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BIFURCATION OF OPERATOR EQUATIONS WITH UNBOUNDED LINEARIZED PART

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The bifuraction problem for the operator equation $x = \lambda Lx + G(\lambda, x)$ is considered, where L is a *closed* linear operator with characteristic value λ_0, and $G(\lambda, x)$ is a continuous higher order term. If $I - \lambda_0 L$ is a closed Fredholm operator and either L is self-adjoint and G is a continuously differentiable gradient operator or λ_0 is of odd algebraic multiplicity, then λ_0 is shown to be a bifurcation point.

Introduction. Several authors have considered the bifurcation problem for nonlinear operator equations with closed linearized part. J. MacBain [7] considered the case where the nonlinear term is compact and obtained global results, similar to those gotten by P. H. Rabinowitz [8] for compact operator equations. Other results in specialized instances were obtained, among others, by M. G. Grandall and P. H. Rabinowitz [3], M. Reeken [10], and R. Böhme [2] who also considered gradient operator equations.

In this note we extend known local bifurcation results, to nonlinear operator equations with linearized parts closed Fredholm operators and continuous higher order terms, dependent on λ, and where characteristic value is of odd algebraic multiplicity.

Bifurcation results are also obtained for variational equations, though except for the dependence of the higher order term on λ they are not as strong as those of Böhme in [2].

1. Preliminary lemmas and definitions. To solve the bifurcation problem for a large class of nonlinear operators with noncontinuous linearized part we must introduce several preliminary definitions and technical lemmas.

The domain of a closed linear operator T of a Banach space $X \to X$ will be denoted $D(T)$. If T and B are two closed linear operators of $X \to X$ by TB we will mean the operator defined by $T(Bx)$ for $x \in D(TB) = \{x \mid x \in D(B) \text{ and } Bx \in D(T)\}$. The null space and range of T will be denoted $N(T)$ and $R(T)$ respectively. For convenience we write $D(T^k) = D_k(T)$, $N(T^k) = N_k(T)$, $R(T^k) = R_k(T)$ and $\bigcup_{k=1}^{\infty} N_k(T) = N_\infty(T)$. The smallest integer $k > 0$ such that $N_k(T) = N_{k+1}(T)$ is called the ascent of T and is denoted by $\alpha(T)$. If there is no such k we say that $\alpha(T) = \infty$. Similarly the smallest integer k such that $R_k(T) = R_{k+1}(T)$ is called the descent of T and is denoted by $\delta(T)$ and we say $\delta(T) = \infty$ if there is no such k.

For the ascent and descent of an operator one can show

LEMMA 1. *$N_n(T) \subseteqq N_{n+1}(T)$, $n = 0, 1, \cdots$. If $N_k(T) = N_{k+1}(T)$ for some k then $N_{k+n}(T) = N_k(T)$. $R_{n+1}(T) \subseteqq R_n(T)$, $n = 0, 1, \cdots$. If $R_{k+1}(T) = R_k(T)$ then $R_{k+n}(T) = R_k(T)$. If $\alpha(T) = p < \infty$ and $\delta(T) < \infty$ then $\alpha(T) = \delta(T)$, $R_p(T) \cap N_p(T) = \{0\}$ and $D_p(T) = N_p(T) \oplus \{R_p(T) \cap D_p(T)\}$.*

Proof. See [12, pp. 271-273].

For any Banach space X, we denote its conjugate by X^* and the conjugate of a linear operator T by T^*. A closed linear operator T of a real or complex Banach space X into itself is said to be a Fredholm operator if $\overline{D(T)} = X$, $R(T)$ is closed and both $N(T)$ and $N(T^*)$ are finite. The index of a Fredholm operator T, written $\kappa(T)$, is $\dim N(T) - \dim N(T^*)$.

THEOREM 2. *Let T and B be two closed Fredholm operators of X into X, then TB is a closed Fredholm operator and $\kappa(TB) = \kappa(T) + \kappa(B)$.*

Proof. See [5, p. 103].

THEOREM 3. *Let T be a closed Fredholm operator of a real or complex Banach space X into X, T^* its conjugate and suppose $\dim N_\infty(T) < \infty$ and $\dim N_\infty(T^*) < \infty$. Then*

(i) *$\alpha(T) < \infty$ and $\delta(T) < \infty$,*

(ii) *$R_p(T)$ is closed and $X = N_p(T) \oplus R_p(T)$ where $p = \alpha(T)$,*

(iii) *T is a one-one map of $R_p(T)$ onto $R_p(T)$ with bounded inverse,*

(iv) *$\dim N_\infty(T) = \dim N_\infty(T^*)$ and $N_p(T^*) = N_\infty(T^*)$ and*

(v) *$\kappa(T) = 0$.*

Proof. The proof of $\alpha(T) < \infty$ is immediate. That $R_p(T)$ is closed follows from Theorem 2 by an induction. To complete the proof of statements (i) and (ii) we first show $\delta(T) < \infty$. Suppose $\delta(T)$ is not finite. Then $R_{i+1}(T)$ is a proper subset of $R_i(T)$ for $i = 0, 1, \cdots$. For each $i > 0$, choose an $x_i \in R_{i-1}(T) - R_i(T)$ and a $y_i \in X^*$ such that, $y_i(x_i) = 1$, $y_i(x_j) = 0$ for $j < i$ and $y_i(R_i(T)) = 0$. Each $y_i \in N_i(T^*)$ [5, p. 59] and as the x_i are linearly independent so are the y_i. Hence $\dim N_\infty(T^*) = \infty$, contradicting our hypothesis and so $\delta(T) < \infty$.

As both the ascent and descent of T are finite, if $p = \alpha(T)$ then $N_p(T) \cap R_p(T) = \{0\}$. Therefore as $\dim N_p(T) < \infty$, $N_p(T) \oplus R_p(T)$

is closed [5, p. 16]. Suppose $X \neq N_p(T) \oplus R_p(T)$. Then, since $\operatorname{codim} R_p(T) = \dim N_p(T^*) < \infty$, there exists a finite dimensional subspace M such that $X = N_p(T) \oplus R_p(T) \oplus M$, [5, p. 103]. Moreover, as T^p is the product of closed Fredholm operators

$$\overline{D_p(T)} = X . \tag{1}$$

Thus

$$\overline{D_p(T)} = N_p(T) \oplus \{R_p(T) \cap \overline{D_p(T)}\} \oplus M . \tag{2}$$

On the other hand by Lemma 1

$$D_p(T) = N_p(T) \oplus \{R_p(T) \cap D_p(T)\} . \tag{3}$$

Clearly

$$N_p(T) \oplus \{R_p(T) \cap D_p(T)\} \subseteqq N_p(T) \oplus \{R_p(T) \cap \overline{D_p(T)}\} . \tag{4}$$

Thus Eqs. (1), (2), (3) and (4) imply $M = \{0\}$ and we have $X = N_p(T) \oplus R_p(T)$. Statement (iii) is now immediate from statement (ii), Lemma 1 and the bounded inverse theorem [12, p. 179].

Lastly, we show statement (iv). From statement (ii) it follows that $X^* = R_p(T)^\perp \oplus N_p(T)^\perp$ [5, p. 100] where for $U \subseteqq X$, $U^\perp = \{y \in X^* \mid y(x) = 0, x \in U\}$. The $\dim R_p(T)^\perp = \dim N_p(T)$, $R_p(T)^\perp = N_p(T^*)$ [5, p. 51] and $N_p(T)^\perp = R_p(T^*)$ [5 p. 95]. Thus $X^* = N_p(T^*) \oplus R_p(T^*)$ and so $N_p(T^*) = N_\infty(T^*)$, which in turn implies

$$\dim N_\infty(T) = \dim N_p(T) = \dim N_p(T^*) = \dim N_\infty(T^*) .$$

To show statement (v) we note that statement (iv) implies $\kappa(T^p) = 0$. But by an induction it follows from Theorem 2 that $\kappa(T^p) = p\kappa(T)$. Hence $\kappa(T) = 0$.

2. Odd multiplicity results. Let X be a real Banach space and suppose L is a densely defined *closed* linear map of X into X with a real characteristic value λ_0, that is there exists a nonzero $x_0 \in X$ such that $\lambda_0 L x_0 = x_0$. The algebraic multiplicity of λ_0 is defined to be $\dim N_\infty(I - \lambda_0 L)$. Suppose further $G(\lambda, x)$ is a continuous map of a neighborhood of $(\lambda_0, 0) \in \boldsymbol{R} \times X$ into X satisfying

$$\| G(\lambda, x_1) - G(\lambda, x_2) \| = h(x_1, x_2) \| x_1 - x_2 \| \tag{5}$$

for (λ, x) near $(\lambda_0, 0)$ and where $h(a, b)$ is a function independent of λ tending to zero as both a and b tend to zero. We shall be concerned with finding nontrivial solutions, that is points $(\lambda, x) \in \boldsymbol{R} \times X$, $x \neq 0$, satisfying the equation

$$x = \lambda L x + G(\lambda, x) . \tag{6}$$

The closure of the set of nontrivial solutions of (6) will be denoted by S. We will call λ_0 a bifurcation point of Eq. (6) if every neighborhood of $(\lambda_0, 0)$ contains a nontrivial solution of (6). Using a topological degree argument, P. H. Rabinowitz [8] proved the following bifurcation result:

THEOREM 4. *If L is completely continuous, Ω is a bounded open set in $\boldsymbol{R} \times X$ containing $(\lambda_0, 0)$, $G(\lambda, x)$ is completely continuous on $\bar{\Omega}$ and λ_0 is a characteristic value of odd algebraic multiplicity, then there is a maximal closed connected subset C of S such that $C \subseteqq \bar{\Omega}$, $(\lambda_0, 0) \in C$ and C either meets the boundary of Ω or meets $(\hat{\lambda}, 0)$, where $\hat{\lambda}$ is another characteristic value of L.*

By methods somewhat similar to those of [7] we can obtain a partial extension of Theorem 4 to those instance where Eq. (6) is not completely continuous and indeed L is not even bounded.

THEOREM 5. *Let L and G be as described above. Suppose $I - \lambda_0 L$ is a closed Fredholm operator and λ_0 is a characteristic value of odd algebraic multiplicity of L and a characteristic value of finite algebraic multiplicity of L^*. Then there exists a maximal closed connected subset of S meeting $(\lambda_0, 0)$ and λ_0 is a bifurcation point.*

Proof. By Theorem 3, $X = N \oplus R$ where $N = N_\infty(I - \lambda_0 L)$ and $R = R_\infty(I - \lambda_0 L)$. Thus $x \in X$ can be uniquely expressed as $x = u + v$ where $u \in N$ and $v \in R$. Moreover for all $\lambda \in \boldsymbol{R}$, $I - \lambda L: N \rightarrow N$, $R \rightarrow R$ and $G(\lambda, x) = G_N(\lambda, x) + G_R(\lambda, x)$ where $G_N(\lambda, x) \in N$ and $G_R(\lambda, x) \in R$. Thus our problem is equivalent to that of finding solutions$(\lambda, u, v) \in \boldsymbol{R} \times N \times R$ of the system of equations

$$u - \lambda L u = G_N(\lambda, u + v) \tag{6a}$$

$$v - \lambda L v = G_R(\lambda, u + v) \,. \tag{6b}$$

Since $I - \lambda L$ has a bounded inverse on R for λ near λ_0 and $(I - \lambda L)^{-1}$ is continuous in λ for all $1/\lambda$ in the resolvent of L (as a mapping of $R \rightarrow R$) [12, p. 257], (λ, u, v) is a solution of (6a), (6b) if and only if (λ, u, v) is a solution of the system

$$u = \lambda L u + G_N(\lambda, u + v) \tag{7a}$$

$$v = (I - \lambda L)^{-1} G_R(\lambda, u + v) \,. \tag{7b}$$

An application of the contraction mapping principle [4, p. 260] to Eq. (7b) shows the existence of a uniquely determined continuous function, $v(\lambda, u) = v$ such that $v(\lambda, u) = (I - \lambda L)^{-1}G_R(\lambda, u + v(\lambda, u))$ for all (λ, u) in a neighborhood of $(\lambda_0, 0)$. Consequently it suffices to find solutions in $\boldsymbol{R} \times N$ of the equation

$$u = \lambda Lu + G_N(\lambda, u + v(\lambda, u)) . \tag{8}$$

By continuity Eq. (8) satisfies the hypothesis of Theorem 4 near $(\lambda_0, 0)$ thus there exists a closed connected subset C' of S meeting $(\lambda_0, 0)$ and λ_0 is a bifurcation point. As the union of connected sets containing a common point is connected an application of Zorn's lemma [6, p. 62] will show that S contains a unique maximal closed connected subset C meeting $(\lambda_0, 0)$. Thus $C' \subseteqq C$ and our theorem is proven.

REMARK. Rabinowitz [9, p. 17] has proven a result similar to Theorem 5 for L bounded.

3. Gradient operators. Let X be a real Hilbert space and suppose L is a densely defined *closed self-adjoint* linear operator of X into X with a characteristic value λ_0. Suppose further $G(\lambda, x)$ is a twice continuously differentiable map of a neighborhood of $(\lambda_0, 0) \in \boldsymbol{R} \times X$ into X such that for fixed λ, $G(\lambda, x)$ is a *gradient* map [13, p. 54] and $G(\lambda, 0) \equiv 0$ and $G_x(\lambda, 0) \equiv 0$ for all λ near λ_0. We shall be concerned with solving the bifurcation problem for the equation

$$x = \lambda Lx + G(\lambda, x) . \tag{9}$$

For L bounded, M. S. Berger [1] has shown

THEOREM 6. *If L is bounded and $I - \lambda_0 L$ is a Fredholm operator then λ_0 is a bifurcation point of Eq.* (9).

The same result may be obtained for L unbounded but closed.

THEOREM 7. *If $I - \lambda_0 L$ is a Fredholm operator then λ_0 is a bifurcation point of Eq.* (9).

Proof. As in the proof of Theorem 5 it suffices to solve the bifurcation problem for the system of equations

$$\begin{aligned} u &= \lambda Lu + G_N(\lambda, u + v) \\ v &= (I - \lambda L)^{-1}G_R(\lambda, u + v) \end{aligned}$$

for $(\lambda, u, v) \in \boldsymbol{R} \times N \times R$, $G_N \in N$ and $G_R \in R$, where $N = N(I - \lambda_0 L)$ and $R = R(I - \lambda_0 L)$. By the implicit function theorem [4, p. 265] there exists a uniquely determined, twice continuously differentiable function $v(\lambda, u)$ such that

$$v(\lambda, u) = (I - \lambda L)^{-1} G_R(\lambda, u + v(\lambda, u))$$

for all (λ, u) near $(\lambda_0, 0)$. Thus our problem is reduced to solving the operator equation

$$u = \lambda L u + G_N(\lambda, u + v(\lambda, u)) \tag{10}$$

for $(\lambda, u) \in \boldsymbol{R} \times N$. Moreover, if for fixed λ, $\mathscr{G}(\lambda, x)$ is the potential of $G(\lambda, x)$ (that is $\mathscr{G}_x(\lambda, x) = G(\lambda, x)$) then arguing as in the proof of Theorem 1 in [14] one can readily verify that for fixed λ

$$\frac{1}{2}\langle (I - \lambda L)(u + v(\lambda, u)), u + v(\lambda, u)\rangle - \mathscr{G}(\lambda, u + v(\lambda, u))$$

is a potential for $(I - \lambda L)u - G_N(\lambda, u + v(\lambda, u))$. ($\langle\ ,\ \rangle$ is the inner product on X.)

Therefore Eq. (10) is a gradient operator equation. Hence, as (10) satisfies the hypothesis of Theorem 6, λ_0 must be a bifurcation point of Eq. (9).

4. Remarks and two counterexamples. By Theorem 3 we could have replaced the hypothesis on the multiplicity of λ_0 of Theorem 5 by the equivalent hypothesis λ_0 is a characteristic value of the same odd algebraic multiplicity of both L and L^*. Moreover we could have alternately assumed λ_0 a characteristic value of odd algebraic multiplicity and $I - \lambda_0 L$ is a Fredholm operator of index zero since we can show

THEOREM 8. *Suppose T is a closed linear Fredholm operator of X into X. If* $\dim N_\infty(T) < \infty$ *and* $\kappa(T) = 0$ *then* $\dim N_\infty(T^*) = \dim N_\infty(T)$.

Proof. Let $p = \alpha(T)$. Then by Theorem 2, $\kappa(T^p) = p\kappa(T) = 0$. As in the proof of Theorem 3, $N_p(T) \oplus R_p(T) \oplus M = X$. If $M \neq \{0\}$ then $\dim N_p(T^*) > \dim N_p(T)$ which is impossible as $\kappa(T^p) = 0$. Hence $X = N_p(T) \oplus R_p(T)$ which implies $X^* = N_p(T^*) \oplus R_p(T^*)$ and so $\dim N_\infty(T^*) = \dim N_p(T^*) = \dim N_p(T)$.

We give an example to show that if λ_0 is a characteristic value of odd algebraic multiplicity of L but $\kappa(I - \lambda_0 L) < 0$ then λ_0 may fail to be a bifurcation point.

Let $H = \ell_2 \times \boldsymbol{R}$ and let S be the shift operator on ℓ_2, that is $S\colon (a_1, a_2, \cdots) \rightarrow (0, a_1, a_2, \cdots)$ and consider the equations

$$x = \lambda[(S + I)x + (y^2, 0, \cdots)]$$
$$y = \lambda y$$

for $(\lambda, x, y) \in \boldsymbol{R} \times H = \boldsymbol{R} \times (\ell_2 \times \boldsymbol{R})$. $\lambda_0 = 1$ is a characteristic value of linear part of odd multiplicity and as

$$\begin{pmatrix} I - \lambda_0(S + I) & 0 \\ 0 & I - \lambda_0 I \end{pmatrix} = -\begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} = L$$

we see $\kappa(L) < 0$. Moreover a simple examination of cases for $y = 0$ or $y \neq 0$ shows that $\lambda_0 = 1$ is not a bifurcation point of the system.

Lastly we show that if λ_0 is a characteristic value of odd geometric multiplicity (that is $\dim N(I - \lambda_0 L)$ is odd) but $\kappa(I - \lambda_0 L) > 0$ then λ_0 may fail to be a bifurcation point.

Let $x, y \in \ell_2$ and consider the system of equations

$$\begin{aligned} x &= \lambda(S^2 + I)x + \lambda(\|x\|^2 + \|y\|^2, 0, \cdots) \\ y &= \lambda(S^{*3} + I)y \end{aligned}$$

where S^* is the adjoint of S, that is the left shift operator $S^*\colon (y_1, y_2, \cdots) \to (y_2, y_3, \cdots)$.

If $\lambda_0 = 1$ then the dimension of the null space of the linearized part of the system, $I - \lambda_0 L$, is equal to 3 and $\dim N(I - \lambda_0 L^*) = 2$. Thus $\kappa(I - \lambda_0 L) = 3 - 2 > 0$. Moreover for λ near λ_0 the system has no solutions other than trivial ones. Indeed, suppose (λ, x, y) is a nontrivial solution and $x = (x_1, x_2, \cdots)$. Then we have the equality

$$(I - \lambda)(x_1, x_2, \cdots) = \lambda(0, 0, x_1, x_2, \cdots) + \lambda(\|x\|^2 + \|y\|^2, 0, \cdots) .$$

Thus $x_1 = \lambda(1 - \lambda)^{-1}(\|x\|^2 + \|y\|^2)$ and for $n = 1, \cdots, x_{2n} = 0$ and $x_{2n+1} = \lambda(1 - \lambda)^{-1} x_{2n-1}$. Therefore

$$x = (\|x\|^2 + \|y\|^2)(\lambda(1 - \lambda)^{-1}, 0, \lambda^2(1 - \lambda)^{-2}, \cdots) .$$

However as $\lambda^n(1 - \lambda)^{-n} \not\to 0$ as $n \to \infty$ for $1/2 \leqq \lambda$, $x \notin \ell_2$. Hence the system does not have any nontrivial solutions for λ near $\lambda_0 = 1$.

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