\text{supp }\hat{u})\right\} \subset Z(G')\ .$$

Proof. Relation (4.7) entails that

$$\langle u^A,\ \psi*v\rangle = 0$$

for all ψ in $S(R^n)$. Lemma 4.1 now tells us that $\hat{u}\nabla F''\cdot\hat{v}$ vanishes in $R^n\backslash Z(G')$ for all F'' in $C_c^\infty(G')$. Lemma 3.2 now leads us to the conclusion that

$$\text{supp }\hat{v} \cap \text{supp }\hat{u}^{A'^{-1}} \subset Z(G')$$

for each A in G. Since

$$\text{supp }\hat{u}^{A'^{-1}} = A'(\text{supp }\hat{u})$$

the desired result follows.

5. **A density theorem.** Let E be a subspace of $S'(R^n)$. If S

is a subset of R^n, we say that S is a set of uniqueness for E relations

$$\text{supp } \hat{v} \subset S, \ v \in E$$

entail that $v = 0$.

THEOREM 5.1. *Let G be a closed subgroup of* $\text{Gl}\,(n, R)$ *of dimension $m \geq n$. Suppose that E is a barrelled G-space and that the set $Z(G')$ is a set of uniqueness for E'. Let u be an element of E. Then $T_C[u]$ coincides with the whole of E if and only if*

(5.1) $\bigcup \{A'(\text{supp } u) \colon A \in G\}$

is dense in R^n.

Proof. The density of the set (5.1) is clearly necessary for $T_C[u]$ to coincide with E. For suppose that some nonvoid open set U in R^n did not intersect the set (5.1). Consider a nonzero ϕ in $C_c^\infty(R^n)$ with support in U. Then its Fourier transform $\hat{\phi}$ is a nonzero element of E' which annihilates $T_C[u]$.

To prove the converse, suppose that (5.1) is dense. Let x be in $R^n \backslash Z(G')$. Then, as in the proof of Lemma 3.2, we deduce that $\{A'x \colon A \in G\}$ is a neighborhood of x. Thus, there exists a point in the set (5.1) which is also in $\{A'x \colon A \in G\}$. In other words, there is a point z in supp $\hat{u}$, and elements A, B in G, such that $A'z = B'x$. Hence, $x = B'^{-1}A'z$ is in the set (5.1). By Theorem 4.2, x cannot belong to the support of the Fourier transform of any distribution v in E' which annihilates $T_C[u]$. It follows that if v annihilates $T_C[u]$, then we must have supp $\hat{v}$ contained in $Z(G')$. Since $Z(G')$ is a set of uniqueness for E' by assumption, we infer that $v = 0$. The Hahn-Banach theorem now tells us that $T_C[u]$ is the whole of E.

The problem considered above has been studied by Gosselin [1] in the case when E is the space Lebesgue square integrable functions, but with no restriction on the dimension of G. The approach in [1] is a measure theoretic one.

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