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# A COUNTEREXAMPLE IN THE THEORY OF DEFINABLE AUTOMORPHISMS

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**As it is well known, the groups of definable automorphisms of two elementary equivalent relational structures satisfy the same $\forall_1$-statements. We show that this does not hold in general for $\forall_2$-statements, thus correcting an error in the literature.**

**0.** An automorphism $\varphi$ of a model $\mathfrak{M}$ is said to be definable if there is a formula $H$ of the (first-order) language of $\mathfrak{M}$ and elements $a_1, \cdots, a_n \in M$, such that for all $x, y \in M$

$$\mathfrak{M} \models H(x, y, a_1, \cdots, a_n) \quad \text{iff} \quad \varphi(x) = y.$$

Let Def Aut ($\mathfrak{M}$) denote the group of definable automorphisms of $\mathfrak{M}$ (see [5]).

In [1] it is remarked that if $\mathfrak{M}$ and $\mathfrak{N}$ are elementary equivalent, then Def Aut($\mathfrak{M}$) and Def Aut($\mathfrak{N}$) are universally equivalent. In this note we show that this is the best possible result. We give an example of an $\forall\exists$-statement, which holds in Def Aut($\mathfrak{M}$) but not in Def Aut($\mathfrak{M}'$), where $\mathfrak{M}$ and $\mathfrak{M}'$ are two elementary equivalent models. In fact our $\mathfrak{M}'$ is an elementary extension of $\mathfrak{M}$. This disproves Theorems 1,2 in [3] (p. 109).

We construct our example from the Prüfer group $\mathbf{Z}(3^\infty)$ and investigate definability using the method of Ehrenfeucht games.

**1.** Our example is as follows. $H$ is the (group theoretical) statement

$$\forall x \exists y \; x = y^2.$$

We define $\mathfrak{M}$ to be $(M, Z, \omega, <, f)$, where $M$ is the disjoint union of $Z$ and $\omega$. $Z$ is the underlying set of the Prüfergroup $\mathbf{Z}(3^\infty)$, which is defined as

$$\left\{ \frac{n}{3^m} \,\middle|\, n, m \in \mathbf{Z} \right\} / \mathbf{Z}.$$

$<$ is the natural ordering of $\omega$, the set of natural numbers. $f$ is a binary function defined by

$$f(n,z) := z + \frac{1}{3^n}\mathbf{Z} \quad \text{if} \quad n \in \omega \quad \text{and} \quad z \in Z$$

$$(+\text{ stands for the addition in } \mathbf{Z}(3^\infty))$$

$$:= 0 \ (\in \omega) \quad \text{otherwise.}$$

We shall denote by $f_n$ the function

$$\lambda z f(n,z) \colon Z \to Z \qquad (n \in \omega).$$

Every automorphism of $\mathfrak{M}$ operates on $\omega$ as the identity and therefore commutes with each $f_n$. The $f_n$ generate the group of all translations of $\mathbf{Z}(3^\infty)$, and so it is easily seen (see e.g. **[4]** p. 43) that the automorphisms of $\mathfrak{M}$ are just those permutations of $M$, which leave $\omega$ fixed and operate on $Z$ like a translation. Since the $f_n$ are definable, all automorphisms are definable and hence

$$\text{Def Aut}(\mathfrak{M}) \cong \mathbf{Z}(3^\infty) \models H.$$

Let $\mathfrak{M}' = (M', Z', W', <', f')$ be an elementary extension of $\mathfrak{M}$ such that $W' \neq \omega$. We claim that Def Aut$(\mathfrak{M}') \not\models H$.

**2.** First we look at definability in $\mathfrak{M}$.

Every element $x$ of $\mathbf{Z}(3^\infty)$ has an unique representation

$$x = \sum_{i=1}^{\infty} \frac{k_i}{3^i}\mathbf{Z},\ k_i \in \{-1,0,1\}, \text{ . almost all } k_i = 0.$$

We define

$$|x| := \sum_{i=1}^{\infty} |k_i|,\ v(x) := \max\{i \mid k_i \neq 0\} \quad \text{and}$$

$$\bar{v}(x) := \min\{\mathrm{i} \mid k_i \neq 0\}$$

We note that

(i) $|-x| = |x|$
(ii) $|x+y| \leqq |x| + |y|$
(iii) $|x+y| = |x| + |y|$ if $v(x) < \bar{v}(y)$
(iv) $v(x+y) \leqq \max(v(x), v(y))$

Let $I_n$ be the set of all partial functions $\varphi$ from $Z$ in $Z$ with the following property:

dom $\varphi$ is finite and for all $a, b \in \mathrm{dom}\,\varphi$

$$|a-b| \leqq 2^n \quad \text{iff} \quad |\varphi(a) - \varphi(b)| \leqq 2^n$$

and in this case $a - b = \varphi(a) - \varphi(b)$.

Clearly $I_{n+1} \subset I_n$ and if $\varphi \in I_0$, $a, b \in \operatorname{dom} \varphi$ and $f_m(a) = b$ then $f_m(\varphi(a)) = \varphi(b)$.

We show that the family $I$ has the back and forth property: Let $\varphi \in I_{n+1}$ and $a \in Z \backslash \operatorname{dom} \varphi$. We want to extend $\varphi$ to $\varphi' \in I_n$ with $\operatorname{dom} \varphi' = \operatorname{dom} \varphi \cup \{a\}$.

There are two possible cases

(1) There is $b \in \operatorname{dom} \varphi$ such that $|a - b| \leqq 2^n$. Define $\varphi'(a) := \varphi(b) + (a - b)$. Then $\varphi' \in I_n$. For let e.g. $c \in \operatorname{dom} \varphi$ and $|c - a| \leqq 2^n$. It follows from (i) and (ii)

$$|b - c| \leqq |a - b| + |c - a| \leqq 2^n + 2^n = 2^{n+1}$$

hence

$$\varphi(b) - \varphi(c) = b - c. \quad \text{It follows}$$

$$\varphi(c) - \varphi'(a) = c - a.$$

(2) For all $b \in \operatorname{dom} \varphi$ $|a - b| > 2^n$.
Choose $a' \in Z$ such that $|a'| > 2^n$ and $\bar{v}(a') > v(\varphi(b))$ for all $b \in \operatorname{dom} \varphi$. Define $\varphi'(a) := a'$. From (iii) it follows that for all $b \in \operatorname{dom} \varphi$

$$|\varphi'(a) - \varphi(b)| > 2^n.$$

Since $\varphi^{-1} \in I_n$ iff $\varphi \in I_n$ it is clear that for all $\varphi \in I_{n+1}$ and $a \in Z$ there is an extension $\varphi'$ of $\varphi$ such that $\varphi' \in I_n$, $a \in rg\varphi'$.

Let $H_n$ be the set of all formulas (of the language of $\mathfrak{M}$), which contain at most $n$ quantifiers and where all function symbols are applied to variables only. It is shown in [2] that, if $\varphi \in I_n$, $a_1, \cdots, a_k \in \operatorname{dom} \varphi$ $b_1, \cdots, b_e \in \omega$ and $H \in H_n$

$$\mathfrak{M} \models H(a_1, \cdots, a_k, b_1, \cdots, b_e) \quad \text{iff}$$

$$\mathfrak{M} \models H(\varphi(a_1), \cdots, \varphi(a_k), b_1, \cdots, b_e).$$

This is a consequence of the back and forth property.

Let now $z \in Z$, $|z| > 2^n$ and $g$ be the translation

$$\lambda x (x + z) \colon Z \to Z.$$

We show that $g$ is not definable using a formula in $H_n$.

Assume that there is a $H \in H_n$, $a_1, \cdots, a_k \in Z$, $b_1, \cdots, b_e \in \omega$ such that for all $a, b \in Z$

$$\mathfrak{M} \models H(a, b, a_1, \cdots, a_k, b_1, \cdots, b_2) \quad \text{iff} \quad g(a) = b.$$

Choose $c \in Z$ such that $|c| > 2^n$ and $\bar{v}(c)$ and $\bar{v}(2c)$ are greater than all $v(a_i)$ and $v(z)$. (Choose a $c$ of the form $\Sigma_{i=m}^{m'} 1/3^i$). Define $\varphi(a_i) := a_i$ $(i = 1, \cdots, k)$, $\varphi(c) := c$ and $\varphi(z + c) := z - c$. It is easily seen that $\varphi \in I_n$. For

$$\left|\begin{array}{l} |(z+c) - a_i| = |z - a_i| + |c| > 2^n \quad \text{by (iii),(iv))} \\ |(z-c) - a_i| = |z - a_i| + |-c| > 2^n \quad \text{(by (i))} \\ |(z+c) - c| > 2^n \\ |(z-c) - c| = |z - 2c| = |z| + |2c| > 2^n \quad \text{(by (iii)).} \end{array}\right.$$

Therefore we have, $\mathfrak{M} \models H(c, z - c, a_1, \cdots, b_e)$

$$\text{since} \quad \mathfrak{M} \not\models H(c, z + c, a_1, \cdots, b_e)$$

Whence $z - c = z + c$ and we have the contradiction $c = 0$.

We prove now that $\mathcal{M}' \models H$. First note that

$$\left|\frac{1}{2} \cdot \frac{1}{3_m}\right| = \left|\sum_{i=1}^{m} \frac{-1}{3^i}\right| = m.$$

This and the last result imply that for all $m > 2^n$, $H(x, y, x_1, \cdots, x_r) \in H_n, a_1, \cdots, a_r \in M$ $H(x, y, a_1, \cdots, a_r)$ does not define an automorphism $\psi$ of $\mathfrak{M}$ such that $\psi^2 = f_m \cup id_\omega$. This is expressible by a set of sentences which hold also in $\mathfrak{M}'$. If we choose $m \in W' \backslash \omega$, we have for all $n \in \omega$ $m' > 2^n$, hence $f'_m \cup id_W$, is a definable automorphism such that there is no definable automorphism $\psi$ $\psi^2 = f'_m \cup id_{W'}$. Whence $\mathcal{M} \not\models H$.

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