PACIFIC JOURNAL OF MATHEMATICS
Vol. 57, No. 1, 1975

GROUPS WHOSE IRREDUCIBLE CHARACTER DEGREES ARE ORDERED BY DIVISIBILITY

R. GOW

This paper concerns the class of finite groups whose complex irreducible character degrees can be linearly ordered by divisibility. It is known that such a group has a Sylow tower. By analyzing the structure of a group in the class whose order is divisible by just two primes, we are able to obtain information on the Sylow subgroups of any group in the class. We classify the two-prime groups in the class Except for certain exceptional pairs of primes, one of which is always 2. Such a group G of order $p^a q^b$ either has a normal abelian Sylow q-subgroup or $H = G/O_p(G)$ has a non-abelian Sylow q-subgroup Q and each p-element of H induces a fixed-point-free automorphism of Q'.

Let L denote the class of finite groups G with the property that the degrees of the complex irreducible characters of G can be linearly ordered by divisibility. The class L was considered in two previous papers, [4] and [6], and the existence of a Sylow tower for any member of L was shown in both papers. While we have not succeeded in describing L purely in group-theoretic terms, the results of this paper indicate that the possible structure of a group in L should be extremely limited.

We consider a non-abelian group G in L and let π be the set of distinct primes dividing the degrees of the irreducible characters of G. Then if π contains r primes, it follows from [6], Theorem 2, that G has a Sylow tower

$$1 \lhd H_r \cdots \lhd H_1 \lhd G$$

where H_r is a normal abelian Hall π'-subgroup of G. We may consider the case where $r = 1$ in the above description to be the trivial example of a group in the class L. The degrees of the irreducible characters of G are all powers of some prime p and G has a normal abelian p-complement. Conversely, by a theorem of Ito, [3], Theorem 17. 10, p. 570, if G has a normal abelian p-complement, the degrees of the irreducible characters of G are all powers of p and so $G \in L$.

It is our intention in this paper to analyze the structure of a group G in L whose order is divisible by precisely two primes. This analysis, although of a restricted nature, can be applied to an arbitrary group in L. For if H_i is any of the normal subgroups of G occurring in the Sylow tower described in the previous paragraph, H_i is in L, and as L is evidently factor-group-closed, our two-prime analysis may be applied to suitable factor-groups of the H_i. In particular our main theorem will yield information about the Sylow subgroups of any group in L.

1. Two-prime analysis. Let $G \in L$ and suppose that $|G| = p^a q^b$, where p and q are distinct primes. Suppose that p divides $\chi(1)$ for each nonlinear irreducible character, χ, of G. G must have a normal Sylow q-subgroup Q. We will show that provided p and q are not certain exceptional primes, then there are only two possible structures for G. Let $O_p(G)$ be the largest normal p-subgroup of G. We note that $O_p(G)$ centralizes Q, and thus if P is a Sylow p-subgroup of G, Q admits $P/O_p(G)$ as a group of automorphisms. Using this notation we will prove

THEOREM 1. *Let $G \in L$ and assume that $|G| = p^a q^b$ where p and q are distinct primes. Assume further that if $p = 2$, q is neither a Fermat nor a Mersenne prime, or if p is a Mersenne prime, q is not* 2. *Then we have either*

(i) *Q is abelian or* (ii) *Q is non-abelian, each element of $P/O_p(G)$ induces a fixed-point-free automorphism of Q', the derived group of Q, and $O_p(G)$ is abelian.*

Thus we see that either the trivial situation occurs or $G/O_p(G)$ is almost a Frobenius group. The second theorem of this paper will show that, apart from the prime exceptions, we have completely classified two-prime groups in L. However, we have been unable to ascertain whether the prime exceptions described in Theorem 1 can give rise to exceptions to the theorem or whether the theorem holds for all pairs of primes. Before we begin the proof of Theorem 1, we state as a lemma a theorem due to Ito, [5], upon which our proof rests.

LEMMA 1. *Let G be a group in which $O_p(G) = 1$ and suppose that G has a normal nilpotent p-complement H. Then if one of the following holds*

(i) *G has odd order.*

(ii) *H has even order and p is not a Mersenne prime.*

(iii) *$p = 2$ and $|H|$ is not divisible by any Fermat or Mersenne primes, G has an irreducible character, χ, of defect* 0 *modulo p* (that is, if $|G| = p^a|H|$, p^a divides $\chi(1)$).

2. Proof of Theorem 1. We can assume that Q is non-abelian. We will need the following facts about a solvable group G: if $F(G)$

is the Fitting subgroup of G and $\Phi(G)$ the Frattini subgroup of G, then $F(G/\Phi(G)) = F(G)/\Phi(G)$ and $F(G)/\Phi(G)$ is the direct product of the minimal normal subgroups of $G/\Phi(G)$, [3], p. 277–9. In particular $F(G)/\Phi(G)$ is abelian. Our initial arguments will apply to $H = G/O_p(G)$. H has a normal p-complement isomorphic to Q, so we may as well assume that Q is a normal p-complement of H. Let us put $|H| = p^c q^b$.

Now $H/\Phi(H)$ has a normal abelian p-complement and so the degrees of its irreducible characters are powers of p. As $O_p(H) = 1$, our remarks of the opening paragraph show that $O_p(H/\Phi(H))$ is also trivial. We can thus apply Lemma 1 to deduce that $H/\Phi(H)$ has an irreducible character, θ, of degree p^c. Thus H also has an irreducible character of degree p^c and as $H \in L$, if χ is an irreducible character of H with $q \mid \chi(1)$, then $p^c \mid \chi(1)$.

As we are assuming Q is non-abelian, Q has nonlinear irreducible characters. Let χ be a nonlinear irreducible character of Q and let T be the stabilizer of χ in H (see [3], p. 569, 17.6b). By a theorem of Gallagher, [3], p. 572, there is an extension, ψ of χ to T and ψ^H is an irreducible character of H, [3], p. 571. We have $\psi^H(1) = \chi(1)|H: T|$. As q divides $\chi(1)$, it also divides $\psi^H(1)$ and so by our result of the previous paragraph p^c divides $\psi^H(1)$. It follows that $|G: T| = p^c$ and so $T = Q$. Thus each nonlinear irreducible character of Q has p^c conjugates in H. It is this statement which implies that, if P_1 is a Sylow p-subgroup of H, P_1 acts without fixed points on Q'.

For, suppose $x \in P_1$ and has order p. We consider the action of x on the elements and characters of Q'. Only linear characters of Q are fixed by x, for we have just seen that the stabilizer of a nonlinear irreducible character of Q is Q itself. By Corollary 2 of [2]. x fixes no elements of Q' other than the identity. Returning to the general case of the group G, we see that $P/O_p(G)$ acts fixed-point-free on Q'.

The final step of the argument is to show that $O_p(G)$ is abelian. We accomplish this by giving further consideration to the character θ, previously constructed for H. This character can be thought of as a character of both H and G. We first consider it as a character of H and examine its restriction to Q. We have by Clifford's theorem, $\theta_Q = u(\lambda_1 + \cdots + \lambda_t)$ where the λ_i are a complete set of H-conjugate irreducible characters of Q and u is some integer. As the degree of θ is a power of p and Q is a q-group, each character λ_i must be linear. If S is the stabilizer of λ_1 in H, Gallagher's theorem shows that there is an extension μ of λ_1 to S, and θ has the form $(\mu\sigma)^H$ where σ is an irreducible character of S/Q. As $\deg\theta = |H: Q| = p^c$, $\deg\sigma$ must equal $|S: Q|$. Thus σ is an irreducible character of S/Q whose degree equals the order of S/Q. This is only possible if $S = Q$ and so λ_1 has p^c conjugates in H. Since $O_p(G)$ centralizes Q in G, it must be in the stabilizer subgroup of λ_1 in G, and it follows from our deduction above that the stabilizer of λ_1 in G must be $O_p(G) \times Q$. Considering θ as a character of G now, we know there is an extension ν of λ_1 to $O_p(G) \times Q$ and $\nu^G = \theta$.

Gallagher's theorem implies that if we take any irreducible character ψ of $O_p(G)$ and consider it as a character of $O_p(G) \times Q$, the induced character $(\nu\psi)^G$ will be irreducible. Now if $\psi(1) > 1$, $(\psi\nu)^G$ has degree some power of p greater than p^c. But we have already constructed characters of degree $p^c \times a$ power of q in H and these provide irreducible characters of the same degree in G. As $G \in L$, $\deg(\nu\psi)^G$ must divide $p^c q^d$, for some $d > 0$, and this is impossible if $\deg \psi > 1$. Thus $\deg \psi = 1$, and it now follows that, as all irreducible characters of $O_p(G)$ are linear, $O_p(G)$ is abelian.

3. A converse of Theorem 1. We can prove a converse of Theorem 1 which will show, in particular, that modulo the prime exceptions, we have completely classified the two-prime groups in L.

THEOREM 2. *Let G be a group which possesses a normal p-complement H and let P be a Sylow subgroup of G. Suppose $O_p(G)$ is abelian and $P/O_p(G)$ acts in a Frobenius manner on H'* (assumed distinct from 1). *Suppose, in addition, that $H \in L$. Then $G \in L$.*

Proof. $O_p(G)$ centralizes H and hence if χ is an irreducible character of H, $O_p(G)$ is contained in the stabilizer of χ. Let χ be a nonlinear character of H and let T denote the stabilizer of χ in G. Put $P_1 = P \cap T$. Since $O_p(G) \subseteqq T$, $O_p(G) \subseteqq P_1$. Now as P_1 fixes a nonlinear character of H, P_1 fixes a nontrivial conjugacy class of H, by Corollary 2 of [1], and hence fixes a nonidentity element of H. If $P_1 \neq O_p(G)$, since $P_1/O_p(G)$ acts in a Frobenius manner on H', P_1 fixes no nonidentity element of H'. But by Theorem 1 of [2], if P_1 fixes no nonidentity element of H', P_1 fixes only linear characters of H. Thus we must have $P_1 = O_p(G)$.

By Gallagher's theorem, if ψ denotes an extension of χ to $T = H \cdot O_p$, $\chi^T = \sum_{i=1}^{s} (\psi\omega_i)$, where the ω_i are the irreducible characters of $T/H \cong O_p(G)$. The ω_i are all one-dimensional as $O_p(G)$ is abelian. The characters $(\psi\omega_i)^G$ are all distinct and irreducible, [3], p. 571, Theorem 17.11, and have degree $\deg \chi \, |P\colon O_p|$.

If λ is a linear character of H, and S is the stabilizer of λ, we apply Gallagher's theorem again to show that if $\lambda^S = \sum_{i=1}^{t} \nu_i(1)(\nu_i\mu)$, where μ is an extension of λ to S, and the ν_i are the irreducible characters of S/H, then each character $(\mu\nu_i)^G$ is irreducible and has degree $\nu_i(1)\,|G\colon S|$. We put $P_2 = P \cap S$, and again we have $O_p(G) \subseteqq P_2$. Since $S/H \cong P_2$, the degree of any irreducible character ν_i of

S/H divides $|P_2: O_p(G)|$, by Ito's theorem, [3], p. 570. Hence deg $(\nu_i\mu)^G$ divides $|P: P_2|\,|P_2: O_p| = |P: O_p|$. If the distinct degrees of the non-linear character of H are $a_1, \cdots, a_r$ where a_1 divides a_2, a_2 divides a_3 etc., the degrees of the characters of G are $a_1|P: O_p|, \cdots, a_r|P: O_p|$, and powers of p which divide $|P: O_p|$. Hence $G \in L$.

4. Notes on the theorems. In Theorem 1, apart from the prime exceptions, we have shown that either Q is abelian or P/O_p acts as a fixed-point-free automorphism group on Q'. In the latter case, it is well known that if p is an odd prime, P/O_p must be cyclic, and if $p = 2$, P/O_p is cyclic or generalized quaternion. Thus, in particular, if both p and q are odd, either Q is abelian or P is an extension of an abelian group by a cyclic group. When we have an arbitrary group G of odd order in L, our two-prime analysis can always be applied to the two-prime composition factors described in the introduction. We can deduce that for about half the primes p dividing $|G|$, the Sylow p-subgroups of G are either abelian or metabelian.

In Theorem 2, H' admits P/O_P as a fixed-point-free group of automorphisms, and it follows from a theorem of Thompson, [3], p. 505, Theorem 8.14, that H' is nilpotent. We believe that the structure of a group in L whose derived group is nilpotent should be a restricted nature and analysis of this situation may provide possible inductive hypotheses for investigating the structure of an arbitrary group in L.

REFERENCES

1. G. Glauberman, *Correspondence of characters for relatively prime operator groups*, Canad J. Math., (1968), 1465-88.

2. R. Gow and J. F. Humphreys, *Normal p-complements and irreducible representations*, submitted to J. London Math. Soc.

3. B. Huppert, *Endliche Gruppen*, Springer, Berlin, 1967.

4. I. M. Isaacs and D. S. Passman, *A characterization of groups in terms of the degrees of their irreducible characters*, Pacific J. Math., **24** (1968), 467-510.

5. N. Ito, *Note on the characters of solvable groups*, Nagoya Math. J., **39** (1970), 23-28.

6. J. G. Thompson, *Normal p-complements and irreducible characters*, J. Algebra, **14** (1970), 129-34.

Received July 5, 1974.

CARLETON UNIVERSITY, OTTAWA