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THE CONVERSE TO THE SMITH THEOREM FOR Z_p-HOMOLOGY SPHERES

STEPHEN J. WILLSON

Let X be a finite CW complex with the Z_p homology of an n-sphere. Let Z_p act cellularly on X. The Smith theorem asserts that the fixed point set X^{Z_p} has the Z_p homology of an m-sphere for $-1 \leqq m \leqq n$. A converse to this Smith theorem is proved.

Suppose X is a finite CW complex, p is a prime, and $\alpha: X \to X$ is a homeomorphism of period p (i.e., α^p is the identity map). Let X^{Z_p} denote the set of points in X left fixed by α. The well-known Smith theorem states that, if X has the Z_p homology of a disk (respectively, an n-sphere), then X^{Z_p} has the Z_p homology of a disk (respectively, some m-sphere where $-1 \leqq m \leqq n$ and the (-1)-sphere is the empty set). The converse to this theorem for the case where X has the Z_p homology of a disk appears in a paper of Lowell Jones **[2]**.

This current paper shows how to extend Jones' methods to obtain the converse for the case where X has the Z_p homology of an n-sphere. Specifically, we prove the following theorem:

THEOREM 1. *Let p be a prime integer and n a positive integer. Let K be a connected finite CW complex satisfying $H_n(K; Z_p) = Z_p$ and for which, if $i \neq n$ and $i \neq 0$, $H_i(K; Z)$ is a finite group of order prime to p.*

Then there exist a finite, simply connected, connected CW complex X containing K as a subcomplex and a cellular homeomorphism $\alpha: X \to X$ of period p so that

(1) $X^{Z_p} = K$

(2) *For some $m > 0$, $H_i(X; Z) = 0$ if $i \neq 0$, $i \neq n + 2m$.*

(3) *If $H_n(K; Z) = Z \oplus A$ where A is a finite abelian group of order prime to p, then $H_{n+2m}(X; Z) = Z$.*

(4) *If $H_n(K; Z) = Z_{p^s} \oplus A$ where A is a finite abelian group of order prime to p, and $s \geqq 1$, then $H_{n+2m}(X; Z) = Z_{p^s}$.*

Here Z denotes the ring of integers and Z_{p^s} denotes the cyclic group of order p^s. It is well-known that $H_n(K; Z)$ must satisfy the hypotheses of either (3) *or* (4) *since $H_n(K; Z_p) = Z_p$.*

The proof is similar to Jones' proof of [**2**; Theorem 1.1], but utilizes some further algebraic lemmas. The algebraic lemmas are given in §1, and their topological analogues are given in §2. The proof of the theorem appears in §3.

If p is not prime, the methods still apply and yield a CW complex X possessing a semi-free Z_p action α with fixed point set K. The cases (3) and (4) are, however, not exhaustive.

I wish to thank the referee for strengthening the original version of Theorem I.

1. Algebraic lemmas. Let $R = Z[Z_p]$, the integral group ring for the group Z_p with generator g. Elements of R will be written $\Sigma\, a_i g^i$ where $a_i \in Z$. All summations run over $i = 0, \cdots, p-1$. The element g^0 is the identity, often written e. In some formulas we shall use the identifications $a_p = a_0$, $a_{p-1} = a_{-1}$, $a_{p+1} = a_1$. Denote by σ the element $\sigma = \Sigma\, g^i$. If A and B are left R modules and $f\colon A \to B$ is a homomorphism, denote by Ker f the kernel of f; by Coker f the cokernel of f: by Image f the image of f. A left R module M is said to be trivial provided $g^i m = m$ for $m \in M$ and $g^i \in Z_p$.

LEMMA 1. *Let $\epsilon\colon R \to Z_{p^s}$ be the augmentation map which takes $\Sigma\, b_i g^i$ to $\Sigma\, b_i$ mod p^s. View Z_{p^s} as a trivial left R module. There is an exact sequence of left R modules*

$$R \oplus R \xrightarrow{\mu} R \xrightarrow{\epsilon} Z_{p^s} \to 0$$

and a homomorphism $\lambda\colon R \to \operatorname{Ker} \mu$ such that

(1) *λ is monic;*

(2) Coker $\lambda = Z_{p^s}$.

Proof. Define μ, if $(a, b) \in R \oplus R$, by

$$\mu(a, b) = (e - g)a + p^{s-1}\sigma b$$

where

$$\sigma = e + g + g^2 + \cdots + g^{p-1} \in R.$$

Define $\lambda\colon R \to R \oplus R$, if $a \in R$, by

$$\lambda(a) = (p^{s-1}\sigma a,\ (g - e)a).$$

We now verify that these maps have the properties asserted above:

Claim 1. $\epsilon\mu = 0$.
This follows since

$$\epsilon\mu(a, b) = \epsilon((e-g)a + p^{s-1}\sigma b) = a\epsilon(e-g) + p^{s-1}b\epsilon(\sigma)$$
$$= a \cdot 0 + p^{s-1}b \cdot p = 0,$$

using the left R module structure of Z_{p^s}.

Claim 2. $\operatorname{Ker}\epsilon \subset \operatorname{Image}\mu$.
If $\epsilon(\Sigma a_i g^i) = 0$, then $\Sigma a_i \equiv 0 \bmod p^s$. Let

$$\sum b_i g^i = \sum a_i g^i - \left(\sum a_i\right) p^{-1}\sigma.$$

Then $\Sigma b_i = 0 \in Z$, and it is easy to see that $\Sigma b_i g^i = (e-g)c$ for some $c \in R$. Hence

$$\mu\left(c, \left(\sum a_i\right)p^{-s}e\right) = (e-g)c + \left(\sum a_i\right)p^{-1}\sigma = \sum a_i g^i.$$

Claim 3. $\operatorname{Image}\lambda \subset \ker\mu$.
To see this, if $a \in R$, note

$$\mu\lambda(a) = (e-g)p^{s-1}\sigma a + p^{s-1}\sigma(g-e)a = 0.$$

Claim 4. λ is monic.
To see this, note $\ker\lambda = \ker(g-e) \cap \ker(p^{s-1}\sigma)$ where $(g-e)$ denotes the homomorphism of multiplication by $(g-e)$, and $p^{s-1}\sigma$ denotes multiplication by $p^{s-1}\sigma$. Then

$$\ker\lambda = \{a\sigma : a \in Z\} \cap \left\{\sum a_i g^i : p^{s-1}\left(\sum a_i\right) = 0 \in Z\right\} = 0.$$

Claim 5. $\operatorname{Coker}\lambda = Z_{p^s}$.
To see this, note

$$\operatorname{Ker}\mu = \left\{\left(\sum a_i g^i, \sum b_i g^i\right) : (e-g)\sum a_i g^i + p^{s-1}\sigma \sum b_i g^i = 0\right\}$$
$$= \left\{\left(\sum a_i g^i, \sum b_i g^i\right) : a_i - a_{i-1} + p^{s-1}\left(\sum b_j\right) = 0 \text{ for all } i\right\}.$$

Summing these latter conditions over i, we obtain $\Sigma a_i - \Sigma a_i + p^s \Sigma b_j = 0$. Hence $\Sigma b_j = 0$ and $a_i = a_{i-1}$ for all i. Thus

$$\operatorname{Ker}\mu = \left\{\left(a\sigma, \sum b_i g^i\right) : a \in Z, \sum b_i = 0\right\}.$$

Define $\gamma : \operatorname{Ker}\mu \to Z_{p^s}$ by

$$\gamma\left(a\sigma, \sum b_i g^i\right) = a + p^{s-1}[pb_0 + (p-1)b_1 + (p-2)b_2 + \cdots + b_{p-1}] \bmod p^s.$$

Then γ is surjective. Moreover, $\gamma\lambda = 0$, which may be seen as follows:

$$\gamma\lambda\left(\sum a_i g^i\right) = \gamma\left(p^{s-1}\left(\sum a_i\right)\sigma, \sum (a_{i-1} - a_i)g^i\right)$$
$$= p^{s-1}\left(\sum a_i\right) + p^{s-1}[p(a_{p-1} - a_0) + (p-1)(a_0 - a_1) + \cdots + (a_{p-2} - a_{p-1})]$$
$$= p^{s-1}\Big[\sum a_i + a_0(-p + p - 1) + a_1(-(p-1) + p - 2) + \cdots$$
$$+ a_{p-2}(1-2) + a_{p-1}(p-1)\Big]$$
$$= p^{s-1}pa_{p-1} \equiv 0 \bmod p^s.$$

Thus to prove Claim 5 there remains to show only that $\operatorname{Ker}\gamma \subset \operatorname{Image}\lambda$. But if $\gamma(a\sigma, \Sigma\, b_i g^i) = 0$, then

$$a + p^{s-1}[pb_0 + (p-1)b_1 + \cdots + b_{p-1}] \equiv 0 \bmod p^s. \tag{1}$$

For arbitrary $c_0 \in Z$, define $c_{i+1} = c_i - b_{i+1}$ for $i = 0, 1, \cdots, p-2$. Then $b_0 = c_{p-1} - c_0$ since $\Sigma\, b_i = 0$, and

$$\lambda\left(\sum c_i g^i\right) = \left(p^{s-1}\left(\sum c_i\right)\sigma, \sum (c_{i-1} - c_i)g^i\right)$$
$$= \Big(p^{s-1}[c_0 + (c_0 - b_1) + (c_0 - b_1 - b_2) + \cdots$$
$$+ (c_0 - b_1 - \cdots - b_{p-1})]\sigma, \sum b_i g^i\Big)$$
$$= \left(p^{s-1}[pc_0 - (p-1)b_1 - (p-2)b_2 - \cdots - b_{p-1}]\sigma, \sum b_i g^i\right).$$

By (1) we may choose c_0 so

$$p^s c_0 = a + p^{s-1}[(p-1)b_1 + (p-2)b_2 + \cdots + b_{p-1}].$$

But then $\lambda(\Sigma\, c_i g^i) = (a\sigma, \Sigma\, b_i g^i)$.

LEMMA 2. *Let $\epsilon: R \to Z$ be the augmentation map. There is a map $\lambda: R \to R$ so*

$$0 \to Z \to R \xrightarrow{\lambda} R \xrightarrow{\epsilon} Z \to 0 \qquad \textit{is exact.}$$

Proof. Let $\lambda(a) = (e - g)a$ for $a \in R$. Then $\epsilon\lambda = 0$ and $\ker \epsilon = \operatorname{Image}\lambda$ easily. Moreover

$$\begin{aligned}\operatorname{Ker}\lambda &= \left\{\sum a_i g^i : \sum (a_i - a_{i-1})g^i = 0\right\}\\ &= \left\{\sum a_i g^i : a_0 = a_1 = a_2 = \cdots = a_{p-1}\right\}\\ &= \{b\sigma : b \in Z\} \cong Z.\end{aligned}$$

LEMMA 3. *If q is an integer prime to p, and $\epsilon: R \to Z_q$ is the augmentation map, then there is an exact sequence*

$$0 \to R \to R \xrightarrow{\epsilon} Z_q \to 0.$$

Proof. This is Lowell Jones' Lemma 1.1 [**2**; p. 53].

2. Topological lemmas. The major steps in the proof of Theorem I consist of applications of the following lemmas, which may be regarded as topological analogues of the lemmas of §1.

We shall let R be $Z[Z_p]$. Unless otherwise indicated, all homology groups have integer coefficients. Note that if X is a CW complex and $\alpha: X \to X$ is a homeomorphism of period p, then $H_i(X; Z)$ inherits the structure of a left R-module.

LEMMA A. *Suppose X is a connected, simply connected, finite CW complex with a cellular Z_p action given by $\alpha: X \to X$ such that $X^{Z_p} = K$. Suppose $H_i(X; Z) = 0$ for $0 < i < m$. Assume $H_m(X; Z)$ contains a finite subgroup A of order prime to p such that A is a trivial left R-submodule of $H_m(X)$. Then there exists a connected, simply connected, finite CW complex Y containing X as a subcomplex and possessing a cellular Z_p action extending α such that*

(1) $Y^{Z_p} = K$

(2) *$H_i(Y; Z) = 0$ for $0 < i < m$.*

(3) *$H_m(Y; Z) = H_m(X; Z)/A$ as an R-module.*

(4) *The inclusion induces an isomorphism of $H_i(X; Z)$ onto $H_i(Y; Z)$ for $i > m$.*

Proof. This is essentially the proof of Theorem 1.1 in [**2**]. We note that it suffices by induction to assume $A = Z_q$ where q is prime to p. Obtain, by the Hurewicz theorem, a map $k: S^m \to X$ which realizes a generator of $Z_q \subset H_m(X; Z)$. We shall attach p cells of dimension $(m+1)$ to X along the maps $k, \alpha k, \alpha^2 k, \cdots, \alpha^{p-1}k: S^m \to X$. Call the resulting CW complex Y_1; clearly we obtain a cellular Z_p action on Y_1 by extending α to permute the points in the added cells. Then

$H_i(Y_1; Z) = H_i(X; Z)$ for $i \neq m$, $m+1$, and the long exact sequence of the pair (Y_1, X) yields the exact sequence of R modules

$$0 \to H_{m+1}(X) \to H_{m+1}(Y_1) \to R \xrightarrow{\epsilon} H_m(X) \to H_m(Y_1) \to 0.$$

Since Z_q is a trivial R module, the map denoted ϵ may be identified with the augmentation map from R onto Z_q. It follows that $H_m(Y_1) = H_m(X)/Z_q$ and

$$0 \to H_{m+1}(X) \to H_{m+1}(Y_1) \to \operatorname{Ker} \epsilon \to 0 \qquad \text{is exact.}$$

By Lemma 3, $\operatorname{Ker} \epsilon \cong R$ and hence is projective. Thus $H_{m+1}(Y_1) = H_{m+1}(X) \oplus R$. The Hurewicz map $h: \pi_{m+1}(Y_1) \to H_{m+1}(Y_1)$ is surjective (see Hu [1; p. 167] or G. W. Whitehead [3]). Hence we may represent the element $e \in R \subset H_{m+1}(Y_1)$ by a map $j: S^{m+1} \to Y_1$. As before, attach p cells of dimension $(m+2)$ to Y_1 along the maps $j, \alpha j, \alpha^2 j, \cdots, \alpha^{p-1} j$ to obtain a CW complex Y; we may extend the map α over Y. Then $H_i(Y) = H_i(Y_1)$ for $i \neq m+2$, $m+1$, and

$$0 \to H_{m+2}(Y_1) \to H_{m+2}(Y) \to R \to H_{m+1}(Y_1) \to H_{m+1}(Y) \to 0$$

is exact. By construction the map of R into $H_{m+1}(Y_1)$ is an isomorphism onto the summand isomorphic to R. Hence

$$H_{m+2}(Y) = H_{m+2}(Y_1) = H_{m+2}(X), \quad H_{m+1}(Y) = H_{m+1}(X).$$

The complex Y satisfies the conclusions of the lemma.

LEMMA B. *Suppose X is a connected, simply connected, finite CW complex with a cellular Z_p action given by $\alpha: X \to X$ such that $X^{Z_p} = K$. Suppose $H_i(X) = 0$ if $0 < i < m$. Assume $H_m(X)$, $H_{m+1}(X)$, and $H_{m+2}(X)$ all are trivial as R modules, and that $H_m(X) = Z$, $H_{m+1}(X; Z_p) = 0$, $H_{m+2}(X; Z_p) = 0$. Then there exists a connected, simply connected, finite CW complex Y which contains X as a subcomplex and possesses a cellular Z_p action extending α such that*

(1) $Y^{Z_p} = K$

(2) $H_i(Y; Z) = 0$ *for* $0 < i \leqq m+1$

(3) $H_{m+2}(Y; Z) = Z$ *as a trivial R module*

(4) *The inclusion induces isomorphisms from $H_i(X; Z)$ onto $H_i(Y; Z)$ for $i > m+2$.*

Proof. Obtain by the Hurewicz theorem a map $k: S^m \to X$ which represents the generator of $H_m(X)$. Attach p cells of dimension

$(m+1)$ along the maps $k, \alpha k, \cdots, \alpha^{p-1}k$ to obtain a CW complex Y_1; and extend the map α over Y_1 via the obvious permutation of points on the added cells. Then $H_i(Y_1) = H_i(X)$ for $i \neq m,\ m+1$; and

$$0 \to H_{m+1}(X) \to H_{m+1}(Y_1) \to R \xrightarrow{\epsilon} H_m(X) \to H_m(Y_1) \to 0$$

is exact. By construction, ϵ may be regarded as the augmentation map from R onto Z. Hence $H_m(Y_1) = 0$ and

$$0 \to H_{m+1}(X) \to H_{m+1}(Y_1) \to \operatorname{Ker} \epsilon \to 0$$

is exact. Since $H_{m+1}(X; Z_p) = 0$ and $H_{m+1}(X)$ is a trivial R module, by Lemma A we may obtain a complex $Y_2 \supset Y_1$ with an action extending α so $H_i(Y_2) = 0$ for $0 < i < m+1$, $H_{m+1}(Y_2) = \operatorname{Ker} \epsilon$, and $H_i(Y_2) = H_i(Y_1) = H_i(X)$ by the inclusion map for $i > m+1$. Let λ be the homomorphism of Lemma 2. By the Hurewicz theorem we represent $\lambda(e) \in H_{m+1}(Y_2)$ by a map $j\colon S^{m+1} \to Y_2$. Adjoin cells to Y_2 along the maps $j, \alpha j, \cdots, \alpha^{p-1} j$ to obtain a complex Y_3 with action α. Then $H_i(Y_3) = H_i(Y_2)$ for $i \neq m+2,\ m+1$, and

$$0 \to H_{m+2}(Y_2) \to H_{m+2}(Y_3) \to R \xrightarrow{\lambda} H_{m+1}(Y_2) \to H_{m+1}(Y_3) \to 0$$

is exact. Since Image $\lambda = \operatorname{Ker} \epsilon$, $H_{m+1}(Y_3) = 0$ and

$$0 \to H_{m+2}(Y_2) \to H_{m+2}(Y_3) \to \operatorname{Ker} \lambda \to 0$$

is exact. Since $H_{m+2}(Y_2) = H_{m+2}(X)$ is a trivial R module and $H_{m+2}(X; Z_p) = 0$, we may apply Lemma A to the subgroup $H_{m+2}(Y_2) \subset H_{m+2}(Y_3)$ to obtain a complex $Y \supset Y_3$ so $H_i(Y) = 0$ for $i < m+2$ and $H_{m+2}(Y) = \operatorname{Ker} \lambda = Z$. This Y satisfies the conclusions of the lemma.

LEMMA C. *Suppose X is a connected, simply connected, finite CW complex with a cellular Z_p action given by $\alpha\colon X \to X$ such that $X^{Z_p} = K$. Suppose $H_i(X) = 0$ if $0 < i < m$. Assume $H_m(X)$, $H_{m+1}(X)$, and $H_{m+2}(X)$ are all trivial as R modules and $H_m(X) = Z_{p^s}$ for some $s \geqq 1$; and both $H_{m+1}(X)$ and $H_{m+2}(X)$ are finite groups of order prime to p. Then $H_{m+2}(X; Z_p) = 0$. Then there exists a connected, simply connected, finite CW complex Y containing X and with a cellular Z_p action extending α such that*

(1) $Y^{Z_p} = K$

(2) $H_i(Y; Z) = 0$ *for* $0 < i \leqq m+1$

(3) *$H_{m+2}(Y; Z) = Z_{p^s}$ as a trivial R module*

(4) *The inclusion induces isomorphisms from $H_i(X)$ onto $H_i(Y)$ for $i > m+2$.*

Proof. Obtain by the Hurewicz theorem a map $k: S^m \to X$ representing a generator for $Z_{p^s} = H_m(X)$. Attach p cells of dimension $(m+1)$ along the maps $k, \alpha k, \cdots, \alpha^{p-1}k$ to obtain a complex Y_1 with action α extending the previous α. Then $H_i(Y_1) = H_i(X)$ for $i \neq m$, $m+1$, and

$$0 \to H_{m+1}(X) \to H_{m+1}(Y_1) \to R \xrightarrow{\epsilon} H_m(X) \to H_m(Y_1) \to 0$$

is exact. By construction, the map ϵ may be identified with the augmentation map of Lemma 1. Then $H_m(Y_1) = 0$ and

$$0 \to H_{m+1}(X) \to H_{m+1}(Y_1) \to \operatorname{Ker} \epsilon \to 0.$$

Apply Lemma A to the subgroup $H_{m+1}(X)$ of $H_{m+1}(Y_1)$ to obtain a complex $Y_2 \supset Y_1$ so that $H_i(Y_2) = 0$ for $0 < i < m+1$; $H_{m+1}(Y_2) = \operatorname{Ker} \epsilon$; $H_i(Y_2) = H_i(X)$ for $i > m+1$.

Let $\mu: R \oplus R \to \operatorname{Ker} \epsilon$ be the homomorphism in Lemma 1. By the Hurewicz theorem we may represent $\mu(e, 0)$ by a map $j: S^{m+1} \to Y_2$ and we may represent $\mu(0, e)$ by a map $l: S^{m+1} \to Y_2$. Adjoin p cells of dimension $(m+2)$ via $j, \alpha j, \cdots, \alpha^{p-1}j$ and also p cells via $l, \alpha l, \cdots, \alpha^{p-1}l$; call the resulting complex Y_3 and extend α over Y_3 in the obvious fashion. Then $H_i(Y_3) = H_i(Y_2)$ for $i \neq m+1$, $m+2$; and

$$0 \to H_{m+2}(Y_2) \to H_{m+2}(Y_3) \to R \oplus R \xrightarrow{\mu} H_{m+1}(Y_2) \to H_{m+1}(Y_3) \to 0$$

is exact. Since Image $\mu = \operatorname{Ker} \epsilon$, $H_{m+1}(Y_3) = 0$; and

$$0 \to H_{m+2}(Y_2) \to H_{m+2}(Y_3) \to \operatorname{Ker} \mu \to 0$$

is exact. Apply Lemma A to the complex Y_3 and the subgroup $H_{m+2}(Y_2) \subset H_{m+2}(Y_3)$; this is possible since $H_{m+2}(X; Z_p) = 0$ and $H_{m+2}(X)$ is a trivial R module. We obtain a complex Y_4 so $H_i(Y_4) = 0$ for $0 < i < m+1$, $H_{m+2}(Y_4) = \operatorname{Ker} \mu$, $H_i(Y_4) = H_i(X)$ for $i > m+2$. Let λ be the homomorphism of Lemma 1, and represent $\lambda(e)$ by a map $r: S^{m+2} \to Y_4$. Attach p cells of dimension $(m+3)$ to Y_4 along $r, \alpha r, \cdots, \alpha^{p-1}r$ to obtain a complex Y. Then $H_i(Y) = H_i(Y_4)$ for $i \neq m+2$, $m+3$ and

$$0 \to H_{m+3}(Y_4) \to H_{m+3}(Y) \to R \xrightarrow{\lambda} H_{m+2}(Y_4) \to H_{m+2}(Y) \to 0$$

is exact. By Lemma 1, λ is monic, so $H_{m+3}(Y_4) = H_{m+3}(Y)$; and $H_{m+2}(Y) = \text{Coker } \lambda = Z_{p^s}$. The complex Y satisfies the conclusions of the lemma.

3. Proof of Theorem I. We must first deal with $\pi_1(K)$. Assume that $n > 1$. Choose a finite generating set $b_1, \cdots, b_q$ for $\pi_1(K)$ by Van Kampen's theorem. We may kill b_1 by adjunction of 2-cells along $b_1, \alpha b_1, \cdots, \alpha^{p-1} b_1$, yielding a CW complex W. Since the image of b_1 in $H_1(K; Z)$ has order prime to p, we find $H_2(W; Z) = H_2(K; Z) \oplus R$, and we may proceed as in Lemma A to remove the R summand by adjunction of 3-cells. Deal similarly with $b_2, b_3, \cdots, b_q$. In this manner we obtain a simply-connected finite CW complex X_1 with cellular action $\alpha: X_1 \to X_1$ of period p so $H_i(X_1; Z) = H_i(K; Z)$ for $i > 1$, and $X_1^{Z_p} = K$. Apply Lemma A to the group $H_2(X_1; Z)$. Continuing inductively in this manner, we obtain a simply-connected, finite CW complex X_n such that $H_i(X_n) = H_i(K)$ for $i > n$; $H_i(X_n) = 0$ for $i < n$; $H_n(X_n) = Z$ if Case (3) of Theorem I is pertinent; and $H_n(X_n) = Z_{p^s}$ if Case (4) of Theorem I is pertinent. Now we apply repeatedly Lemma B for Case (3) and Lemma C for Case (4). After finitely many steps, the process terminates since $H_i(K) = 0$ for sufficiently large i.

If $n = 1$, we modify the above proof slightly. We first kill $H_1(K; Z)$ except for the summand Z or Z_{p^s} by Lemma A. Call the resulting complex W_1, and choose a finite generating set $b_1, \cdots, b_q$ for $\pi_1(W_1)$. We may assume that the image of b_1 in $H_1(W_1)$ is a generator of $H_1(W_1) = Z$ or Z_{p^s}. If the image of b_j is represented by $m_j \in Z$ for $j = 2, \cdots, q$, then $b_j b_1^{-m_j}$ has image 0 in $H_1(W_1)$, and the elements $b_1, b_2 b_1^{-m_2}, \cdots, b_q b_1^{-m_q}$ generate $\pi_1(W_1)$. Kill $b_2 b_1^{-m_2}, \cdots, b_q b_1^{-m_q}$ as in the case where $n > 1$; we obtain a complex W_2 for which $\pi_1(W_2) = Z$ or Z_{p^s}, and $H_i(W_2; Z) = H_i(K; Z)$ for $i \geqq 2$. Apply Lemma B or C to W_2. The remainder of the proof follows as for the case $n > 1$.

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Iowa State University