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TWO THEOREMS ON GROUPS OF CHARACTERISTIC 2-TYPE

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D. Gorenstein has made the following conjecture: suppose that G is a finite simple group which is simultaneously of characteristic 2-type and characteristic 3-type. Then G is isomorphic to one of $PSp(4, 3)$, $G_2(3)$ or $U_4(3)$. In this paper, we prove two results which, taken together, yield a proof of this conjecture under the additional assumption that G has 2-local 3-rank at least 2.

1. Introduction. In this paper we study finite simple groups, all of whose 2-local and 3-local subgroups are 2-constrained and 3-constrained respectively. The results we obtain are extensions of Thompson's theorem ES, and their relation to simple groups of characteristic 2-type is entirely analogous to the relation of theorem ES to simple N-groups.

The two Main Theorems are actually slight extensions of a conjecture of Gorenstein [10], and we refer the reader to [10] for a more detailed discussion of these ideas.

It will be convenient, before stating our main results, to develop some notation, most of which is standard.

Let X be a group, Y a subgroup of X, and π a set of primes. Then $\mathcal{M}_X(Y; \pi)$ denotes the set of Y-invariant π-subgroups of X. In particular, if the only Y-invariant π-subgroup of X is 1, we write $\mathcal{M}_X(Y; \pi) = \{1\}$.

For a finite group X, $\pi(X)$ is the set of prime divisors of $|X|$. As in [26], the subdivision of $\pi(X)$ into π_1, π_2, π_3 and π_4 will be important. We recall that $p \in \pi_3 \cup \pi_4$ if a S_p-subgroup P of G has a normal abelian subgroup of rank at least 3, which we write as $SCN_3(P) \neq \emptyset$. Moreover,

$$p \in \pi_3 \text{ if } SCN_3(P) \neq \emptyset \text{ and } \mathcal{M}_X(P; p') \neq \{1\}$$
$$p \in \pi_4 \text{ if } SCN_4(P) \neq \emptyset \text{ and } \mathcal{M}_X(P; p') = \{1\} .$$

If p is a prime, X a group, and P a S_p-subgroup of $O_{p',p}(X)$, we say that X is p-constrained if $C_X(P) \leqq O_{p',p}(X)$.

For p a prime and X a group, a p-local subgroup of X is the normalizer of some nonidentity p-subgroup of X.

We say that X is of characteristic p-type if $p \in \pi_4$ and every p-local subgroup of X is p-constrained.

With these definitions we can now state Gorenstein's conjecture: Suppose that G is a finite simple group, p an odd prime, and suppose further that G is simultaneously of characteristic 2-type and characteristic p-type. Then G is isomorphic to one of $G_2(3)$, $PSp(4, 3)$ or $U_4(3)$.

A little more notation is required to state our own results: if $p \in \pi_3 \cup \pi_4$, $\mathfrak{A}(p)$ denotes the set of elementary abelian subgroups of type (p, p) which are contained in elementary abelian subgroups of type (p, p, p).

Suppose that P is a S_p-subgroup of X and that $SCN_3(P) \neq \emptyset$. We write

$$p \in \pi_4^* \quad \text{if} \quad \mathsf{M}_X(P; 2) = \{1\} .$$

Hence we require that P should not normalize any nontrivial 2-group, though it may well normalize some proper $\{2, p\}'$-group.

We can finally state the main results of the present paper.

THEOREM 1. *Suppose that G is a finite simple group, p an odd prime, and that the following conditions hold:*

(a) *G is of characteristic 2-type.*

(b) *$p \in \pi_4^*$ and all p-local subgroups of G are p-constrained.*

(c) *Some 2-local subgroup of G contains an element of $\mathfrak{A}(p)$.*

Then $p = 3$ and G is isomorphic to $G_2(3)$, $PSp(4, 3)$ or $U_4(3)$; or $p = 5$ and G satisfies the conditions of part (c) *of Theorem* D *of* [19].

THEOREM 2. *Suppose that G is a finite simple group and that the following conditions hold:*

(a) *G is of characteristic 2-type.*

(b) *$3 \in \pi_4^*$ and all 3-local subgroups of G are 3-constrained.*

(c) *Some 2-local subgroup of G contains an elementary abelian subgroup of type* (3, 3).

Then G is isomorphic to $G_2(3)$, $PSp(4, 3)$ or $U_4(3)$.

A word about the hypotheses of Theorems 1 and 2. Evidently hypothesis (b) is a little weaker than requiring G to be of characteristic p-type (respectively, of characteristic 3-type). The point here is that the only relevant elements of $\mathsf{M}_G(P; p')$, for P a S_p-subgroup of G, are the 2-groups. It is condition (c) which provides an initial hold on the subgroup structure of G. The strength of this assumption has previously been demonstrated in [19], and we will make use of the results obtained there in the present paper. Of course, elimination of (c) in Theorem 1 would provide a complete proof of Gorenstein's conjecture.

Finally we remark that although Theorems 1 and 2 are of a very similar nature, the proofs of the two theorems are completely different. Indeed if G is a finite simple group of characteristic 2- and p-type, the truth of Gorenstein's conjecture would imply that $p = 3$. Under the assumptions of Theorem 1, the results of [19] already show that $p = 3$, but in Theorem 2 we must assume at the outset that $p = 3$. Moreover in Theorem 2 we have to account for the possibility that G has a 2-local subgroup containing an elementary abelian subgroup $V \cong (3, 3)$ with $V \notin \mathfrak{A}(3)$, and this causes difficulties. In view of Thompson's work in [4] and [26] this is not unexpected.

Finally, we emphasize that all groups considered in this paper are finite. Our notation is standard and usually follows that of [26]. We use the notation Z_n, D_n, Q_2n, SD_2n to denote the cyclic group of order n, dihedral group of order n, generalized quaternion group of order 2^n, and semidihedral group of order 2^n respectively. $A \wr B$ is the regular wreathed product of A with B, A^*B the central product of A and B, and $A \subsetneq B$ means A is isomorphic to a subgroup of B. Our notation for simple groups also follows [26], in particular A_n, $\sum_n$ are the alternating and symmetric groups of degree n respectively. The solvability of groups of odd order [4] is assumed throughout.

2. **Some preparatory lemmas.** In this section we collect together some results which we shall need in the sequal. Most of the results are already in the literature.

LEMMA 2.1 (*Generalized $P \times Q$-lemma*): *Suppose that P is a p-group, Q a q-group with p and q distinct primes and p odd. Suppose further that $Q \lhd PQ$ and that PQ normalizes the p-group M. Then $[Q, M] = 1$ if, and only if, $[Q, \Omega_1(C_M(P))] = 1$.*

Proof. This is contained in a result of Bender [2], and is a slight extension of a result of Thompson [9, Lemma 5.3.4] which we shall also need.

LEMMA 2.2. *P is a p-group which admits a fixed-point-free automorphism of order 3. Then P has class at most 2.*

Proof. This is an old result of Burnside. A proof can be found in [12, Theorem 8.1].

LEMMA 2.3. *G is a solvable group such that $O(G) = 1$ and $|G: O_2(G)| = 2.3^a$, $a \geqq 1$. Suppose that a S_3-subgroup R of G has a subgroup R_0 of order 3 such that R_0 is weakly closed in R and*

$C(R_0) \cap O_2(G) = 1$. *Then one of the following holds*:

(a) $G \cong \sum_4$

(b) $O_2(G)$ *is characteristic in a* S_2-*subgroup of* G.

Proof. Let $Q = Q_2(G)$ with T a S_2-subgroup of G, so that $|T: Q| = 2$. If $|Q| = 4$ then (a) obviously holds, so we may assume that $|Q| \geqq 8$. Now as $C_Q(R_0) = 1$ then Q has class at most 2 by Lemma 2.3. We will show that Q is the only subgroup of T of its isomorphism class, in which case (b) is immediate.

Now by the Frattini argument we have $G = QN_G(R)$, so because of the existence of R_0 we get $|N_G(R)| = 2|R|$, so $N_G(R) = R\langle t\rangle$ with $T = Q\langle t\rangle$ for some involution t. Since R_0 is weakly closed in R then $R_0 \lhd R\langle t\rangle$ so t either inverts or centralizes R_0. If t inverts R_0 then the desired result is proved by Higman in [12, Theorem 8.1], so we may assume that $[R_0, t] = 1$.

Suppose to begin with that Q is abelian. If there is a second subgroup of T which is isomorphic to Q then t must centralize a subgroup of index 2 in Q. As R_0 normalizes $C_Q(t)$ we get $Q \cap C(R_0) \neq 1$, a contradiction. Now suppose that Q has class 2. Higman shows in [12, Theorem 8.1] that every subgroup Q_0 of Q with $|Q: Q_0| = 2$ is such that $Q' = Q'_0$. If there is a subgroup $Q_1 < T$ with $Q_1 \cong Q$ and $Q_1 \neq Q$ then $Q_0 = Q_1 \cap Q$ has index 2 in Q, so we get $Q'_0 = Q'$. But $Q'_0 \leqq Q'_1 \cong Q'$, so $Q'_1 = Q'$. Finally, set $\bar{G} = G/Q'$. As $Q' \leqq \phi(Q)$ then $\bar{G}/\bar{Q}$ acts faithfully on $\bar{Q}$ and $\bar{Q}$ is abelian. Since $Q_1 = Q_0\langle tq\rangle$ for some $q \in Q$ and $\bar{Q}_1$ is abelian then $\bar{t}$ centralizes $\bar{Q}_0$. But $|\bar{Q}: \bar{Q}_0| = 2$, so $\bar{t}$ centralizes a subgroup of $\bar{Q}$ of index 2 and we obtain a contradiction as before. This completes the proof of Lemma 2.3.

The next result, though apparently of an elementary nature, requires deep results of Gorenstein and Gilman [5] and Walter [27] for its proof.

LEMMA 2.4. *G is a $3'$-group which admits an automorphism a of order 3 such that $C_G(a)$ has odd order. Then G is solvable of 2-length 1.*

Proof. Since $(|G|, |\langle a\rangle|) = 1$ then G has an $\langle a\rangle$-invariant S_2-subgroup T. By Lemma 2.2 T has class at most 2.

First we show that G is solvable, so suppose that this is not the case. Proceeding by induction, we may assume that G is characteristically simple, hence is the direct product of isomorphic groups $G_1, \cdots, G_r$. As $\langle a\rangle$ permutes the G_i among themselves we get $r = 1$ or 3 and if $r = 3$ and then $\langle a\rangle$ is transitive on $\{G_1, G_2, G_3\}$. But in this case $\langle a\rangle$ must centralize the involution $tt^a t^{a^2}$ whenever t is an involution of G_1, against the assumption that $C_G(a)$ has odd order.

So in fact G is simple. As T has class at most 2 we may identify G by the results of Gorenstein-Gilman and Walter mentioned above. As G is $3'$-group the only possibility $G \cong Sz(q)$ for some $q \geqq 8$. But every outer automorphism of $Sz(q)$ has a fixed-point subgroup of even order (see [25]), which contradiction completes the proof that G is solvable.

In proving that G has 2-length 1 we may assume that $O(G) = 1$ and try to prove that $T \lhd G$. As G is solvable it has an $\langle a \rangle$-invariant Hall $2'$-subroup H, and $H\langle a \rangle$ acts faithfully $Q = O_2(Q)$. Next we show $[a, H] = 1$, so suppose this is not the case and choose V to be an $\langle a \rangle$-invariant subgroup of H minimal subject to $[V, a] \neq 1$. As $V\langle a \rangle$ is faithful on Q, it is well known that $C_Q(a) \neq 1$, a contradiction. So V does not exist and hence $[H, a] = 1$. It follows that $Q\langle a \rangle \lhd G\langle a \rangle$, and hence $T = QN_T(\langle a \rangle)$. But $N_T(\langle a \rangle) = C_T(a) = 1$, so $Q = T$ as required.

LEMMA 2.5. *G is a simple group with a cyclic S_p-subgroup P, p an odd prime. Suppose that $C(P)$ has odd order and $|N(P): C(P)| = 2$. Then G has only one class of icvolutions.*

Proof: This is a result of Brauer's [3].

LEMMA 2.6. *G is a group, p an odd prime, $G = O^p(G)$, and P a S_p-subgroup of G. Suppose that P is non-cyclic and abelian, and that $|N(P): C(P)| = 2$. Then G is p-solvable.*

Proof. This is a recent result of Smith and Tyrer [24].

LEMMA 2.7. *G is a simple group of characteristic 2-type. Then $O(N) = 1$ for each 2-local subgroup N of G.*

Proof. This is a well-known result of Gorenstein [7] which we will frequently use without specific reference to it.

LEMMA 2.8. *G is a simple group of characteristic 2-type with a maximal 2-local subgroup N such that $O_2(N)$ is of sympletic type. Then G has a non-solvable 2-local subgroup.*

Proof. This is contained in a result of Lundgren [20]. (Observe that $O(N) = 1$ by Lemma 2.7.)

LEMMA 2.9. *G is a simple group such that $2 \in \pi_3 \cup \pi_4$ and $C_G(x)$ is solvable of 2-length 1 for each involution x of G. Then G is isomorphic to one of the following groups: $L_2(q)$, $Sz(q)$ or $U_3(q)$ for $q = 2^n \geqq 8$.*

Proof. This is a combination of results of Bender, Goldschmidt and Suzuki. The result is discussed in [8].

LEMMA 2.10. *G is a simple group and T a S_2-subgroup of G. Suppose that T has an abelian subgroup of index at most 2. Then the following hold*:

(a) *T is either abelian or isomorphic to D_2n, SD_2n or $Z_2n \wr Z_2$.*

(b) *G is isomorphic to one of the following groups*: *$L_2(q)$, $q \geqq 4$, $L_3(q)$ or $U_3(q)$, $q \geqq 3$, a group of Ree-type, A_7, M_{11} or J_1.*

Proof. This is the combination of the work of a number of authors. For a fuller discussion of the result we refer the reader to [21].

3. The Proof of Theorem 1. In this section we will present a proof of Theorem 1. So for the balance of this section G will denote a simple group satisfying the hypotheses of Theorem 1, and $p \neq 5$.

We have already made an initial investigation of the consequences of the hypotheses of Theorem 1 in some joint work with Klinger [19]. We obtained there the following result which represents the first major reduction in the proof of Theorem 1.

PROPOSITION 3.1. *Under the assumptions of Theorem 1, the following conditions hond.*

(a) $p = 3$

(b) *No 2-local subgroup of G contains an elementary abelian subgroup of type* $(3, 3, 3)$.

(c) *We can choose $B \in \mathfrak{A}(3)$ and a maximal element F of $\mathsf{M}(B; 2)$ such that F is extra-special of width $w \leqq 4$. Moreover F is the central product of w B-invariant quaternion subgroups. We have $Z = Z(F) = C_F(B)$, and $C_F(B_0)$ has rank 1 for $1 < B_0 < B$.*

We shall retain the notation of Proposition 3.1 throughout. Moreover we set $\hat{B} = \{B_0 \mid 1 < B_0 < B$ and $C_F(B_0) \neq Z\}$. Thus $C_F(B_0) \cong Q_8$ for $B_0 \in \hat{B}$, $|\hat{B}| = w$, and if $\hat{B} = \{B_1, \cdots, B_w\}$ with $C_F(B_i) = Q_i$, $1 \leqq i \leqq w$, then $F = Q_1^* \cdots {}^*Q_w$. We also set $Z = \langle z \rangle$.

We also obtained in [19] the following useful result.

LEMMA 3.2. *Suppose that $D \in \mathfrak{A}(3)$, $H \in \mathsf{M}(D; 2)$, and $C_H(D) \neq 1$. Then H is of symplectic-type.*

LEMMA 3.3. *Let x be an involution of G. Then exactly one of the following holds.*

(a) *$C_G(x)$ has cyclic S_3-subgroups.*
(b) *$x \sim z$ in G.*

Proof. Suppose that (a) is false, in which case $C(x)$ contains a noncyclic elementary 3-subgroup D. By Proposition 3.1(b) we have $|D| = 9$. Suppose to begin with that $D \in \mathfrak{A}(3)$. Then if $\langle x \rangle \leqq F_0 \in \mathsf{M}^*(D; 2)$ we get that F_0 is of symplectic-type by Lemma 3.2, in particular $\langle x \rangle = \Omega_1(Z(F_0))$. But if T is a S_2-subgroup of G containing F_0 we must have, since $O_2(N(F_0)) = F_0$, that $Z(T) < F_0$. Hence $\langle x \rangle = \Omega_1(Z(T))$ and so $x \sim z$.

Finally, suppose that $D \notin \mathfrak{A}(3)$. We will show that this case cannot occur: namely, if $D \notin \mathfrak{A}(3)$ then D contains every element of order 3 in $C(D)$, in particular if R is a S_3-subgroup of G containing D then $Z_0 = \Omega_1(Z(R)) < D$. Set $L = C_G(Z_0) > R$. As $SCN_3(R) \neq \emptyset$ then $O_{3'}(L)$ has odd order, so x acts faithfully $O_{3',3}(L)/O_{3'}(L)$. Set $P = O_{3',3}(L)/O_{3'}(L)$. $\langle x \rangle \times D$ acts on P and we have $[x, \Omega_1(C_P(D))] \leqq [x, D] = 1$, hence $[x, C_P(D)] = 1$, hence $[x, P] = 1$ by the $P \times Q$-lemma. This contradiction proves the lemma.

We now set $C = C_G(Z)$. T will denote some fixed S_2-subgroup of C, and R is a S_3-subgroup of C which contains B. We collect the facts we shall need about C in the following lemma:

LEMMA 3.4. *The following conditions hold:*
(a) *$C = N(F)$, and C is a maximal 2-local subgroup of G.*
(b) *T is a S_2-subgroup of G and $Z = Z(T)$.*
(c) *Either $R = B$ is elementary of order 9 or one of the following holds:*
(i) *$w = 3$ and R is non-abelian of order 27.*
(ii) *$w = 4$ and R is non-abelian metacyclic of order 27.*
(d) *$\mathsf{M}^*_C(B; 3') = \{F\}$.*

Proof. We have $|Z| = 2$, so clearly $N(F) \leqq C$. On the other hand if $F_0 = O_2(C)$ we must have $F_0 \leqq F$ since otherwise we get $F < F_0F \in \mathsf{M}(B; 2)$, against the maximality of F. Now as C is 2-constrained we must have $Z < F_0$, hence $C_{F_0}(B) \neq 1$ and F_0 is of symplectic-type by Lemma 3.2. We deduce that F_0 is a product of the subgroups $Q_1, \cdots, Q_w$. If $F_0 \neq F$ then some Q_i, say Q_1, satisfies $Q_1 \cap F_0 = Z$. But then $[Q_1, F_0] = 1$, against the 2-constraint of C, and so we must have $F_0 = F$ and $N(F) = C$. The same proof now yields that C is also a maximal 2-local subgroup of G, so (a) is proved. (b) is a straight forward consequence of (a).

As for (c), suppose that $B < R$. In this case, we must have $w = 3$ or 4, so assume to begin with that $w = 3$. Hence, we get $C/F \subsetneq \mathrm{Out.}(F) \cong O_6^-(2)$, so $R \subsetneq Z_3 \wr Z_3$. Since R has rank 2 by

Proposition 3.1(b) we get that R is non-abelian of order 27. Now suppose that $w = 4$ and assume without loss that $B_1 \leqq B \cap Z(R)$. Hence R normalizes $Q_1 = C_F(B_1)$ and $C_R(Q_1) = R_1$ is a cyclic subgroup of R index 3, in particular R is metacyclic. Moreover R_1 is faithful on $Q_2^* Q_3^* Q_4$ so $R_1 \subsetneq Z_3 \wr Z_3$ and $|R_1| = 9$. This completes the proof of (c).

Now we certainly have $\{F\} = \mathcal{N}_C^*(B; 2)$ by part (a), so to prove (d) it suffices to show that $\mathcal{N}_C(B; q) = \{1\}$ for each prime $q \in \{2, 3\}'$. We get, since $q \geqq 5$, that $[C_Q(B), Q_i] = 1$ for $1 \leqq i \leqq w$, hence $[C_Q(B), F] = 1$, hence $C_Q(B) = 1$ by 2-constraint. Now if $w = 3$ we get $|Q| = 5$ and hence $[Q, B] = 1$, a contradiction. Thus $w = 4$. Now choose $b \in B^\#$ with $C_Q(b) \neq 1$. We have $C_F(b) \cong Q_8$ so $[C_Q(b), C_F(b)] = 1$, hence $|C_Q(b)| = 5$ and we get $[C_Q(b), B] = 1$ which is again impossible. This completes all parts of the lemma.

As a corollary of Lemmas 3.4(a) and 2.8 we obtain immeditely

LEMMA 3.5. *G has a non-solvable 2-local subgroup.*

Next we prove

LEMMA 3.6. *One of the following occurs.*

(a) *C is solvable.*

(b) *$w = 4$ and C/F is isomorphic to a subgroup of* $\mathrm{Aut}(A_6)$ *containg* $\mathrm{Inn.}(A_6)$.

Proof. Suppose that C is nonsolvable. Then we obviously have $w \geqq 3$. Suppose that $w = 3$, and let E/F be a minimal normal subgroup of C/F. By Lemma 3.4(d) E has order divisible by 3, so as all 3-local subgroups of $O_6^-(2) \cong \mathrm{Out.}(F)$ are solvable whilst C is nonsolvable we must have that E/F is simple. Now $O_6^-(2)$ has 3-rank 3, whilst C has 3-rank 2 by Proposition 3.1(b). We deduce that $E/F \not\cong O_6^-(2)$, so the only possibilities are $E/F \cong A_5$ or A_6. But if $E/F \cong A_5$ some element of $B^\#$ centralizes E/F, an impossibility. Suppose that $E/F \cong A_6$. In this case B is a S_3-subgroup of E and there is a cyclic subgroup of order 4 in E/F normalizing and acting irreducibly on B. On the other hand there is exactly one subgroup of B of order 3 not in $\hat{B}$, so $N_E(B)$ cannot act irreducibly on B. This contradiction establishes that fact that $w = 4$ if C is nonsolvable.

Again let E/F be a minimal normal subgroup of C/F, where we are now assuming that $w = 4$. As before E/F is either simple or a 3-group. If the latter case occurs then we can choose $b \in (E \cap B)^\#$. Setting $\bar{C} = C/F$ we deduce that $C_{\bar{C}}(\bar{b})$ is nonsolvable and that $C_{\bar{C}}(\bar{b})$ has a normal subgroup $\bar{A} \cong A_5$. But then $\bar{B}$ normalizes a S_3-subgroup of $\bar{A}$, against $F \in \mathcal{N}^*(B; 2)$, so we have shown that E/F is simple.

Next, let S be a S_7-subgroup of E and suppose that $S \neq 1$. Thus $|S| = 7$, and a simple computation shows that $C_F(S) \cong D_8$. Now as no 3-element of $\bar{C}^{\#}$ contralizes a four-group of F and as $O_8^+(2)$ has no elements of order 35 we deduce that $N_C(S)$ is a $\{2, 7\}$-group. A Frattini argument now yields $R < E$, so $R = B$ by Lemma 3.4(c) and a theorem of Huppert [14]. Sylow's theorem now tells us that $|E: F| = 2^a 3^2 7$ or $2^a 3^2 5^2 7$ for some a. However, $O_8^+(2)$ has no such simple subgroup (with elementary S_3-subgroups), so we have shown that E is a $7'$-group. Being simple E/F is a $\{2, 3, 5\}$-group, and surveying the possibilities we find that $E/F \cong A_5$ or A_6. However if $E/F \cong A_5$ then some element of $B^{\#}$ contralizes E/F, and we have already shown that this cannot occur. Hence $E/F \cong A_6$, and the conclusions of Lemma 3.6 now follows easily.

LEMMA 3.7. *Suppose that* $w = 3$ *and that* B_0 *is the subgroup of* B *of order* 3 *satisfying* $B_0 \notin \hat{B}$. *Then the following hold*:

(a) $C_C(B_0)$ *has cyclic* S_2-*subgroups*.

(b) C *contains a* S_2-*subgroup of* $C_G(B_0)$.

Proof. Let I be a S_2-subgroup of $C_C(B_0)$. Thus $Z \leqq I$. Since $B_0 \notin \hat{B}$ we have $I \cap F = Z$. Now if (a) is false then there is an involution $x \in I - Z$; x must invert a subgroup of $\hat{B}$ so we may suppose that x inverts B_1. Thus $\langle x \rangle \times B_0$ normalizes $Q_1 = C_F(B_1)$, so $[x, Q_1] = 1$ since B_0 is faithful on Q_1.

Next, suppose that either x or xz (say x for definiteness) is conjugate to z, and set $L = C_G(x)$. We have $\langle x \rangle = \phi(O_2(L))$ and $B_0 Q_1 < L$. Since $L \cong C$ and $Q_1 = [Q_1, B_0]$ we get $Q_1 < O_2(L)$, hence $Z = \phi(Q_1) = \langle x \rangle$. This is false, so we deduce that neither x or xz is conjugate to z.

Finally consider $N = N_G(B_1) > B_1(Q_1 \times \langle x \rangle)$. If P is an $\langle x, z \rangle$-invariant S_3-subgroup of $O_{3',3}(N)$ then $P = \langle C_P(y) \mid y \in \langle x, z \rangle^{\#} \rangle$. But by Lemma 3.3 and the provious paragraph we get that both $C_P(x)$ and $C_P(xz)$ are cyclic, hence are both centralized by z. But then z centralizes P, contradiction. This proves part (a) of the lemma, and (b) is a straightforward consequence of it.

LEMMA 3.8. *Suppose* $w \geqq 3$. *Then* C *controls fusion of its subgroups of order* 3.

Proof. As a consequence of Lemma 3.7(b) we get (with the notation of that lemma) that B_0 is conjugate to no element of $\hat{B}$ in case $w = 3$. Now suppose that B_5, B_6 are two subgroups of R of order 3 with $B_5 \sim B_6$ in G. It follows that $C_F(B_i)$ is a quaternion group, $i = 5, 6$.

Let $Q_i = C_F(B_i)$ with I_i a S_2-subgroup of $C_C(B_i)$, $i = 5, 6$. Thus $Q_i \trianglelefteq I_i$. A simple computation shows that I_i is either quaternion or semidihedral (of order 16), so in any case we have $Z = Z(I_i)$ and I_i is a S_2-subgroup of $C_G(B_i)$, $i = 5, 6$. As $B_5 \sim B_6$ we can choose $g \in G$ satisfying $B_5^g = B_6$, $I_5^g = I_6$ by Sylow's theorem. Hence $Z^g = Z$, that is $g \in C$, as required.

LEMMA 3.9. *Suppose $w \geqq 3$ and x is a involution with $x \in C - F$, Then $C_C(x)$ is a 2-group.*

Proof. We only need show, after Lemmas 3.4 and 3.6, that x centralizes no nontrivial 3-subgroup of C, so suppose that this is false. By Lemma 3.7 we can assume that x centralizes B_1. We calculate that $\langle x, Q_1\rangle$ is semidihedral of order 16.

Now set $N = N_G(B_1)$, and let P be a $\langle x, Q_1\rangle$-invariant S_3-subgroup of $O_{3',3}(N)$. Thus $P = \langle C_P(y) \mid y \in \langle x, z\rangle^{\#}\rangle$. If $x \not\sim z$ in G then $C_P(x) = C_P(xz)$ is cyclic by Lemma 3.3, hence $B_1 = \Omega_1(C_P(x)) = \Omega_1(C_P(xz))$, hence z centralizes P, contradiction. It follows that $x \sim xz \sim z$ in G. Setting $L = C_G(x)$ and $x = z^g$, we get $B_1, B_1^g < L$, so $B_1 = B_1^{g1}$ for some $1 \in L$ by Lemma 3.8. We have thus shown that $x \sim xz \sim z$ in N.

Finally, since $C_P(Z)$ admits Q_1 we must have $B_1 = \Omega_1(C_P(Z))$, so $B_1 = \Omega_1(C_P(y))$ for any $y \in \langle x, z\rangle^{\#}$. This is absurd, and the lemma is proved.

LEMMA 3.10. *The following conditions hold.*

(a) *If $B_i \in \hat{B}$ then $C_F(B_i)$ is a S_2-subgroup of $C_C(B_i)$.*

(b) *If C is solvable and $w \geqq 3$ then C/F has cyclic S_2-subgroups.*

(c) *If C is nonsolvable then C/F has 2-rank 2.*

Proof. To prove (a), let $B_i \in \hat{B}$ with $Q_i = C_F(B_i)$ and suppose that Q_i is not *a* S_2-subgroup of $C_C(B_i)$. Let I_i be a S_2-subgroup of $C_C(B_i)$ which contains Q_i. By Lemma 3.9 I_i is generalized quaternion of order at least 16, so $C_C(B_i)$ has a normal 2-complement. But then we get $[B, Q_i] \leqq Q_i \cap O(C_C(B_i)) = 1$, a contradiction which proves (a).

(b) is a simple consequence of (a) together with Lemma 3.7(a).

Finally, suppose that C is nonsolvable and that C/F has 2-rank at least 3. By Lemma 3.6 we must have that C/F contains a subgroup isomorphic to $\sum_6$, and that $w = 4$. But $\sum_6$ has elements of order 6 and every subgroup of B of order 3 lies $\hat{B}$ as $w = 4$. This contradicts (a), and the lemma is proved.

LEMMA 3.11. *Suppose that $w \geqq 3$. Then Z is weakly closed in F.*

Proof. Suppose false. Then there is $g \in G - C$ such that $Z^g = Y < F$. Set $C^g = L = C_G(Y)$ with $D = O_2(L)$. Now $C_F(Y) = Y \times F_0$ where F_0 is extra-special of width $w - 1$ and $F_0 < L$. It is a simple consequence of Lemma 3.10 that $F_0 \cap D \neq 1$, so $Z < D$, so F_0D/D is elementary abelian. By Lemma 3.10 again, we deduce that $|F_0: F_0 \cap D| \leqq 4$, and $|F_0: F_0 \cap D| \leqq 2$ if $w = 3$ (for C is solvable in this case). Thus $F_0 \cap D$ is not elementary abelian, so we obtain the contradiction $Z = \phi(F_0 \cap D) = \phi(D) = Y$. This proves the lemma.

It is now easy to prove

PROPOSITION 3.2. *The case $w = 3$ cannot occur.*

Proof. Let N be a nonsolvable 2-local subgroup of G, the existence of which is guaranteed by Lemma 3.5, and set $D = O_2(N)$ with N_2 a S_2-subgroup of N. As $w = 3$ then C is solvable by Lemma 3.6, so $N \neq C$. Lemma 3.4(b) allows us to assume that $N_2 < C$, in which case $Z < Z(D)$.

Let $V = \Omega_1(Z(D))$, so that $Z < V \lhd N$. As $w = 3$ Lemma 3.10(b) shows that a S_2-subgroup of C has rank at most 4, in particular $|V| \leqq 16$. So if $\mathfrak{X}$ is the N-orbit of $V^\#$ which contains z then $|\mathfrak{X}| \leqq 9$, as follows from Lemma 3.11.

Now we have $|N| = |\mathfrak{X}||C_N(Z)|$. Since $C_N(Z)$ is a $\{2, 3\}$-group, N is nonsolvable, and $|\mathfrak{X}| \leqq 9$, we deduce that $|\mathfrak{X}| = 5$ or 7. In either case there is a subgroup $J < N$ of order 3 centralizing Z, so $J < C$. Since $V = C_V(J)[V, J]$, Lemmas 3.9 and 3.10 yield $V < F$, so $\mathfrak{X} \subseteqq F$ against Lemma 3.11. This completes the proof of Proposition 3.2.

PROPOSITION 3.3. *The case $w = 4$ cannot occur.*

Proof. Our proof in this case is a little different to that of Proposition 3.2, since if C is nonsolvable, then Lemma 3.5 is of no help. We break the proof into a number of steps. We start with

(1) Suppose that x is an involution of $C - F$, and that $x \in O^2(C)$ if C is nonsolvable. Then $C_F(x)$ is elementary of order 16.

Observe that since $w = 4$ then every subgroup of order 3 in B lies in $\hat{B}$. Now as $x \notin O_2(C)$ then x inverts a subgroup of C of prime order by the Baer-Suzuki theorem [1]. If C is solvable such a subgroup must have order 3 by Lemma 3.4(c). If C is not solvable then in $\bar{C} = C/F$, $\bar{x}$ must invert a subgroup of order 3 by the structure of A_6. Let B_0 be a subgroup of C such that $\bar{x}$ inverts $\bar{B}_0$. Thus $H = FB_0\langle x \rangle$ is a group. If H is 2-closed we get that $|C_H(B_0)|_2 = 16$, so $C_F(B_0)$ is not a S_2-subgroup of $C_C(B_0)$, against Lemma 3.10(a). So H is not 2-closed, hence $x \notin O_2(H)$, hence x inverts a subgroup of H of order 3. So in any case x inverts a subgroup of C of order 3, and we may assume that such a subgroup is $B_1 \in \hat{B}$.

The same argument also shows that x inverts a subgroup of order 3 in $C_c(B_1)/B_1$, so that x inverts an elementary subgroup of C of order 9. We may therefore assume that x inverts B. Hence x normalizes Q_i and $\langle Q_i, x\rangle \cong SD_{16}$ for $1 \leqq i \leqq 4$. A simple computation now proves (1).

(2) Let y be an involution of $F - Z$ such that C contains a S_2-subgroup of $C_G(y)$. Then $C_G(y) < C$.

Set $Y = \langle y\rangle$, $L = C_G(Y)$, $D = O_2(L)$. Since $D < C$ we get $Z < V = \Omega_1(Z(D))$. Proceeding under the assumption that $L \not< C$ we get $Z \not< L$, so if $\mathfrak{X}$ is the L-orbit of $V^{\#}$ which contains z we get that $|\mathfrak{X}|$ is odd and $|\mathfrak{X}| \geqq 3$. By Lemma 3.11 we get $\mathfrak{X} \cap F = \{z\}$.

We claim next that $D \cap F$ is elementary abelian. If C is solvable or C is nonsolvable and $(\mathfrak{X} - \{z\}) \cap O^2(C) \neq \emptyset$, this is an immediate consequence of step (1), so we may suppose that neither of these conditions hold. Choose $x \in \mathfrak{X} - \{z\}$, Thus by Lemma 3.6(b) and 3.10(c) we have $C = O^2(C)\langle x\rangle$. In this case we have that $x \notin \phi(K)$ whenever K is a 2-subgroup of C containg x, in particular $x \notin \phi(D)$. As $x \sim z$ in L then also $z \notin \phi(D)$, so $z \notin \phi(D \cap F)$ so $D \cap F$ is elementary abelian, as required.

Now $C_F(Y) = Y \times F_0$ where F_0 is extra-special of width 3. By the last paragraph $D \cap F_0$ is elementary, so $|F_0 : D \cap F_0| \geqq 8$, so L/D has 2-rank at least 3.

Next we show that $C_L(Z)$ is a 2-group. If false, the structure of C yields only one possibility, namely that $C_L(Z)$ is a $\{2, 5\}$-group and C is nonsolvable. Let K be a S_5-subgroup of $C_L(Z)$, so that $V = C_V(K) \times [V, K]$. Now it is a simple consequence of step (1) that $|V| \leqq 2^6$, so if $[V, K] \neq 1$ then $|[V, K]| = 2^4$, $C_V(K) = \langle Y, Z\rangle$, so $V < F$. This is false, so $[V, K] = 1$ and $K < C_L(V) \lhd L$. However $[F_0, K]$ is nonabelian and $[F_0, K] \leqq O_2(C_L(V)) = O_2(L)$. This is impossible as $D \cap F$ is elementary, so we have shown that $C_L(Z)$ is a 2-group.

Since $|L| = |\mathfrak{X}|\,|C_L(Z)|$ we deduce that $|\mathfrak{X}| = |L|_{2'} < 63$. As $3.5.7 > 63$ it follows that $\pi(L)$ contains at most 3 distinct primes. Next, suppose that L is solvable. As L/D has 2-rank at least 3 and L has cyclic S_3-subgroups by Lemma 3.3, we gain a simple contradiction using Lemma 5.34 of [26]. So L is nonsolvable. Hence L/D is a nonsolvable $\{2, 3, p\}$-group with cyclic S_3-subgroups, 2-rank at least 3, $O_2(L/D) = 1$, $|L : D|_{2'} < 63$, and $L/D \subsetneq GL(5, 2)$. There are no such groups, and step (2) is proved.

(3) If y is an involution of $F - Z$ then $C_G(y) < C$. If C contains a S_2-subgroup of $C_G(y)$ we are done by step (2). In any case by Sylow's theorem y is conjugate to an involution x of C such that C contains a S_2-subgroup of $C_G(x)$. Suppose that $x^g = y$ for

some $g \in G$. Evidently $x \in F$, so we get $C(y) < C^g$ by step 2. Now if $C_F(y) = \langle y \rangle \times F_0$ then F_0 is extra-special of width 3. By the structure of C^g we get $F_0 \cap O_2(C^g) \neq 1$, so $Z < O_2(C^g)$, so $Z = Z(O_2(C^g))$ by Lemma 3.11. Hence $C^g = C$, and step (3) follows.

(4) Z is weakly closed in C with respect to G. For suppose that $g \in G$ and $Z^g < C$, $Z^g = Y \neq Z$. Set $L = C_G(Y)$, $D = O_2(L)$. A straightforward calculation shows that $F \cap D \neq 1$, so let f be an involution with $f \in F \cap D$. By step 3 we get $C_G(f) \leqq L$, in particular $C_F(f) < L$. This leads to $Z < D$, so $Z = Z(O_2(D)) = Y$ by Lemma 3.11. This is a contradiction, so (4) is proved.

Finally, since C contains a S_2-subgroup of G, step 4 and the Z^*-theorem [6] yield $G = O(G)C$. As $2 \in \pi_4$ then $O(G) = 1$, so $Z \lhd G$. But then a S_3-subgroup of G normalizes a nontrivial 2-group, contradiction. Proposition 3.3 is thus proved.

Thus in order to proved Theorem 1, we are reduced to studying the case $w = 2$. In this case a simple computation shows that the S_2-subgroup T of G has section 2-rank at most 4, hence the theorem is a consequence of the results in Part II of [11]. However, we can avoid direct appeal to this important result by making use of prior characterizations of the groups $G_2(3)$, $PSp(4, 3)$ and $U_4(3)$ by the centralizers of their central involutions.

We retain our previous notation, so that $F = Q_1^* Q_2$ with $Q_i \cong Q_8$ and $B_i = C_B(Q_i)$, $i = 1, 2$. We set $Z = Z(F)$ with $C = C(Z) = N(F)$. As $w = 2$ then B is a S_3-subgroup of C and moreover $C/F \subsetneq \sum_3 \wr Z_2$, in particular $C = TB$ and $T/F \subsetneq D_8$.

Now as a simple consequence of the Z^*-theorem [6] we cannot have $T = F$. Because $SCN_3(T) \neq \emptyset$ we also calculate that $T/F \not\cong D_8$ and $T/F \not\cong Z_4$. Hence $|T: F| = 2^n$ and T/F is elementary abelian, $n = 1$ or 2.

LEMMA 3.12. *Suppose that A is an elementary abelian subgroup of T with $A \lhd T$ and $|A| = 8$. Then $A < F$.*

Proof. Since $A \lhd T$ then $Z \leqq A_0 = A \cap F$. Suppose that $Z = A_0$. Then $[A, F] \leqq Z$ so A stabilizes the chain: $F \rhd Z \rhd 1$, so $A < F$, a contradiction. Now suppose that $|A_0| = 4$ and choose $a \in A - A_0$. As a normalizes F it either fixes Q_1 and Q_2 or interchanges them. If $Q_i^a = Q_i$, $i = 1, 2$, we get $[Q_i, a] \leqq Q_i \cap A = Z$, so a stabilizes and hence $a \in F$, contradiction. On the other hand if $Q_1^a = Q_2$ we get $|[F, \langle a \rangle]| = 8$ against $]F, \langle a \rangle] \leqq A_0$. The lemma is thus proved.

Now by Lemma 2.8 G has a nonsolvable 2-local subgroup N. Lemma 3.4(b) allows us to assume that T contains a S_2-subgroup of N. Set $D = O_2(N)$, $V = \Omega_1(Z(D))$, and retain this notation for the remainder of this section. As T has 2-rank at most 4 than $|V| = 8$ or 16. We consider these two possibilities separately. First we have

PROPOSITION 3.4. *If* $|V| = 8$ *then* $G \cong G_2(3)$.

Proof. Since $C(D) = Z(D)$ we get $Z < V$, and a simple computation yields $C(V) = D$. As N is not solvable we get $N/D \cong L_3(2)$, so N has a Frobenius subgroup K of order 21 transitive on $V^{\#}$. Thus K has a subgroup J of order 3 such that $Z = C_V(J)$. It follows that $J < C$ and that $V = Z \times [V, J] < F$.

Now F has exactly 6 elementary subgroups of order 8, falling into two conjugacy classes of length 3 under the action of C. It follows that $N_C(V)$ contains a S_2-subgroup of C, hence $T < N$. Thus, $|T| = 8|D|$. As $|T| \leqq 2^7$ then $|D| \leqq 2^4$ and so D is abelian. Evidently D cannot be nonelementary, so in fact $|D| = 8$, $V = D$, and $|T| = 2^6$.

Since $F < N$, $F \not< D$, there are involutions in $N - D$, so there is an involution $x \in N - D$ such thst $DJ\langle x\rangle$ is a group with $DJ\langle x\rangle/D \cong \sum_3$. Hence we may assume that x inverts J, in which case x centralizes Z. As $x \in C$ and $x \notin F$ we get $T \cong F\langle x\rangle$ and $C = FB\langle x\rangle$. Now we may assume that $J < B$. The structure of N yields $J \in \hat{B}$ (that is, $C_F(J) = Z$), so x normalizes $B = O_3(C(J) \cap FB)$. Let J_1 be an $\langle x\rangle$-invariant complement to J in B.

If x centralizes J_1, then x must interchange Q_1 and Q_2. In this case $F\langle x\rangle$ contains elementary abelian subgroups of order 16, and some such elementary subgroup of C must contain V. But as $V = C(V)$ this is impossible. Hence, x inverts J_1. Now we check that C is isomorphic to the centralizer of a (central) involution of $G_2(3)$. By a theorem of Janko [15] we get $G \cong G_2(3)$ as required.

PROPOSITION 3.5. *If* $|V| = 16$ *and* $|T| = 2^6$ *then* $G \cong PSp(3, 4)$.

Proof. As F has 2-rank 3 we get $F \cap V < V$, so if $v \in V - F$ then $T = F\langle v\rangle$ and $C = FB\langle v\rangle$. Since v centralizes an elementary abelian subgroup of F of order 8 (namely $F \cap V$) then v must interchange Q_1 and Q_2.

Hence C is isomorphic to the centralizer of a central involution of $PSp(4, 3)$. By a theorem of Janko [16] we get $G \cong PSp(4, 3)$ as required.

PROPOSITION 3.6. *If* $|V| = 16$ *and* $|T| = 2^7$ *then* $G \cong U_4(3)$.

Proof. As $V < T$ a simple computation proves that in fact $V \lhd T$ so that T is a S_2-subgroup of N. Moreover $C_T(V) = V$, hence $V = C(V)$, $V = D$, and $T/D \cong D_8$.

As in Proposition 3.5 we can choose $v \in V - F$. Then v interchanges Q_1 and Q_2 and centralizes a subgroup J of C of order 3. We may assume that $J < B$ in which case $J \not\in \hat{B}$. Moreover J normalizes $V \cap F = C_F(v)$, hence $J < N$. Let U be a S_2-subgroup of $N_C(J)$, so that U contains $\langle v, z \rangle$ as a subgroup of index 2. We next show that U is either elementary abelian or dihedral.

Since N/D is nonsolvable with a dihedral S_2-subgroup of order 8 then one of the following occurs: $N/D \cong L_2(7)$, $N/D \cong A_6$, or N/D contains a subgroup isomorphic to $\sum_5$. Suppose to begin with that the latter case occurs. By Lemma 2.6 of [11, Part II] the extension N/D splits, so $N = DH$ with $H \cong \sum_5$. Hence, $N \cap C = D(H \cap C)$ with $H \cap C \cong \sum_4$. Now J is a S_3-subgroup of $N \cap C$, hence is conjugate to a S_3-subgroup of $H \cap C$, hence is inverted by a involution x of $N \cap C$. We thus get that $U \cong \langle v, z, x \rangle$ has the desired isomorphism type in this case. Now suppose that N/D is isomorphic to either $L_2(7)$ or A_6. If the A_6 case occurs then J must correspond to (123) (456) since $C_V(J) \neq 1$ (c.f. [13, p. 157] so in either case $N_N(J)$ has S_2-subgroups of order 8. Moreover, since $F < N$, $F \not< D$ there are involutions in $N - D$, so there is an involution $x \in N - D$ which inverts J. Hence, $W = \langle v, z, x \rangle$ is a S_2-subgroup of $N_N(J)$. But N contains a S_2-subgroup of $N_C(J)$, so $U \cong W$ and again U has the required isomorphism type.

Suppose to begin with that U is elementary of order 8. Then we find, since $C = FBU$, that the extension C/F splits. Following a paper of Phan [22] we check that C is isomorphic to the centralizer of a central involution of $L_4(3)$. Phan goes on to show that G has a second class of involutions with nonconstrained centralizers. As G is of characteristic 2-type this cannot occur, so we deduce that U is not abelian.

Finally, suppose that $U \cong D_8$. Now we check that C is isomorphic to the centralizer of an involution in $U_4(3)$. A second result of Phan [23] yields $G \cong U_4(3)$ as required. This completes the proof of Theorem 1.

4. The proof of Theorem 2. In this section we will present a proof of Theorem 2. In a sense our proof is unsatisfactory: for one thing we must assume at the outset that the relevant odd prime p for which the p-locals are p-constrained is 3. Moreover our proof utilizes several deep characterization theorems whose relevance, at least superficially, would appear to be small.

From now on, we use the following notation: G is a finite simple group of characteristic 2-type and R is a S_3-subgroup of G. We assume that all 3-local subgroups of G are 3-constrained and that

$И(R; 2) = \{1\}$. B is an elementary subgroup of R of order 9 such that $И(B; 2) \neq \{1\}$.

Now if some 2-local subgroup of G contains an element of $\mathfrak{A}(3)$ then by Theorem 1 we have $G \cong G_2(3)$, $PSp(4, 3)$, or $U_4(3)$ as required. So in trying to prove Theorem 2 we may, and shall, assume

(*) No 2-local subgroup of G contains an element of $\mathfrak{A}(3)$. We will eventually show that (*) leads to a contradiction, in which case Theorem 2 will be proved.

Our first lemma gives a number of properties of B which we shall use in the sequel. First observe, since $B \notin \mathfrak{A}(3)$, that $Z(R)$ is cyclic and that $Z = \Omega_1(Z(R)) < B$. We fix this notation for the remainder of the paper.

LEMMA 4.1. *The following conditions hold.*

(a) *If* $F \in И(B; 2)$ *then* $C_F(Z) = 1$.

(b) *If* $F \in И(B; 2)$ *then* F *has class ot most* 2.

(c) $C(B)$ *has odd order.*

(d) Z *is weakly closed in* B.

(e) *If* $1 < B_0 < B$, $B_0 \neq Z$, *then* $C(B_0)$ *is solvable.*

Proof. To prove (a), let $F \in И(B; 2)$ and suppose that $E = C_F(Z) \neq 1$. Set $L = C_G(Z) > EB$. Since $Z \leqq Z(R)$ then $R < L$, hence $O_{3'}(L)$ has odd order since $И(R; 2) = \{1\}$. On the other hand L is 3-constrained, so there is an EB-invariant S_3-subgroup P of $O_{3',3}(L)$ on which E acts faithfully. We get $[E, \Omega_1(C_P(B))] \leqq P \cap [E, B] \leqq P \cap E = 1$, so $[E, P] = 1$ by the generalized $P \times Q$-lemma. This is a contradiction, so (a) is proved. Part (b) follows from (a) and Lemma 2.2. As for (c), since $C(B)$ is 3-constrained and B contains all elements of order 3 in $C(B)$, we find that $C(B)$ has a normal 3-complement. Now (c) follows immediately from (a).

Next choose $1 \neq F \in И(B; 2)$. As B is noncyclic there is a subgroup $1 \neq B_0 < B$ satisfying $C_F(B_0) \neq 1$. It follows from (a) that B_0 is not conjugate to Z in G. On the other hand, since $N_R(B) > C_R(B)$, all subgroups of order 3 in B distinct from Z are conjugate in R. Thus Z is weakly closed in B as required.

Finally let $1 \neq B_0 < B$ with $B_0 \neq Z$. Set $L = C(B_0)$. By (d) we can assume that $C_R(B_0)$ is a S_3-subgroup of L. Thus $C_R(B_0) = B_0 \times R_1$ where R_1 is cyclic and $Z = \Omega_1(R_1)$. Since L is 3-constrained we get that L is 3-solvable and $L/O_{3'}(L)$ is solvable. But by (a), $C(Z) \cap O_{3'}(L)$ has odd order. So $O_{3'}(L)$ is solvable by Lemma 2.4, hence L is solvable, as required.

LEMMA 4.2. *A* S_2*-subgroup of* $N(B)$ *has order at most* 2.

Proof. Let T be a S_2-subgroup of $N(B)$. We have $C_T(B) = 1$ by Lemma 4.1(c), so T acts faithfully on B. By Lemma 4.1(d), we have $Z \lhd N(B)$, so T is actually both faithful and reducible on B. Now assume that $|T| > 2$. The only possibility is $|T| = 4$ and T is a four-group.

Set $L = N_G(Z) > TB$. As usual we have $O_{3'}(L)$ of odd order, so there is a TB-invariant S_3-subgroup P of $O_{3',3}(L)$ such that T is faithful on P. Let K be a critical subgroup of P of exponent 3 (that is, K is a characteristic subgroup of P and every 3′-element of $L/O_{3',3}(L)$ is faithful on K. The existence of K is proved in [9, Theorem 5.3.13]). Now as a consequence of (*) and Lemma 4.1(c) we find that if x is an involution of G then $C_G(x)$ has cyclic S_3-subgroups. Hence $|K| \leqq 27$.

Suppose first that $|K| = 9$. Then $K \cong (3, 3)$, so as $Z \lhd L$ then L is solvable and T covers a Hall 3′-subgroup of $L/O_{3'}(L)$. Thus L has a normal 2-complement and L has 3-length 1, so we may assume that $P = R$. Thus $K \in U(R)$, so that K is contained in an element of $SCN_3(R)$. On the other hand $K \neq B$, $[K, B] \neq 1$, so there is an involution of T acting without fixed points on $C_R(K)$. This forces $C_R(K)$ to be abelian of rank 2, a contradiction. So we have $|K| = 27$.

Suppose K is abelian. Then $K \cong (3, 3, 3)$, hence

$$K = \langle \Omega_1(C_K(t)) \mid t \in T^\# \rangle > B \,.$$

But then $B \in \mathfrak{A}(3)$, against (*). So K must be extra-special of order 27 and exponent 3. It follows, since $\mathrm{Inn.}(K) = O_3(\mathrm{Aut.}(K))$, that $P = K * C_P(K)$ and $L/O_{3',3}(L) \subsetneq GL(2, 3)$. Next, notice that $C_P(K)$ is cyclic: this follows since $K \cap C_P(K) = Z$, $K = \langle \Omega_1(C_K(t)) \mid t \in T^\# \rangle$, and $C_P(K)$ admits T. Hence, we get $K = \Omega_1(P)$. Since $SCN_3(R) \neq \varnothing$, whilst $SCN_3(P) = \varnothing$, we get that $P < R$. As L also contains a four-group, it follows that $L/O_{3',3}(L) \cong GL(2, 3)$. Now K contains exactly four subgroups of order 9, including B and some $U \in U(R)$. Since $B \not\sim U$ we deduce that B has exactly 3 conjugates in L, so $N_L(B)$ contains a S_2-subgroup of L. This contradicts the first paragraph of the proof, so the lemma is proved.

Now set $R_0 = C_R(B)$. As we have observed, we can assume that R_0 is a S_3-subgroup of $C_G(B)$. The next lemma is crucial.

LEMMA 4.3. *Suppose that* $1 \neq F \in \mathcal{N}^*(R_0; 2)$. *Let* $N = N_G(F)$, *with* T *a* S_2-*subgroup of* N. *Then the following hold*:

(a) T *is a* S_2-*subgroup of* G.

(b) $|T : F| \leqq 2$.

Proof. Let S be a S_3-subgroup of N which contains R_0. We may assume without loss that $S \leqq R$. Now by (*) we have that S

contains no element of $\mathfrak{A}(3)$, so if $U \in \mathscr{U}(R)$ we have $S \cap C_R(U)$ cyclic. Hence S is metacyclic. Set $K = O^3(N)$. By a theorem of Huppert [14] K has abelian S_3-subgroups. Set $S_0 = K \cap S$.

Case 1. S_0 *is non-cyclic*. We get $B = \Omega_1(S_0)$, so by Lemma 4.2 it follows that $|N_K(B): C_K(B)| \leqq 2$. If $N_K(B) = C_K(B)$ then K has a normal 3-complement by a transfer theorem of Burnside. If $|N_K(B): C_K(B)| = 2$ then K is 3-solvable by Lemma 2.6. So in either case K is 3-solvable, hence $K = O_{3'}(K)N_K(B)$. As F is a S_2-subgroup of $O_{3'}(K)$ we get $T = FN_T(B)$, so $|T: F| \leqq 2$ by Lemma 4.2. Finally, TB is a group satisfying all the conditions of Lemma 2.3, so F is characteristic in T, so (a) follows and the result is proved in Case 1.

Case 2. S_0 *is cyclic*. If $S_0 = 1$ then $K = O_{3'}(N)$ is the normal 3-complement of N, so $F = T$ and there is nothing to prove. Hence, we may assume that $S_0 \neq 1$. Set $B_1 = \Omega_1(S_0) < B$.

Now set $\bar{N} = N/O_{3'}(N)$. $C_{\bar{K}}(\bar{B}_1)$ has a normal 3-complement which admits $\bar{B}$, so if $C_{\bar{K}}(\bar{B}_1)$ has even order then there is a 2-group $\bar{U} \neq 1$ admitting $\bar{B}$. But then B normalizes a S_2-subgroup of U which properly contains F, contradiction. So $C_{\bar{K}}(\bar{B}_1)$ has odd order. If $\bar{K}$ has a minimal normal subgroup of order 3 then $\bar{K}$, and hence K, is 3-solvable. In this case we complete the proof as in Case 1, so we may suppose that every minimal normal subgroup of $\bar{K}$ is non-solvable. As $\bar{K}$ has cyclic S_3-subgroup we deduce that $\bar{L} = \bar{K}^\infty$ is simple.

Next we have $C_{\bar{L}}(\bar{B}_1)$ of odd order and $|N_{\bar{L}}(\bar{B}_1): C_{\bar{L}}(\bar{B}_1)| = 2$. By Lemma 2.5 we get that $\bar{L}$ has one class of involutions. Moreover, as $\bar{B}$ normalizes $N_{\bar{L}}(\bar{B}_1)$ there is a subgroup B_2 of order 3 in B such that $\bar{B}_2$ centralizes an involution $\bar{x}$ of $N_{\bar{L}}(\bar{B}_1)$. Thus $\bar{B}_2 \not\leq \bar{L}$ and $\bar{B}_2$ normalizes $C_{\bar{L}}(\bar{x})$. We show next that $\langle \bar{x} \rangle$ is a S_2-subgroup of $C_{\bar{L}}(\bar{B}_2)$. For this, it is enough to show that $C_{\bar{L}}(\bar{B}_2)$ is solvable: for $C_{\bar{L}}(\bar{B}_2)$ has a cyclic S_3-subgroup, so if $C_{\bar{L}}(\bar{B}_2)$ is solvable and has a S_2-subgroup of order at least 4 then $И(\bar{B}; 2) \neq \{1\}$, a contradiction. Now if $B_2 \neq Z$ the solvability of $C_{\bar{L}}(\bar{B}_2)$ follows from Lemma 4.1(e). On the other hand if $B_2 = Z$ then $C_G(Z)$ has S_2-subgroups of 2-rank at most 1, so the solvability of $C_{\bar{L}}(\bar{B}_2)$ follows easily in this case also. So we have indeed shown that $\langle \bar{x} \rangle$ is a S_2-subgroup of $C_{\bar{L}}(\bar{B}_2)$.

Consider now the group $C_{\bar{L}}(\bar{x})$. It is a $3'$-group, and moreover $C_{\bar{L}}(\bar{x}) \cap C(\bar{B}_2)/\langle \bar{x} \rangle$ has odd order by the last paragraph. By Lemma 2.4 we find that $C_{\bar{L}}(\bar{x})/\langle \bar{x} \rangle$ is solvable of 2-length 1, so $C_{\bar{L}}(\bar{x})$ has the same property. So we have shown that $\bar{L}$ is a simple group such that every involution of $\bar{L}$ has a centralizer which is solvable of 2-length 1. Let $\bar{Y}$ be a $\bar{B}_2$-invariant S_2-subgroup of $\bar{L}$ with $\langle \bar{x} \rangle \leqq$

$Z(\bar{Y})$. We claim that $SCN_3(\bar{Y}) \neq \varnothing$. Otherwise, since $\langle \bar{x} \rangle = C_{\bar{Y}}(\bar{B}_2)$, we find easily that $\bar{Y}$ is of symplectic type, and even extra-special. But as is well-known, no simple group has such a S_2-subgroup. Thus we may now identify $\bar{L}$ using Lemma 2.9. As $\bar{L}$ has a cyclic S_3-subgroup with an odd-order centralizer, and as $\bar{L}$ admits an automorphism (induced by $\bar{B}_2$) with fixed-point subgroup having twice odd order, we find that the only possibility is $\bar{L} \cong L_2(8)$. Hence a S_3-subgroup of N is isomorphic to a S_3-subgroup of Aut. $(L_2(8))$, hence is metacylic of order 27 and exponent 9. But then $B = R_0 = \Omega_1(S)$ char S and as $|N_R(B): C_R(B)| = 3$ we get $S = N_R(B)$. As $SCN_3(R) \neq \phi$ and $B \notin \mathfrak{A}(3)$ this is impossible. So the analysis of Case 2 is completed, and the lemma is proved.

We can now prove

PROPOSITION 4.1. *$O_{3'}(C(R_0))$ is transitive on $\mathcal{M}^*(R_0; 2)$.*

Proof. First notice that $\mathcal{M}(R_0; 2) \neq \{1\}$. For by assumption we have $\mathcal{M}(B; 2) \neq \{1\}$. So if $1 \neq F \in \mathcal{M}(B; 2)$ and $1 < B_0 < B$ satisfies $F_0 = C_F(B_0) \neq 1$ we get, since $F_0 \neq Z$ by Lemma 4.1(a), that $C(B_0)$ is solvable. Thus $F_0 \leqq \langle \mathcal{M}_{C(B_0)}(B; 2) \rangle \leqq O_{3'}(C(B_0))$. As $R_0 < C(B_0)$ then R_0 must normalize a (nontrivial) S_2-subgroup of $O_{3'}(C(B_0))$, as required.

Supposing the proposition false, choose elements D_1, D_2 in $\mathcal{M}^*(R_0; 2)$ such that D_1 and D_2 are not conjugate in $O_{3'}(C(R_0))$ and such that $|D_1 \cap D_2|$ is maximal subject to this condition. Set $D = D_1 \cap D_2$. We next show that $D \neq 1$. Namely, since $D_i \neq 1$ then $D_i = O_2(N(D_i))$ for $i = 1, 2$, and so B is faithful on D_1 and D_2. Thus if $B_i^* = \{1 < B_0 < B \mid C_{D_i}(B_0) \neq 1\}$, then $|B_i^*| \geqq 2$ for $i = 1, 2$. As $Z \notin B_1$, $Z \notin B_2$ by Lemma 4.1(a) it follows that $B_1^* \cap B_2^* \neq \varnothing$. Choose $B_0 \in B_1^* \cap B_2^*$, with $D_i^* = C_{D_i}(B_0) \neq 1$, $i = 1, 2$. As before, we get $\langle D_i^*, D_2^* \rangle \leqq O_{3'}(C(B_0))$, so there is $x \in O_{3'}(C(R_0))$ such that $\langle D_1^*, (D_2^*)^x \rangle$ is a 2-group. It follows that $D \neq 1$.

Now set $N = N(D)$. Maximality of $|D|$ ensures that $D = O_2(N)$. As N is 2-constrained then $C(D) = Z(D)$, in particular $Z(D_i) \leqq D$ for $i = 1, 2$. As D_i has class at most 2 by Lemma 4.1(b), then $D \lhd D_i$, so $D_i \leqq N$ for $i = 1, 2$. Let T_1 be a S_2-subgroup of N which contains D_1. By Lemma 4.3, we have $|T_1: D_1| \leqq 2$. Moreover D_1/D is abelian, so T_1/D has an abelian subgroup of index at most 2.

As in Lemma 4.3, we may argue that N has metacyclic S_3-subgroup. Suppose that N is 3-solvable. Then we get

$$\langle N_{D_1}(D), N_{D_2}(D) \rangle \leqq \langle \mathcal{M}_N(R_0; 2) \rangle \leqq O_{3'}(N) ,$$

so $\langle N_{D_1}(D), N_{D_2}(D)^x \rangle$ is a 2-group for some $x \in O_{3'}(C(R_0))$, and the maximality of $|D|$ is contradicted. Hence, N is not 3-solvable. The argument of Case 1 of the previous lemma now yields that $K = O^3(N)$ has a nontrivial cyclic S_3-subgroup, and moreover if $\bar{N} = N/O_{3'}(N)$ then $\bar{L} = \bar{N}^\infty$ is simple. Now as $\bar{N}$ has S_2-subgroups which have abelian subgroups of index at most 2, $\bar{L}$ has the same property, so we may identify $\bar{L}$ using Lemma 2.10. In particular, if $T = T_1 \cap L$, so that $\bar{T}$ is a S_2-subgroup of $\bar{L}$, then $\bar{T}$ is either elementary abelian, or isomorphic to D_2n, SD_2n or $Z_2n \wr Z_2$. Now B normalizes $D_1 \cap L$ and $|T: D_1 \cap L| \leqq 2$. As $B \cap L$ is cyclic it follows that there is a subgroup B_0 of B of order 3 such that $B_0 \not< L$ and B_0 normalizes a S_2-subgroup of L. Hence we can assume that B_0 normalizes T.

Now if T is non-abelian then $[T, B_0] = 1$. Since $C_L(B)$ has odd order by Lemma 4.1(c) it follows in this case that $C_{\bar{L}}(\bar{B}_1)$ has odd order, where $B_1 = \Omega_1(B \cap L)$. It follows that $\bar{L} \not\cong L_3(q)$ or $U_3(q)$ (q odd), A_7 or M_{11}, for these groups have nonabelian S_2-subgroups and elements of order 6.

Suppose we have $\bar{L} \cong L_2(q)$ for some q. We may choose $B_2 < B$, $|B_2| = 3$, $B_2 \neq Z$, $B_2 \not< L$. Then B_2 induces a field automorphism of $\bar{L}$ of order 3. Since $C(B_2)$ is solvable by Lemma 4.1(e), the only possibilities are $q = 8$ or $q = 27$. In the latter case $\bar{L}$ has a noncyclic S_3-subgroup, a contradiction. We may eliminate the former possibility as in Lemma 4.3, and so $\bar{L} \not\cong L_2(q)$ for any q.

As J_1 has no outer automorphisms of order 3 [17], and as groups of Ree-type have non-cyclic S_3 subgroups [18], $\bar{L}$ can be isomorphic to none of these groups. Having exhausted the possibilities given by Lemma 2.10, we deduce that $\bar{L}$ does not exist, so the proof of Proposition 4.1 is completed.

The Proof of Theorem 2. We retain the notation of the previous lemma, so that $B \leqq R_0 = C_R(B)$. As $SCN_3(R) \neq \emptyset$ then there is $U \in \mathscr{U}(R)$ such that $R_0 U = N_R(B)$. So $|R_0 U: R_0| = 3$, so $R_0 \lhd R_0 U$. Hence, U must permute the elements of $\mathsf{N}^*(R_0; 2)$ among themselves. By Proposition 4.1 $\mathsf{N}^*(R_0; 2)$ contains a $3'$ number of elements, so U must fix some element $F \in \mathsf{N}^*(R_0; 2)$. As $F \neq 1$ then $\mathsf{N}(U; 2) \neq \{1\}$. However, $U \in \mathfrak{A}(3)$. This contradicts our basic assumption (*), so Theorem 2 is proved.

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