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A LEBESGUE DECOMPOSTION FOR VECTOR VALUED ADDITIVE SET FUNCTIONS

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Our purpose is to show that several recent results dealing with a Lebesgue decomposition of vector valued set functions can be verified by using an earlier result due to R. B. Darst [3].

I. Introduction. In 1963 R. B. Darst established a result giving the Lebesgue decomposition of s-bounded elements in a normed Abelian group with respect to an algebra of projection operators. As a result, one can establish the decomposition of s-bounded additive functions defined on an algebra of sets.

New results have emerged since then for the decompositions of finitely additive and countably additive set functions defined on an algebra of sets and with values lying in a Banach space. In particular, there is a theorem by J. Brooks [1] (1968) and one by J. Uhl [5] (1970). We shall show that both are consequences of the theorem by Darst, thus unifying the three results.

THEOREM (Darst). *Let G be a generalized, complete normed Abelian group, which means*

(1) $\|0\| = 0$,

(2) *if* $g \neq 0$ *then* $0 < \|g\| \leqq \infty$, *and*

(3) *only the subgroup* $\{g \in G: \|g\| < \infty\}$ *need be complete. Let T be a Boolean algebra of projection operators defined on G, with the property that if* $t_1, t_2 \in T$ *with* $t_1 \leqq t_2 (t_1 t_2 = t_1)$ *then* $\|t_1(g)\| \leqq \|t_2(g)\|$ *for all g in G. For $x > 0$ we let $T_x \subset T$ possess the properties*

(1) $t_x \in T_x$ *and* $t \in T$ *implies* $tt_x \in T_x$, *and*

(2) $t_x \in T_x$ *and* $t_y \in T_y$ *implies* $t_x \vee t_y \in T_{x+y}$. *Define a function* $Y: G \to R$ *by* $Y(g) = \lim_{x \to 0^+} [\sup \|t(g)\|: t \in T_x]$. *Let $f \in G$ be bounded and s-bounded, i.e., for every sequence $\{t_i\}$ of pairwise disjoint elements of T, $t_i(f) \to 0$. Then there exists unique elements $h, s \in G$ such that $f = h + s$ and*

(1) $Y(h) = 0$, *and*

(2) *given $\varepsilon > 0$ there exists $t \in T_\varepsilon$ such that* $\|t'(s)\| < \varepsilon$.

II. First result. We now state Brooks' theorem and show that it is a special case of Darst's theorem.

THEOREM (Brooks). *Let X be a Banach space and Σ any σ-*

algebra of sets. Let $u: \Sigma \to X$ *be countably additive, and let* β *be an outer measure on* Σ. *Then there exists unique mappings* u_1, u_2: $\Sigma \to X$, *both countably additive, such that* $u = u_1 + u_2$ *and also*

(1) u_1 *is* β*-continuous* ($\beta(E_n) \to 0$ *implies* $u_1(E_n) \to 0$)

(2) u_2 is β*-singular* (*there exists a set* $\hat{E} \in \Sigma$ *such that* $\beta(\hat{E}) = 0$ *and* $u_2(E) = u_2(E \cap \hat{E})$ *for all sets* $E \in \Sigma$).

Proof. Let $G = \{$countably additive mappings from Σ to $X\}$. Then G is an Abelian group under addition. Define a norm on G by $|||u||| = \sup\{||u(E)||: E \in \Sigma\}$. First we show that G is a complete normed space whose elements are bounded and s-bounded.

Let $u \in G$, and define a set $E \in \Sigma$ to be u-bounded if the set $\{||u(A)||: A \in \Sigma, A \subset E\}$ is a bounded set of real numbers. If $|||u||| \not< \infty$ then there exists a sequence $\{E_n\}$ from Σ such that $||u(E_n)|| > n$. Then $\hat{E} = \bigcup_{n=1}^{\infty} E_n \in \Sigma$ and is not u-bounded. But $u(\hat{E}) \in X$ implies $||u(\hat{E})|| < \infty$. If $N > 2||u(\hat{E})||$ then there exists a set $A \subset \hat{E}$ with $||u(A)|| > N$. Thus $||u(\hat{E} - A)|| = ||u(\hat{E}) - u(A)|| \geqq ||u(A)|| - ||u(\hat{E})|| > (1/2)N$. But $\hat{E}$ being unbounded implies either A or $\hat{E} - A$ is also unbounded, and both of these sets have "large" measure. Continuing the same procedure, with whichever of the above two sets is unbounded, yields a decreasing sequence of sets $\{A_n\} \subset \Sigma$ such that $||u(A_n)|| > n$. But $u(\lim A_n) = \lim u(A_n)$ implies $||u(\lim A_n)|| = \infty$. This is impossible since $u(\lim A_n) \in X$. Hence every element of G is bounded.

Let $\{u_n\} \subset G$ be Cauchy; thus $\sup_{E \in \Sigma} ||u_n(E) - u_m(E)|| \to 0$. So given any $E \in \Sigma$, the sequence $\{u_n(E)\}$ is Cauchy, and thus converges to $u(E)$. To show $u \in G$, first let $\{A_i\} \subset \Sigma$ be pairwise disjoint. Now u is bounded since $|||u_n - u_m||| < \varepsilon$ for all $n, m \geqq N$ implies $|||u||| \leqq |||u_N||| + \varepsilon < \infty$. It can easily be shown that u is finitely additive. Since $u_N(\bigcup_{i=1}^{\infty} A_i) \in X$ this implies $||u_N(\cup A_i)||$ is finite, and thus

$$\left\| \sum_{i=1}^{\infty} u_N(A_i) \right\|$$

is finite. Given $\varepsilon > 0$ there exists a positive integer M such that $||\sum_{i=1}^{n} u_N(A_i) - \sum_{i=1}^{m} u_N(A_i)|| < \varepsilon$ for all $n, m \geqq M$. Then for all $m \geqq M$ we have $||u_N(\bigcup_{i=m+1}^{\infty} A_i)|| \leqq \varepsilon$, and, for large k, it follows that

$$\begin{aligned} \left\|u\left(\bigcup_{i=k}^{\infty} A_i\right)\right\| &\leqq \left\|u\left(\bigcup_{i=k}^{\infty} A_i\right) - u_N\left(\bigcup_{i=k}^{\infty} A_i\right)\right\| + \left\|u_N\left(\bigcup_{i=k}^{\infty} A_i\right)\right\| \\ &\leqq \varepsilon + \left\|u_N\left(\bigcup_{i=k}^{\infty} A_i\right)\right\| \\ &\leqq 2\varepsilon \,. \end{aligned}$$

Finally

$$\begin{aligned}\left\|u\Big(\bigcup_{i=1}^{n} A_i \cup \bigcup_{i=n+1}^{\infty} A_i\Big) - \sum_{i=1}^{n} u(A_i)\right\| &= \left\|u\Big(\bigcup_{i=1}^{n} A_i\Big) + u\Big(\bigcup_{i=n+1}^{\infty} A_i\Big) - \sum_{i=1}^{n} u(A_i)\right\| \\ &= \left\|\sum_{i=1}^{n} u(A_i) + u\Big(\bigcup_{i=n+1}^{\infty} A_i\Big) - \sum_{i=1}^{n} u(A_i)\right\| \\ &= \left\|u\Big(\bigcup_{i=n+1}^{\infty} A_i\Big)\right\| \\ &\longrightarrow 0\end{aligned}$$

by the previous statement. Thus $u(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} u(A_i)$ and G is complete.

The group G thus fits the hypothesis of Darst's theorem. The elements of T are the projection operators induced by the elements of Σ, i.e., $t \in T$ corresponds to some set $E \in \Sigma$, say $t \equiv t_E$, so that $t_E(u)(F) = u(F \cap E)$. Then $t_E \leqq t_F$ if and only if $E \subset F$, and in that case $|||t_E(u)||| \leqq |||t_F(u)|||$ for all u. For $x > 0$, let $T_x = \{t_E \in T\colon \beta(E) \leqq x\}$. Then T_x possesses the two properties

(1) $t_E \in T_x$ and $t_F \in T$ implies $t_E t_F = t_{E \cap F} \in T_x$, and

(2) $t_E \in T_x$ and $t_F \in T_y$ implies $t_E \vee t_F = t_{E \cup F} \in T_{x+y}$.

Now $u \in G$ is s-bounded if for every sequence $\{t_i\} \subset T$ of disjoint elements, i.e., $t_i = t_{E_i}$ with the E_i pairwise disjoint, we have $|||t_{E_i}(u)||| \longrightarrow 0$. To show that every $u \in G$ is s-bounded, first let $\{E_i\} \subset \Sigma$ be pairwise disjoint; so $\bigcup_{i=1}^{\infty} E_i \in \Sigma$. Thus $u(\cup E_i) \in X$ and $u(\cup E_i) = \sum u(E_i)$. The sequence of partial sums $S_n = \sum_{i=1}^{n} u(E_i)$ converges to $u(\cup E_i)$, and hence is Cauchy. Thus $||S_n - S_m|| = ||\sum_{i=m+1}^{n} u(E_i)|| < \varepsilon$ for all n, m greater than some positive integer N. In particular, for $m = n - 1$, $||u(E_n)|| < \varepsilon$ for all $n > N$. Thus $||u(E_i)|| \longrightarrow 0$. Consequently, given any sequence $\{F_i\} \subset \Sigma$, it follows that $||u(E_i \cap F_i)|| \longrightarrow 0$, which implies $\sup_{F \in \Sigma} ||u(E_i \cap F)|| \longrightarrow 0$. Thus u is s-bounded.

By applying Darst's theorem, we know there exist unique elements $u_1, u_2 \in G$ such that $u = u_1 + u_2$ and

(1) $Y(u_1) = 0$, and

(2) given $\varepsilon > 0$ there exists $t_{\hat{E}} \in T_\varepsilon$ such that $|||t_{\hat{E}'}(u_2)||| < \varepsilon$. It remains to show that u_1 is β-continuous, u_2 is β-singular and that they are unique.

Let $\{E_n\} \subset \Sigma$ with $\beta(E_n) \longrightarrow 0$. Then $Y(u_1) = 0$ implies

$$\begin{aligned}0 &= \lim_{x \to 0^+} [\sup |||t(u_1)|||\colon t \in T_x] \\ &= \lim_{x \to 0} [\sup |||t_E(u_1)|||\colon t_E \in T_x] \\ &= \lim_{x \to 0} [\sup ||u_1(E \cap F)||\colon \beta(E) \leqq x,\ F \in \Sigma] \\ &= \lim_{n \to \infty} [\sup ||u_1(E_n \cap F)||\colon F \in \Sigma]\ .\end{aligned}$$

Letting $F = E_n$ gives $||u_1(E_n)|| \longrightarrow 0$, so $u_1(E_n) \longrightarrow 0$ and u_1 is then β-continuous.

From (2), given $\varepsilon_n = 2^{-n}$ there exists $t_{\hat{E}_n} \in T_{\varepsilon_n}$ such that

$$|||t_{\hat{E}'_n}(u_2)||| < 2^{-n} .$$

This means $\sup_{F \in \Sigma} ||u_2(\hat{E}'_n \cap F)|| < 2^{-n}$, so if $H \subset \hat{E}'_n$ then $||u_2(H)|| < 2^{-n}$. Let $E^*_n = \bigcup_{i=1}^n \bigcap_{j=i}^\infty \hat{E}'_j$. Then $E^*_n \subset \hat{E}'_n$ and $||u_2(E^*_n)|| < 2^{-n}$. Note that $E^*_1 \subset E^*_2 \subset E^*_3 \subset \cdots$. Letting $E^* = \bigcup_n E^*_n = \lim E^*_n$ gives $E^* \in \Sigma$ and

$$\begin{aligned} ||u_2(E^*)|| &= ||u_2(\lim E^*_n)|| = ||\lim u_2(E^*_n)|| \\ &= \lim ||u_2(E^*_n)|| \leqq \lim 2^{-n} = 0 . \end{aligned}$$

Let $E \in \Sigma$ be arbitrary. To show $u_2(E) = u_2(E \cap \hat{E})$ where $\hat{E}$ is the desired set need to prove u_2 is β-singular, it suffices to show $u_2(E \cap \hat{E}') = 0$. Our set $\hat{E}$ will be $\hat{E} = E^{*\prime}$. Then $E \cap \hat{E}' = E \cap E^* = E \cap [\bigcup_n E^*_n] = \lim_n (E \cap E^*_n)$. Then

$$\begin{aligned} ||u_2(E \cap \hat{E}')|| &= ||u_2(\lim E \cap E^*_n)|| \\ &= ||\lim u_2(E \cap E^*_n)|| \\ &= \lim ||u_2(E \cap E^*_n)|| \\ &\leqq \lim 2^{-n} \\ &= 0 . \end{aligned}$$

So $u_2(E \cap \hat{E}') = 0$. And finally, $t_{\hat{E}_n} \in T_{\varepsilon_n}$ implies $\beta(\hat{E}_n) \leqq 2^{-n}$ for all n. This implies $\beta(\bigcup_{k \geqq n} \hat{E}_k) \leqq 2^{-n} + 2^{-n-1} + \cdots = 2^{1-n}$. Thus

$$\begin{aligned} \beta(\hat{E}) = \beta(E^{*\prime}) = \beta\Big(\bigcap_{n=1}^\infty \bigcup_{i \geqq n} \hat{E}_i\Big) &\leqq \beta\Big(\bigcup_{i \geqq n} \hat{E}_i\Big) \quad \text{for all} \quad n \\ &\leqq \sum_{i=n}^\infty \beta(\hat{E}_i) \quad \text{for all} \quad n \\ &= 2^{1-n} \quad \text{for all} \quad n . \end{aligned}$$

Therefore $\beta(\hat{E}) = 0$.

Finally, to verify the uniqueness of the decomposition of u, we shall show that if $u = w_1 + w_2$ with w_1 being β-continuous and w_2 being β-singular then $u_1 = w_1$ and $u_2 = w_2$. Now

$$\begin{aligned} Y(w_1) &= \lim_{x \to 0^+} [\sup |||t(w_1)|||: t \in T_x] \\ &= \lim_{x \to 0} [\sup |||t_E(w_1)|||: t_E \in T_x] \\ &= \lim_{x \to 0^+} [\sup ||w_1(E \cap F)||: \beta(E) \leqq x,\ F \in \Sigma] . \end{aligned}$$

But $\beta(E \cap F)$ tends to zero as x tends to zero. Hence $w_1(E \cap F) \to 0$, which implies $Y(w_1) = 0$.

Since w_2 is β-singular, this means there exists a set $E^* \in \Sigma$ such that $\beta(E^*) = 0$ and $w_2(E) = w_2(E \cap E^*)$ for all $E \in \Sigma$. But then $t_{E^*} \in$

T_ε and $|||t_{E^{*\prime}}(w_2)||| < \varepsilon$. The uniqueness of the decomposition from Darst's theorem now implies $u_1 = w_1$ and $u_2 = w_2$ The proof is complete, and one concludes that Brooks' theorem is a special case of Darst's theorem.

III. Second result. We now state the theorem by Uhl and derive the same result as before.

THEOREM (Uhl). *Let $F: \Sigma \to X$ be a finitely additive vector measure defined on a Boolean algebra, and where X is a Banach space. Suppose F satisfies any of the three equivalent condition:*

(1) *F is continuous with respect to some finitely additive nonnegative measure $u: \Sigma \to$ Reals,*

(2) *$F(\Sigma)$ is conditionally weakly compact, or*

(3) *$F(\Sigma)$ is contained in a weakly complete subset of X.*

If λ is a finitely additive nonnegative measure on Σ, then F is uniquely representable as $F = G + H$ where G, H are finitely additive vector measures with G continuous with respect to λ and x^*H and λ mutually singular for all x^* in the dual space X^*. If F and λ are both countably additive, then so are G and H.

Proof. Let $\mathscr{G} = \{F: \Sigma \to X$ where F is finitely additive$\}$. Then $\mathscr{G}$ is Abelian group under addition, and a norm can be defined on $\mathscr{G}$ by $|||F||| = \sup_{E \in \Sigma} ||F(E)||$. To show $(\mathscr{G}, |||\ \ |||)$ is complete, let $\{F_n\}$ be Cauchy in $\mathscr{G}$. Then $\sup_{E \in \Sigma} ||F_n(E) - F_m(E)|| \to 0$, so $\{F_n(E)\}$ is Cauchy in X for each E, and denote its limit by $F(E)$. Letting $\{E_k\} \subset \Sigma$ be pairwise disjoint implies

$$F\left(\bigcup_{i=1}^{k} E_i\right) = \lim_n F_n\left(\bigcup_{i=1}^{k} E_i\right) = \lim_n \sum_{i=1}^{k} F_n(E_i)$$
$$= \sum_{i=1}^{k} \lim F_n(E_i) = \sum_{i=1}^{k} F(E_i) .$$

Thus F is finitely additive, and the completeness is established. The subspace of bounded elements is therefore complete, so $\mathscr{G}$ is a generalized, complete, normed Abelian group. But not every element of $\mathscr{G}$ is bounded. For example [4], let $X =$ Reals and let Σ be the set of all finite disjoint unions of right-hand closed subintervals $(a, b]$ where $0 < a < b \leqq 1$. Define $F(a, b] = g(b) - g(a)$ where

$$g(x) = \begin{cases} 0 & \text{if} \quad x \text{ is irrational} \\ n & \text{if} \quad x = m/n,\ (m, n) = 1 . \end{cases}$$

Extend F by linearity. Then F is well-defined and finitely additive, but $|||F||| = \infty$.

But anyway, $\mathscr{G}$ fits into the hypothesis of Darst's theorem. The elements of T are, as before, the projection operators induced by the elements of Σ, i.e., $t \in T$ corresponds to some set $E \in \Sigma$, say $t \equiv t_E$, so that $t_E(F)(E_1) = F(E \cap E_1)$. Then $t_{E_1} \leqq t_{E_2}$ if and only if $E_1 \subset E_2$, and hence $|||t_{E_1}(F)||| \leqq |||t_{E_2}(F)|||$. For $x > 0$, define $T_x = \{t_E \in T: \lambda(E) \leqq x\}$. Then T_x possesses the desired properties. As before, $F \in \mathscr{G}$ is s-bounded if for every sequence $\{t_{E_i}\} \subset T$ of pairwise disjoint operators we have $t_{E_i}(F) \rightarrow 0$. We now show that if $F \in \mathscr{G}$ is continuous with respect to some finitely additive nonnegative measure $u: \Sigma \rightarrow$ Reals, then F is s-bounded. Let $\{E_i\} \subset \Sigma$ be pairwise disjoint. Then ϕ and $\phi' =$ whole space are in Σ. Since $u: \Sigma \rightarrow$ Reals this implies $u(\phi') = c < \infty$. Thus $E \in \Sigma$ implies $u(E) \leqq c$. So $u(E_1) \leqq u(E_1 \cup E_2) \leqq \cdots \leqq u(\bigcup_{i=1}^n E_i) \leqq \cdots \leqq c$ yields a monotone increasing sequence of positive real numbers bounded above, hence the sequence converges. Thus given $\varepsilon > 0$ there exists a positive integer N such that $n, m \geqq N$ implies $u(\bigcup_{i=1}^n E_i) - u(\bigcup_{i=1}^m E_i) < \varepsilon$. Letting $m = n - 1$ gives $u(E_n) < \varepsilon$, and thus $u(E_n) \rightarrow 0$. But F is continuous with respect to u, so $F(E_n) \rightarrow 0$, and consequently F is s-bounded.

Now we can show that the same conditions on F imply F is bounded. As before, define a set E to be F-bounded if the set $\{||F(A)||: A \subset E, A \in \Sigma\}$ is a bounded set of positive numbers. If we assume $|||F||| = \infty$ then the whole space, call it S, is not F-bounded. If $N > 2||F(S)||$ then there exists a set $E_1 \in \Sigma$ such that $||F(E_1)|| > N$. Hence $||F(S - E_1)|| > (1/2)N$, and with S unbounded, then either E_1 or $S - E_1$ is unbounded, with both of these sets having "large" measure. Assuming $S - E_1$ is unbounded, and letting $N_1 > 2||F(S - E_1)||$, then there exists a set $E_2 \subset S - E_1$ such that $||F(E_2)|| > N_1$ Then $||F((S - E_1) - E_2)|| > (1/2)N_1$, so $S - E_1$ contains two sets of "large" measure, with one of them being unbounded, say E_2. Continuing this procedure yields a sequence of disjoint sets $E_1, (S - E_1) - E_2, \cdots$ with each one of "large" measure. This contradicts the s-boundedness of F.

Hence if $F \in \mathscr{G}$ is continuous with respect to u, then F is a bounded and s-bounded element, Applying Darst's theorem yields unique elements $G, H \in \mathscr{G}$ such that $F = G + H$ and

(1) $Y(G) = 0$, and

(2) given $\varepsilon > 0$ there exists $t_{\hat{E}} \varepsilon T_\varepsilon$ such that $|||t_{\hat{E}'}(H)||| < \varepsilon$.

Condition (1) implies that G is continuous with respect to λ. To see this, let $\{E_n\} \subset \Sigma$ with $\lambda(E_n) \rightarrow 0$. Then $Y(G) = 0$ implies

$$\begin{aligned} 0 &= \lim_{x \to 0^+} [\sup |||t(g)|||: t \in T_x] \\ &= \lim_{x \to 0^+} [\sup |||t_E(G)|||: t_E \in T_x] \\ &= \lim_{x \to 0^+} [\sup ||G(E \cap E^*)||: \lambda(E) \leqq x, E^* \in \Sigma] \end{aligned}$$

$$= \lim_{n\to\infty} [\sup ||G(E^* \cap E_n)||: E^* \in \Sigma] \, .$$

Hence for $E^* = E_n$ we have $||G(E_n)|| \to 0$, so $G(E_n) \to 0$.

Now let $x^* \in X^*$. To show x^*H and λ are mutually singular we must show that given $\varepsilon > 0$ there exists $E^* \in \Sigma$ such that $\lambda(E^*) < \varepsilon$ and $|x^*H(E^{*\prime} \cap E)| < \varepsilon$ for all $E \in \Sigma$. It is impossible for $\varepsilon = 0$ in the finite additivity case [2]. Letting $\varepsilon > 0$ we note that since x^* is continuous at 0 then there exists $\delta_\varepsilon > 0$ such that for $z \in X$, $||z|| < \delta_\varepsilon$ implies $|x^*(z)| < \varepsilon$. Let $\varepsilon_1 < \min(\varepsilon, \delta_\varepsilon)$. From condition (2), given $\varepsilon_1 > 0$ there exists $E^* \in \Sigma$ such that $t_{E^*} \in T_{\varepsilon_1}$ and $|||t_{E^{*\prime}}(H)||| < \varepsilon_1$. But this means $\lambda(E^*) \leqq \varepsilon_1 < \varepsilon$ and $\sup_{E\in\Sigma} ||H(E \cap E^{*\prime})|| < \varepsilon_1$. Thus

$$||H(E^{*\prime} \cap E)|| < \delta_\varepsilon \quad \text{so} \quad |x^*H(E^* \cap E)| < \varepsilon \, .$$

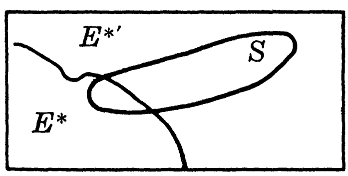
Finally, to verify the uniqueness of the decomposition, let $F = G_1 + H_1$ with G_1 continuous with respect to λ and x^*H_1 and λ mutually singular for all $x^* \in X^*$. Then

$$Y(G_1) = \lim_{x\to 0^+} [\sup ||G_1(E \cap E^*)||: \lambda(E) \leqq x, \, E^* \in \Sigma] \, .$$

But $\lambda(E \cap E^*) \to 0$ as $x \to 0^+$, so $G_1(E \cap E^*) \to 0$ and thus $Y(G_1) = 0$.

We know there exists unique mappings $G^2, H^2: \Sigma \to X$ such that $H_1 = G_2^2 + H^2$, $Y(G^2) = 0$ and given $\varepsilon > 0$ there exists $t_{\hat{E}} \in T_\varepsilon$ such $|||t_{\hat{E}'}(H^2)||| < \varepsilon$. We know that both x^*H_1 and x^*H^2 are singular with respect to λ. One can easily show that $x^*(H_1 - H^2)$ is also λ-singular. To see this, let $\varepsilon > 0$, then there exists sets $E_1, E_2 \in \Sigma$ such that $\lambda(E_i) < \varepsilon/2$ and $|x^*H_1(E_1' \cap E)| < \varepsilon/2$ and $|x^*H^2(E_2' \cap E)| < \varepsilon/2$ for all $E \in \Sigma$. Letting $\hat{E} = E_1 \cup E_2$ gives $\lambda(\hat{E}) < \varepsilon$ and $|x^*(H_1 - H^2)(\hat{E}' \cap E)| < \varepsilon$.

Now $H_1 - H^2 = G^2$. If $G^2 \not\equiv 0$, then there exists a set $S \in \Sigma$ such that $G^2(S) \neq 0$. By the Hahn-Banach theorem there exists $x^* \in X^*$ such that $|x^*G^2(S)| = ||G^2(S)|| > 0$. Let $0 < \varepsilon < (1/4)|x^*G^2(S)|$. Since x^* is continuous at zero, there exists $\hat{\delta}_\varepsilon > 0$ such that if $||x|| < \hat{\delta}_\varepsilon$ then $|x^*(x)| < \varepsilon$. And given $\hat{\delta}_\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\lambda(\hat{E}) < \delta_\varepsilon$ then $\sup_{E\in\Sigma} ||G^2(\hat{E} \cap E)|| < \hat{\delta}_\varepsilon$. Letting $\varepsilon_1 < \min\{\varepsilon, \delta_\varepsilon, \hat{\delta}_\varepsilon\}$ we know there exists a set $E^* \in \Sigma$ such that $\lambda(E^*) < \varepsilon_1$ and $|x^*(H_1 - H^2)(E^{*\prime} \cap E)| < \varepsilon_1$ for all $E \in \Sigma$.



Thus

$$\begin{aligned}|x^*G^2(S)| &\leqq |x^*G^2(S\cap E^*)| + |x^*G^2(S\cap E^{*\prime})| \\ &= |x^*G^2(S\cap E^*)| + |x^*(H_1 - H^2)(S\cap E^{*\prime})| \\ &\leqq \varepsilon + \varepsilon_1 \\ &< \frac{1}{2}|x^*G^2(S)|\,.\end{aligned}$$

This is a contradiction, so $G^2 \equiv 0$ and $H_1 \equiv H^2$. The uniqueness of the decomposition for Darst's theorem then implies $G = G_1$ and $H = H_1$.

This completes the proof that first the part of Uhl's theorem is a special case of Darst's theorem. The second part is when F and λ are both countably additive. But if we let $\mathscr{G}$ = {countably additive maps from Σ to X}, then $(\mathscr{G}, ||| \;|||)$ is complete, and the rest of the proof is as in the first part of Uhl's theorem.

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