PACIFIC JOURNAL OF MATHEMATICS
Vol 58, No. 1, 1975

A MAP OF E^3 ONTO E^3 TAKING NO DISK ONTO A DISK

EDYTHE P. WOODRUFF

An example is given of an u.s.c. decomposition in which no disk in E^3 maps onto a disk under the natural projection map P, and, furthermore, the decomposition space E^3/G is homeomorphic to E^3. Each nondegenerate element is a tame arc. The image $P(H)$ of the set of nondegenerate elements is 0-dimensional, although $\mathrm{Cl}\, P(H)$ is E^3. The basic construction used is called a knit Cantor set of nondegenerate elements.

Bing and Borsuk [3] have given an example of a 3-dimensional absolute retract R containing no disk. They define a particular u.s.c. decomposition of E^3 that yields R as the decomposition space. Hence, their example is a closed map of E^3 taking no disk onto a disk, but, of course, their image is not E^3.

In [8] the author defined a set $X \subset E^3$ to be the P-lift of a set Y contained in the decomposition space E^3/G if and only if X and Y are homeomorphic and the image of X under the natural projection is Y. A disk is said to said to be P-liftable if and only if it has a P-lift. Using this terminology, the example that is constructed in this note has no P-liftable disk in the image space.

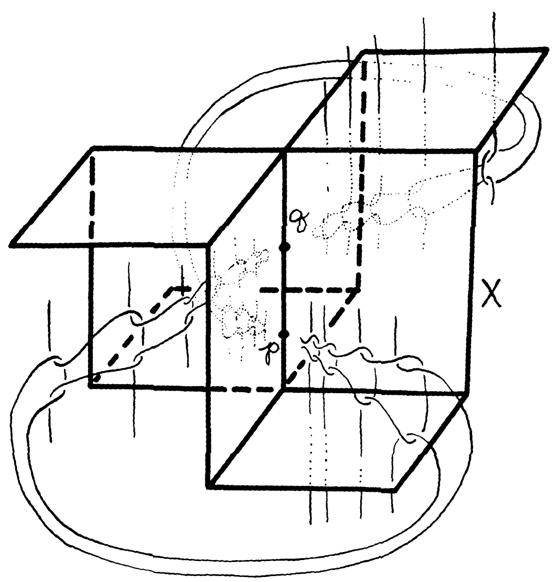
In [1] Armentrout asked whether there exists a pointlike decomposition G of E^3 such that there is a 2-sphere S in E^3/G that can not be approximated by a P-liftable sphere. This was first answered by the author in [9] by giving an example of a space E^3/G containing such a 2-sphere. In the decomposition space of this note no 2-sphere is P-liftable. Hence, this space is another answer to Armentrout's query.

The construction we describe in this note is based on a knit example in the author's papers [6], [7], and [9]. It is assumed that the reader is familiar with this example and the notations in [6]. We also need the following definitions.

DEFINITION. Let J_1' be the circle in the $x—y$ plane with radius 1 and center at the origin, and J_2' be the circle in the $y—z$ plane with radius 1 and center at $y = -1$, $z = 0$. Any two tame simple closed curves J_1 and J_2 in E^3 are said to *simply link* if and only if there is a homeomorphism of E^3 onto itself taking J_1 and J_2 onto J_1' and J_2', respectively. Two disjoint compact sets S_1 and S_2 are said to *simply link* if and only if there exist simple closed curves $J_1 \subset S_1$ and $J_2 \subset S_2$ such that J_1 and J_2 simply link.

DEFINITION. Let $\{M_i\}$ be a defining sequence for a decomposition G. The sequence $M_1^1, M_2^1, M_2^2, M_3^1, M_3^2, M_3^3, M_4^1, \cdots$, denoted by $\{M_i^j\}$, is called a *compound defining sequence* for G if and only if (1) $i = 1,2,3,\cdots$; (2) $1 \leqq j \leqq i$; (3) $\{M_i^j\}$ with the lexicographic order indicated above on the indices is a defining sequence; (4) M_i^{j-1} is a regular neighborhood of M_i^j; and (5) $M_i^i = M_1$. Given any decomposition of E^3 with a defining sequence, there exists a compound defining sequence.

In [6] and [9] the author gave an example of a knit decomposition G_0 of E^3, a 2-complex X in E^3, and an $\varepsilon > 0$ such that $P(X) \subset E^3/G_0$ is a disk D having the property that no disk D_ε which is ε-homeomorphic to D is P-liftable. This decomposition used "knit Cantor sets of nondegenerate elements". In the figure two countably infinite sets of arcs knit from the point p to the point q are indicated. Each arc pictured represents a Cantor set of arcs. These Cantor sets of arcs and the limiting arc g_p containing p and q are the nondegenerate elements of the decomposition G_0. The 2-complex X consists of eight squares that do not form a disk. Notice that each arc except g_p pierces X in a point. Since in E^3/G_0 the arc g_p has an image that is a point, the image of X is a disk D. The decomposition is a modification of two (2,1) toroidal decompositions. The entwining of the nondegenerate elements caused by the (2,1) toroidal decompositions is not indicated in the figure. It would be above and below the portions shown. The result



of the entwining is that Bd T_0 is not homotopic to a point in $E^3 - H^*$. It was shown that E^3/G_0 is homeomorphic to E^3. In the proof only disks that are ε-homeomorphic to D were considered. Using similar arguments, it can be shown that there is no disk δ in E^3 such that $P(\delta)$ is a disk in E^3/G_0 and Bd X is homotopic in $E^3 - M_1$ to Bd δ. (In **[6]** M_1 is the first manifold in a particular defining sequence for G_0.) Hence, for this decomposition G_0, there is a regular neighborhood T_0 of Bd X in E^3 which is an unknotted polyhedral solid torus such that no simple closed curve homotopic in T_0 to its core bounds a disk δ that has an image in E^3/G_0 that is a disk.

Simply linking T_0, there is an unknotted polyhedral solid torus S_0 that contains M_1, which in turn contains all the nondegenerate elements of G_0. Let T_C and S_C be any pair of simply linked unknotted polyhedral solid tori. There is a homeomorphism of E^3 onto itself that takes T_0 and S_0 onto T_C and S_C, respectively. Given $d > 0$, this homeomorphism can be chosen so that the diameter of each nondegenerate element is less than d. (This follows from the proof that E^3/G_0 is homeomorphic to E^3.) Hence, given simply linked unknotted polyhedral solid tori T_C and S_C and given $d > 0$, there is a decomposition G_C of E^3 with nondegenerate elements H_C such that (1) $H_C^* \subset S_C$; (2) for each $g \in H_C$, diam $g < d$; and (3) no disk δ with Bd δ homotopic in T_C to its core has an image in E^3/G_C which is a disk.

We now construct a family $\mathcal{T}$ of solid tori T_C. Associated with each T_C there are an S_C and H_C having the above properties. The family is dense in E^3 and so chosen that for any disk D in E^3 there are a solid torus $T_C \in \mathcal{T}$ and a tame simple closed curve $J \subset D \cap T_C$ such that J is homotopic in T_C to its core. Let G be the union of $H = \{g \in H_C \colon H_C$ is associated with some $T_C \in \mathcal{T}\}$ and points in $E^3 - H$. Then no disk projects onto a disk under the mapping $P\colon E^3 \to E^3/G$.

The family $\mathcal{T}$ is constructed in stages. To define the first stage, we start with the set of points $\mathcal{V}_1 = \{(p/2,\ q/2,\ r/2)\colon p,\ q,$ and r are integers$\}$. Associated with $\mathcal{V}_1$ is the set $\mathcal{C}_1$ of all unknotted polygonal simple closed curves having vertices in $\mathcal{V}_1$ and diameters not greater than one.

For any tame unknotted simple closed curve J, let $L_J = \mathrm{lub}\{d\colon$ there is a polygonal simple closed curve K in the unbounded component of $E^3 - N_d(J)$ such that K simply links $J\}$. (Here $N_d(J)$ denotes the d-neighborhood of J.) For each $C \in \mathcal{C}_1$, choose a polygonal simple closed curve K_C that simply links C and lies in the unbounded component of the complement of the $L_C/2$-neighborhood of C. These can certainly be chosen so that the diameter of each K_C is less than four and each K_C fails to intersect the union of the other such simple closed curves associated with elements of $\mathcal{C}_1$. For each K_C, choose a polyhedral solid torus S_C with core K_C and contained in the $L_C/4$-neighborhood of K_C. This implies that C simply links S_C. The set of these solid tori S_C can be chosen to be mutually disjoint.

The method for choosing the solid torus T_C that simply links S_C depends on the fact that the union of two solid tori with common boundaries and disjoint interiors is the 3-sphere obtained as the union of E^3 and the point at infinity. Since simply linked tori do not have common boundaries, for each C enlarge S_C slightly: choose a solid torus S'_C that satisfies the definition of S_C and contains S_C in its interior. Let N be the $L_C/4$-neighborhood of C. Let A_C be the complement of a polyhedral 3-ball containing $N \cup S_C$ and having diameter less than eight. Let B_C be a polyhedral 3-ball (a tubular neighborhood of a polygonal arc) in $E^3 - (A_C \cup N \cup S'_C)$ connecting A_C and S'_C in such a way that $\mathrm{Cl}(E^3 - (A_C \cup B_C \cup S'_C))$ is a polyhedral unknotted solid torus having C as a core. Denote this solid torus with core C by T_C. Observe that T_C contains the $L_C/4$-neighborhood of C and simply links S_C. Let $\mathscr{T}_1 = \{T_C : C \in \mathscr{C}_1\}$. This is the first stage of the construction of the family $\mathscr{T}$.

For each T_C and S_C, we choose a decomposition G_C, having the above properties with respect to T_C and S_C and having no nondegenerate element with diameter greater than one. It can be assumed that each nondegenerate element is polygonal.

From the definition of a compound defining sequence it follows that each component which is a solid torus is one of a finite nest of solid tori which are regular neighborhoods of the innermost one of the nest. We assume that all solid tori in a nest are tubular neighborhoods of the same polygonal simple closed curve.

To define the nth stage, let $\mathscr{V}_n = \{(p/(2^n), q/(2^n), r/(2^n)) : p, q, \text{ and } r$ are integers$\}$. The family $\mathscr{C}_n$ is the set of all unknotted polygonal simple closed curves having vertices in $\mathscr{V}_n$ and diameters again not greater than one. Complete choices of T_C and S_C as in the first stage with the added requirement that each K_C miss all nondegenerate elements from previous stages.

We next determine the size requirement for nondegenerate elements at the nth stage. For any $C \in \mathscr{C}_n$, the associated solid torus S_C intersects the compound defining sequences of only a finite number of the sets H_C previously defined. Call them $H_k, 1 \leqq k \leqq k_C$, where k_C is the appropriate integer. For each H_k, let $\{(M_k)_i^j\}$ be the compound defining sequence. Because S_C misses each set H_k^*, there are only a finite number of $(M_k)_i^j$ whose boundaries intersect S_C. For each C, let $x_C = \min\{d : d$ is the distance between two sets $\mathrm{Bd}\,(M_k)_i^j \cap S_C$ for some values of $i, j,$ and $k\}$. This x_C is strictly positive. We require that each nondegenerate element in H_C have diameter less than x_C and less than $1/n$. There is a decomposition G_C satisfying this and the conditions above with respect to S_C and the corresponding T_C. Again assume that each image of a nondegenerate element is polygonal and that manifolds in compound defining sequences are unions of prisms. This completes the construction of the nth stage.

Let G be the union of $H = \{g \in H_C : C \in \mathscr{C}_n$ for some positive integer $n\}$ and points in $E^3 - H^*$. This decomposition G defines the map claimed in the title.

The proof is based on McAuley's countably shrinkable theorem [4], as slightly revised by Reed [5], To use the theorem it is necessary to shrink certain elements without permitting others to grow too much. Some of the shrinking is based on Bing's shrinking of the (2,1) toroidal decomposition [2]. Recall that, in the construction, elements at a later stage are not permitted to intersect boundaries of more than two manifold stages in previous compound defining sequences. This allows growth of later stage nondegenerate elements to be controlled during the shrinking of a particular stage. The proof is tedious, but straightforward.

The author wishes to thank Charles H. Goldberg for his comments concerning the construction and proof.

REFERENCES

1. Steve Armentrout, *Monotone decompositions of E^3*, Annals of Math. Studies, **60** (1966), 1–25.

2. R. H. Bing, *A homeomorphism between the 3-sphere and the sum of two horned spheres*, Ann. of Math., (2) **56** (1952), 354–362.

3. R. H. Bing and K. Borsuk, *A 3-dimensional absolute retract which does not contain any disk*, Fund. Math., **54** (1964), 159–175.

4. Louis F. McAuley, *Upper semicontinuous decompositions of E^3 into E^3 and generalizations to metric spaces*, *Topology of 3-Manifolds and Related Topics*, Proc. Univ. of Georgia Inst., 1961, 21–26.

5. Myra Jean Reed, *Decomposition spaces and separation properties*, Doctoral Dissertation, SUNY/Binghamton, May, 1971.

6. Edythe P. Woodruff, *Concerning the condition that a disk in E^3/G be the image of a disk in E^3*, Proc. Conf. on Monotone Mappings and Open Mappings at SUNY/Binghamton (1970).

7. ———, *Concerning the condition that a disk in E^3/G be the image of a disk in E^3*, Doctoral Dissertation, SUNY/Binghamton, May, 1971.

8. ———, *Conditions in which disks are P-liftable*, Trans. Am. Math. Soc., **186** (1973), 403–418.

9. ———, *Examples of disks in E^3/G which cannot be approximated by P-liftable disks*, Fund. Math., **86** (1974), 117–136.

TRENTON STATE COLLEGE
TRENTON, NEW JERSEY 08625