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ON THE FIRST AND THE SECOND CONJUGATE POINTS

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Three properties of conjugate points and extremal solutions of an nth-order linear ordinary differential equation are discussed. Also, a connection between the zero distribution and the factorization of an nth-order differential operator in the interval $(a, \eta_2(a))$ is established.

1. Introduction. We shall be concerned with the nth-order differential equation

$$Ly = \sum_{k=0}^{n} p_k(x)\, y^{(k)} = 0, \tag{1.1}$$

where the coefficients are real-valued functions which are continuous on an interval I and $p_n(x) \neq 0$, $x \in I$. A differential equation of the form (1.1) is called nonsingular on I. A solution y of (1.1) is said to have a zero of order k at $c \in I$ if $y(c) = y'(c) = \cdots = y^{(k-1)}(c) = 0$; if in addition $y^{(k)}(c) \neq 0$, we say that y has a zero of order exactly k at c. A zero of order exactly one is called simple. The mth conjugate point $\eta_m(a)$ of a point $a \in I$ is the smallest number $b > a$, $b \in I$, such that there exists a nontrivial solution of (1.1) which vanishes at a and has $n + m - 1$ zeros (counting multiplicities) on $[a, b]$ **[6]**. Obviously, we have the relation $\eta_1(a) \leqq \eta_2(a) \leqq \cdots$. A nontrivial solution of (1.1) which has n zeros on $[a, \eta_1(a)]$ is called an extremal solution for the interval $[a, \eta_1(a)]$. A nontrivial solution of (1.1) is said to have an $i_1 - i_2 - \cdots - i_j$ distribution of zeros on I if it has a zero of order i_k at $x_k \in I$, $x_1 < x_2 < \cdots < x_j$, $k = 1, 2, \cdots, j$.

So far as the study of zero distribution of solutions [1–5, 8–11, 14] is concerned, it is convenient to divide the problem into two cases: $\eta_1(a) = \eta_2(a)$ and $\eta_1(a) < \eta_2(a)$. In a recent paper, Gustafson **[2]** obtained an interesting result for the case $\eta_1(a) = \eta_2(a)$. Evidently, $\eta_1(a) < \eta_2(a)$ for any second-order differential equation of the form (1.1). However, for higher-order equations both cases $\eta_1(a) = \eta_2(a)$ and $\eta_1(a) < \eta_2(a)$ occur. For example, $\eta_1(a) = \eta_2(a) = \eta_3(a)$ for the equation $y^{(iv)} + 10y'' + 9y = 0$ **[1]**, while $\eta_1(a) < \eta_2(a)$ for

$$(ry'')'' - py = 0, \quad r > 0, \quad p > 0, \quad r \in C'', \quad p \in C,$$

according to a result of Leighton and Nehari **[6]**. Other equations with the property

$$(\mathrm{P}_1) \qquad \eta_1(a) < \eta_2(a)$$

have been observed by Peterson **[10]**.

Suppose Eq. (1.1) has an extremal solution for $[a, \eta_1(a)]$. Then it is well-known that (1.1) has an extremal solution for $[a, \eta_1(a)]$ which does not vanish on $(a, \eta_1(a))$ **[12]**. Of particular interest is the equation which has the property

(P_2) No extremal solution for $[a, \eta_1(a)]$ vanishes on $(a, \eta_1(a))$.

For example, it can be easily shown that $y''' + y = 0$ and $y''' - y = 0$ have the property (P_2). In fact, every extremal solution of $y''' + y = 0$ has a 2–1 distribution of zeros. On the other hand, every extremal solution of $y''' - y = 0$ has a 1–2 distribution of zeros. These two equations also have the property (P_1).

As it turns out, closely connected with (P_1) and (P_2) is the property

(P_3) There do not exist two (not necessarily distinct) extremal solutions for $[a, \eta_1(a)]$ with zero distributions $(n - k) - k$ and $(n - k - 1) - (k + 1)$, respectively, where k is a fixed number, $1 \leqq k \leqq n - 2$.

In §2 we prove that (P_3) implies (P_1) and (P_2). Conversely, (P_1) and (P_2) taken together imply (P_3). Moreover, we shall show that in general (P_1) neither implies nor is implied by (P_2). As the last result of this section we shall exhibit a class of differential equations which has the properties (P_1), (P_2) and (P_3).

In §3 we assume (P_1) and investigate the zero distribution of solutions on the intervals $[a, \eta_1(a)]$ and $(a, \eta_2(a))$, and their consequences. In particular, we discuss a connection between the zero distribution and the factorization of (1.1) on the interval $(a, \eta_2(a))$.

In the sequel we shall have an occasion to use the function $w(x; x_1^{[k_1]}, x_2^{[k_2]}, \cdots, x_p^{[k_p]})$ defined and used in **[5]**. Let $y_1, y_2, \cdots, y_n$ be n linearly independent solutions of (1.1). Then the function w is defined by

$$w(x; x_1^{[k_1]}, x_2^{[k_2]}, \cdots, x_p^{[k_p]})$$

$$\equiv \begin{vmatrix} y_1(x) & y_2(x) & \cdots\cdots & y_n(x) \\ y_1(x_1) & y_2(x_1) & \cdots\cdots & y_n(x_1) \\ y_1'(x_1) & y_2'(x_1) & \cdots\cdots & y_n'(x_1) \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\ y_1^{(k_1-1)}(x_1) & y_2^{(k_1-1)}(x_1) & \cdots & y_n^{(k_1-1)}(x_1) \\ y_1(x_2) & y_2(x_2) & \cdots\cdots & y_n(x_2) \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\ y_1(x_p) & y_2(x_p) & \cdots\cdots & y_n(x_p) \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots} \\ y_1^{(k_p-1)}(x_p) & y_2^{(k_p-1)}(x_p) & \cdots & y_n^{(k_p-1)}(x_p) \end{vmatrix}, \tag{1.2}$$

$1 \leqq p \leqq n-1$, $k_1 + k_2 + \cdots + k_p = n-1$. Obviously, this function w is a solution of (1.1) with a zero of order k_i at x_i, $i = 1, 2, \cdots, p$. Moreover, it is continuous function of $x_1, x_2, \cdots, x_p$.

2. Properties (P_1), (P_2) and (P_3). Suppose (1.1) has an extremal solution Y for $[a, \eta_1(a)]$ with an $i_1 - i_2 - \cdots - i_j$ distribution of zeros, i.e., Y has a zero of order i_k at x_k, $k = 1, 2, \cdots, j$, $i_1 + i_2 + \cdots + i_j = n$, $a = x_1 < x_2 < \cdots < x_j = \eta_1(a)$. Numerous results have been obtained for the zero distribution of Y [2, 5, 9, 10, 12]. Of particular importance in this section is the following result which will be frequently referred to in the proofs.

THEOREM 2.1 **[5]**. *If Y has a zero of order exactly i_m at x_m, $2 \leqq m \leqq j-1$, then* (1.1) *has an extremal solution for* $[a, \eta_1(a)]$ *with an* $i_1 - \cdots - i_{m-1} - (i_m - 1) - i_{m+1} - \cdots - i_j$ *distribution of zeros and an additional zero at an arbitrary point* $\xi \in [a, \eta_1(a)]$.

A simple application of this theorem shows that (P_3) implies (P_2). This result can then be used to prove that (P_3) also implies (P_1). On the other hand, if (1.1) does not satisfy (P_3), it is easily confirmed that (1.1) must violate either (P_1) or (P_2).

THEOREM 2.2. Eq. (1.1) *has the property* (P_3) *if and only if it satisfies* (P_1) *and* (P_2).

We shall illustrate by means of examples that in general (P_1) neither implies nor is implied by (P_2). The nonsingular equation

$$(2.1)\qquad \begin{aligned} &y''' - \frac{6\sin x\cos x\,(\cos^2 x - \sin^2 x)}{3\sin^2 x\cos^2 x - 2}\,y'' \\ &\quad - \frac{9\sin^2 x\cos^2 x + 14}{3\sin^2 x\cos^2 x - 2}\,y' = 0 \end{aligned}$$

has as a fundamental set of solutions $\sin^2 x\cos x$, $\cos^2 x\sin x$, and 1 **[1, 13]**. The Wronskian W of these three solutions is given by $W = 3\sin^2 x\cos^2 x - 2 < 0$, and the corresponding adjoint equation

$$(2.1)^*\qquad \begin{aligned} &v''' + \left(\frac{6\sin x\cos x\,(\cos^2 x - \sin^2 x)}{3\sin^2 x\cos^2 x - 2}\,v\right)'' \\ &\quad - \left(\frac{9\sin^2 x\cos^2 x + 14}{3\sin^2 x\cos^2 x - 2}\,v\right)' = 0 \end{aligned}$$

has a fundamental set of solutions $\sin^2 2x/W$, $(2\sin x - 3\sin^3 x)/W$, and $(3\cos^3 x - 2\cos x)/W$. It is easily confirmed that $\eta_1(0) = \eta_2(0) = \pi/2$ for (2.1)* and no extremal solution for $[0, \eta_1(0)]$ of (2.1)* vanishes in $(0, \eta_1(0))$. This shows that (P_2) does not in general imply (P_1).

To see that (P_1) does not in general imply (P_2), consider the nonsingular equation

$$(2.2)\qquad (6x^2 - 8x + 3)y^{(iv)} - (12x - 8)y''' + 12y'' = 0,$$

for which $1, x, x(1-x)^2$, and $x(1-x)^3$ form a fundamental set of solutions, $\eta_1(0) = 1$, and of which no extremal solution for [0, 1] has a 3–1 distribution of zeros **[4]**. Moreover, no extremal solution for [0, 1] can vanish more than once in (0, 1). It is easily verified that (2.2) has no nontrivial solution with zeros of order 2 and 3 at $x = 0$ and $x = 1$, respectively. From these facts we can readily deduce $\eta_1(0) < \eta_2(0)$. On the other hand, $x(\lambda - x)(1-x)^2$, $0 < \lambda < 1$, is an extremal solution for [0, 1] which vanishes at λ, $0 < \lambda < 1$.

An obvious consequence of these examples is that (P_3) is not in general implied by either (P_1) or (P_2) alone.

In view of Theorem 2.2, it is clear that any differential equation which satisfies (P_3) will also satisfy (P_1) and (P_2). Consider a differential equation of the form

$$(2.3)\qquad L_n y + py = 0,$$

where the operator L_n is successively defined by

$$L_0 y = \rho_0 y, \quad L_k y = \rho_k (L_{k-1} y)', \qquad k = 2, 3, \cdots, n.$$

The functions $\rho_0, \rho_1, \cdots, \rho_n$ are assumed to be positive, $\rho_k \in C^{n-k}$, $k = 0, 1, \cdots, n$, and p is assumed not to vanish. Eq. (2.3) was extensively studied by Nehari **[7]**, who established the following result: If a nontrivial solution of (2.3) has zeros of order k and $n - k$ at $x = a$ and $x = b$, respectively ($a < b$), then $n - k$ is even or odd, according as $p < 0$ or $p > 0$. Evidently, this result implies that Eq. (2.3) satisfies (P_3). Hence, we have the following theorem.

THEOREM 2.3. *Eq.* (2.3) *has the properties* (P_1), (P_2), *and* (P_3).

3. Zero distribution and factorization. In this section we exclusively consider a differential equation of the form (1.1) with property (P_1). Let Y be an extremal solution of (1.1) for $[a, \eta_1(a)]$ with an $i_1 - i_2 - \cdots - i_j$ distribution of zeros, $a = x_1 < x_2 < \cdots < x_j = \eta(a)$, $i_1 + i_2 + \cdots + i_j = n$. Then Y has a zero of order exactly i_k at x_k, $k = 1, 2, \cdots, j$. This is because $\eta_1(a) < \eta_2(a)$. Therefore, by a repeated application of Theorem 2.1, we obtain

THEOREM 3.1. *Suppose* (1.1) *has the property* $\eta_1(a) < \eta_2(a)$ *and has an extremal solution for* $[a, \eta_1(a)]$ *with an* $i_1 - i_2 - \cdots - i_j$ *distribution of zeros,* $i_1 + i_2 + \cdots + i_j = n$. *Let* $k_1, k_2, \cdots, k_p$ *be arbitrary positive integers such that* $k_1 + k_2 + \cdots + k_p = n$, *and let* $a = \xi_1, \xi_2, \cdots, \xi_p = \eta_1(a)$ *be distinct points in* $[a, \eta_1(a)]$. *If* $i_1 \leqq k_1$ *and* $i_j \leqq k_p$, *then* (1.1) *has an extremal solution for* $[a, \eta_1(a)]$ *which has a zero of order exactly* k_m *at* ξ_m, $m = 1, 2, \cdots, p$.

As is clear from Theorem 3.1, the zeros of solutions in $(a, \eta_1(a))$ can be moved to an arbitrary point in $(a, \eta_1(a))$, or can be separated into lower-order zeros in $(a, \eta_1(a))$. However, no such statements can be made in general for the zeros at the end points a and $\eta_1(a)$. On the other hand, the zeros of an extremal solution for $[a, \eta_1(a)]$ can be simultaneously separated into simple zeros in $[a, \eta_1(a) + \epsilon)$, $\epsilon > 0$ **[4, 14]**. By using a slight modification of the arguments given in the proof of Theorem 1 in **[4]**, we shall establish the following result.

THEOREM 3.2. *If* (1.1) *has a nontrivial solution with an* $(n - l) - l$ *distribution of zeros in* $(a, \eta_2(a))$, *then* (1.1) *has a nontrivial solution with the zero distribution*

$$\underbrace{1 - 1 - \cdots - 1}_{i} - j - \underbrace{1 - \cdots - 1}_{k}, \qquad i + j + k = n, \tag{3.1}$$

in $(a, \eta_2(a))$, *provided* $i \geqq n - l$ *or* $k \geqq l$.

Proof. Consider the case $i \geqq n-l$. Let y be a nontrivial solution of (1.1) which has zeros of order $n-l$ and l at $x=b$ and $x=c$, respectively, $a<b<c<\eta_2(a)$ and suppose l is maximal. Consider the function

$$w(x) \equiv \begin{cases} w(x; c^{[n-1]}) & \text{if} \quad l=n-1, \\ w(x; b^{[n-l-1]}, c^{[l]}), & \text{otherwise} \end{cases}$$

defined in §1. This function w cannot vanish identically; for if $w \equiv 0$ it would imply the existence of a nontrivial solution with a zero of order $n-l-1$ at b and a zero of order $l+1$ at c, contrary to the assumption. Therefore, w is a nontrivial solution of (1.1) with a zero of order exactly $n-l$ at b and a zero of order exactly l at c. Consequently, the $n-l$ zeros at b and $l-j$ (out of l) zeros at c can be separated into $n-j$ simple zeros in such a way that there are i simple zeros to the left and k simple zeros to the right of the jth-order zero at c (Cf. The proof of Theorem 1 **[4]**). This proves the theorem for the case $i \geqq n-l$.

The proof for the case $k \geqq l$ is similar.

REMARK. The above theorem can be restated as follows: If (1.1) does not have a nontrivial solution with the zero distribution (3.1) in $(a, \eta_2(a))$, then (1.1) does not have nontrivial solutions in $(a, \eta_2(a))$ with zero distributions $(n-1)-1$, $(n-2)-2, \cdots, (n-k)-k$, $(n-k-j)-(k+j), \cdots, 1-(n-1)$.

We shall see that this result provides a link between the zero distribution and the factorization of the differential operator L in (1.1).

Let $y_1, y_2, \cdots, y_n$ be n linearly independent solutions of (1.1) and define

$$W_k \equiv \begin{vmatrix} y_1 & y_2 & \cdots\cdots & y_k \\ y_1' & y_2' & \cdots\cdots & y_k' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(k-1)} & y_2^{(k-1)} & \cdots\cdots & y_k^{(k-1)} \end{vmatrix}, \quad k = 1, 2, \cdots, n.$$

It is well-known that $W_p > 0$ if and only if the operator L can be written as $L = L_1 L_2$, where L_1 and L_2 are nonsingular differential operators of order $n-p$ and p, respectively **[15]**. We require the following obvious extension of this result.

THEOREM 3.3. *Eq.* (1.1) *has* k *solutions* $y_1, y_2, \cdots, y_k$ *such that* $W_{k_1} > 0$, $W_{k_2} > 0, \cdots, W_{k_l} > 0$, $k_1 < k_2 < \cdots < k_l = k$, *if and only if the differential operator* L *in* (1.1) *can be written as the product of* $l+1$ *nonsingular differential operators, i.e.,* $L = L_{l+1}L_l \cdots L_1$, *where* L_1 *is of order* k_1, L_i *is of order* $k_i - k_{i-1}$, $i = 2, 3, \cdots, l$, *and* L_{l+1} *is of order* $n - k_l$.

Suppose (1.1) does not have a nontrivial solution with an $(n-p)-p$ distribution of zeros in (a, b). Let $y_1, y_2, \cdots, y_n$ be solutions of (1.1) such that $y_i^{(n-j)}(a+\epsilon) = \delta_{ij}$, $\epsilon > 0$, $i, j = 1, 2, \cdots, n$. Then $W_p > 0$ in $(a+\epsilon, b)$. Since $\epsilon > 0$ is arbitrary, we may assume that $W_p > 0$ in (a, b). Hence, we have $L = L_1 L_2$, where L_1 and L_2 are nonsingular differential operators of order $n-p$ and p, respectively.

Likewise, from Theorems 3.2, 3.3, and the above remark we deduce

THEOREM 3.4. *If* (1.1) *does not have a nontrivial solution with the zero distribution* (3.1) *in* $(a, \eta_2(a))$, *the differential operator* L *can be written as the product of nonsingular differential operators,*

$$L = L_{i+k+1} L_{i+k} \cdots L_1$$

in $(a, \eta_2(a))$, *where* L_m, $m \neq k+1$, *is of first order and* L_{k+1} *is of* jth *order.*

Let

$$\mathfrak{L} v = \sum_{k=0}^{n} q_k(\xi)\, v^{(k)} = 0$$

be the differential equation obtained from $Ly = 0$ through the change of variable $\xi = a + \eta_2(a) - x$. Clearly, $Ly = 0$ has a nontrivial solution with an $i_1 - i_2 - \cdots - i_k$ distribution of zeros in $(a, \eta_2(a))$ if and only if $\mathfrak{L} v = 0$ has a nontrivial solution with an $i_k - i_{k-1} - \cdots - i_1$ distribution of zeros in $(a, \eta_2(a))$. In particular, if $Ly = 0$ does not have a nontrivial solution with the zero distribution (3.1), then $\mathfrak{L} v = 0$ does not have a nontrivial solution with the zero distribution

$$\underbrace{1 - 1 - \cdots\cdots - 1}_{k} - j - \underbrace{1 - \cdots\cdots - 1}_{i}, \qquad i + j + k = n,$$

in $(a, \eta_2(a))$. Apply Theorem 3.4 to the nonsingular differential operator $\mathfrak{L}$: $\mathfrak{L}$ can be written as the product of nonsingular differential operators

$$\mathfrak{L} = \dot{\mathfrak{L}}_{i+k+1} \dot{\mathfrak{L}}_{i+k} \cdots \dot{\mathfrak{L}}_1 \tag{3.2}$$

in $(a, \eta_2(a))$, where $\dot{\mathfrak{L}}_p, p \neq i+1$, is of first order and $\dot{\mathfrak{L}}_{i+1}$ is of jth order. Transform the equation $\mathfrak{L}v = \dot{\mathfrak{L}}_{i+k+1}\dot{\mathfrak{L}}_{i+k} \cdots \dot{\mathfrak{L}}_1 v = 0$ back to $Ly = 0$ by substituting $x = a + \eta_2(a) - \xi$. Under this transformation each differential operator $\dot{\mathfrak{L}}_p, p = 1, 2, \cdots, i+k+1$, in (3.2) remains nonsingular. Moreover, the order of each $\dot{\mathfrak{L}}_p$ and the order in which these differential operators appear remain unchanged. We summarize this result in the following theorem.

THEOREM 3.5. *If* (1.1) *does not have a nontrivial solution with the zero distribution* (3.1) *in* $(a, \eta_2(a))$, *the differential operator* L *can be written as the product of nonsingular differential operators,* $L = \mathfrak{L}_{i+k+1}\mathfrak{L}_{i+k} \cdots \mathfrak{L}_1$, *in* $(a, \eta_2(a))$, *where* $\mathfrak{L}_p, p \neq i+1$, *is of first order and* $\mathfrak{L}_{i+1}$ *is of* jth *order.*

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