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CONTINUOUS MEASURE-PRESERVING MAPS ONTO PEANO SPACES

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In this paper we obtain measure-theoretic versions of some topological existence theorems relating to continuous maps, in particular the Hahn-Mazurkiewitz Theorem. Let X be a Peano Space and λ a Borel measure on X with $\lambda(X) = 1$. There is a continuous measure-preserving surjection from the unit interval (with Lebesque measure) to X if and only if the support of λ is X.

1. Introduction. Peano's construction in 1890 of a continuous surjection from $I = [0, 1]$ to $S = I \times I$ led twenty-five years later to the Hahn-Mazurkiewitz theorem, which characterizes those topological spaces which are the continuous image of the unit interval.[1] Since the original paper, there have been many elegant constructions of space-filling curves, including one by Hilbert.[2] Perhaps most striking about Hilbert's construction is its remarkable symmetry: one sees easily that for each of the intervals $A_{i,j} = [i/4^j, (i+1)/4^j]$, the image of $A_{i,j}$ is a square of area 4^{-j}; and the images of distinct intervals $A_{i,j}$ and $A_{k,j}$ intersect in a set of (planar Lebesgue) measure zero. It is easy, in fact, to verify that Hilbert's space-filling curve is measure-preserving. This suggests the possibility that, under suitable restrictions, a Peano space X which is also a measure space might be the image of a continuous measure-preserving map from the unit interval. Clearly a necessary condition for the existence of such a map is that open subsets of X have positive measure. The aim of this paper is to show that this condition is sufficient as well. We will prove

THEOREM 1. *Let $I = [0, 1]$ and μ be Lebesgue measure on I. Let X be a Peano space and λ a Borel measure on X with $\lambda(X) = 1$. Then a* (necessary and) *sufficient condition that there be a continuous measure-preserving surjection $f: \langle I, \mu\rangle \rightarrow \langle X, \lambda\rangle$ is that X be the support of λ*[3].

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1. The Hahn-Mazurkiewitz theorem states:

A topological space X is the continuous image of $I = [0, 1]$ if and only if X is compact, connected, locally connected, and metrizable. Such spaces are called Peano spaces.

2. Pictures of the first three stages of Hilbert's construction may be found in **[4]**, p. 341.

3. After an abstract of this result appeared in the January, 1973 Notices of the American Mathematical Society, R. M. Blumenthal informed me that he had obtained similar results. His proof, entirely different from this, is in **[1]**.

The argument we employ follows essentially the same lines as that given on pp. 129–30 of **[2]** to prove the Hahn-Mazurkiewitz theorem. One obtains a continuous surjection from a Cantor subset of I to X, and extends that map to one defined on I.

DEFINITION 1.1. Let ν be a Borel measure on a compact metric space Y, λ a Borel measure on a compact measure space X, and $F: Y \to X$ a surjection. We say F is *measure-positive* if:

(a) For each Borel set $A \subset X$ with $\lambda(A) > 0$, $\nu(F^{-1}(A)) > 0$.

(b) For each $B \subset Y$ containing a set of positive ν-measure, $F(B)$ contains a set of positive λ-measure.

DEFINITION 1.2. Let $\langle Y, \nu\rangle$ and $\langle X, \lambda\rangle$ be as above. A surjection $F: Y \to X$ is measure-preserving if for each measurable $A \subset X$, $F^{-1}(A)$ is measurable and $\nu(F^{-1}(A)) = \lambda(A)$.

In §2, we demonstrate the existence of continuous measure-positive surjections from Cantor sets (with nontrivial Borel measures) to compact metric measure spaces. We apply these results to obtain continuous measure-positive surjection from a "fat" Cantor subset of I to the Peano space X, and in §3, extend it to a continuous measure-positive surjection $F: I \to X$. A simple application of the Radon-Nikodym theorem then guarantees the existence of the desired continuous measure-preserving map.

2. Almost measure-preserving surjections.

DEFINITION 2.1. A *Cantor set* is a compact, perfect, totally disconnected metrizable space.

It is well known[4] that if K is a Cantor set and X is a compact metric space, there is a continuous surjection from K to X. If μ and λ are Borel measures on a Cantor set K and a compact metric space X respectively, there may not exist a continuous measure-preserving surjection $f: K \to X$. If, for example, μ is Haar measure on the topological group $K = 2^\omega = \Pi_{n=1}^{\infty}\{0, 1\}$, each open-closed subset of K has measure of the form $a/2^b$ (a, b positive integers). Thus if there is an open-closed subset of X of λ-measure $1/3$, f cannot be both continuous and measure-preserving. The most we can expect in general is that the measure of each $A \subset X$ and its inverse in Y be "close".

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4. See **[2]**, pp. 127–128, **[3]**, p. 166 or **[4]**.

DEFINITION 2.2. Let S be a sigma-algebra of subsets of a space Y, and μ and ν measures defined on S. We say μ and ν are *ϵ-close on S* if for each $A \subset S$ we have

$$(1+\epsilon)^{-1}\mu(A) \leqq \nu(A) \leqq (1+\epsilon)\mu(A).$$

DEFINITION 2.3. Let $\langle X, B, \lambda \rangle$ be a measure triple, with λ a measure defined on the Borel sets B of X. Let μ be a measure on the topological space Y, $f\colon Y \to X$ a continuous surjection, and $\epsilon > 0$. We say f is *ϵ-almost measure preserving*, (AMP(ϵ)), if the measures μ and $f^{-1}(\lambda)$ are ϵ-close. Where there is no risk of confusion, we will suppress the ϵ.

The aim of this section is to prove

THEOREM 2. *Let μ be a nonatomic Borel measure on a Cantor set K, with* $\operatorname{supp}(\mu) = K$ *and* $\mu(K) = 1$. *Let λ be a Borel measure on a compact metrizable space X, with* $\operatorname{supp}(\lambda) = X$ *and* $\lambda(X) = 1$. *For each $\epsilon > 0$, there is an* AMP(ϵ) *surjection* $f_\epsilon\colon \langle K, \mu \rangle \to \langle X, \lambda \rangle$.

The proof consists in the construction of sequences of partitions of both K and X, in such a manner that corresponding partition elements of K and X are of "close" measure.

Note. Any AMP surjection is measure-positive.

Partitions of X.

Agreement. Set $I_n = [0,1]$ with metric $\rho_n(a_n, b_n) = n^{-1}|a_n - b_n|$. $I^\omega = \Pi_{n=1}^\infty I_n$, with metric $\rho(\langle a_n \rangle, \langle b_n \rangle) = \sup_n \rho_n(a_n, b_n)$, is the Hilbert cube. If X is the compact metrizable space of Theorem 2, we can embed X in $\Pi_{n=1}^\infty (0,1)$. All appropriate identifications having been performed, we may think of λ as a Borel measure on I^ω, with $\operatorname{supp} \lambda = X$; and we will use ρ as the metric on X.

DEFINITION 2.4. The finite collection $\mathcal{P} = \{P_1, P_2, \cdots, P_n\}$ of subsets of X is called a *λ-null partition of X* if

(a) each P_i is closed (relative to X), and $X = \cup_{i=1}^n P_i$.

(b) $\lambda(P_i) > 0$ for each i.

(c) $i \neq j \Rightarrow \lambda(P_i \cap P_j) = 0$.

Where the context is clear, we will refer to to $\mathcal{P}$ simply as a partition.

DEFINITION 2.5. Let $\mathcal{P} = \{P_1, \cdots, P_n\}$ and $Q = \{Q_1, \cdots, Q_m\}$ be partitions of X. We say *Q is a refinement of $\mathcal{P}$* if, for each P_i, the collection $\{Q_j \in Q : \lambda(P_i \cap Q_j) > 0\}$ forms a partition of P_i.

DEFINITION 2.6. The *diameter* of any set $A \subset X$ is

$$D(A) = \sup\{\rho(x, y): x, y \in A\}.$$

DEFINITION 2.7. The *mesh of a partition* $\mathcal{P} = \{P_1, P_2, \cdots, P_n\}$ is

$$\Delta(\mathcal{P}) = \max\{D(P_i): i = 1, 2, \cdots, n\}.$$

PROPOSITION 2.1. *There is a sequence $\mathcal{P}_1, \mathcal{P}_2, \cdots$, of λ-null partitions of X, where for each n, $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$ and $\Delta(\mathcal{P}_n) < 2^{-n}$.*

Proof. Since $\lambda(X) < \infty$, we have for each i that at most countably many of the 1-co-dimensional hyperplanes of the form $\{x = \langle x_j \rangle \in I^\omega: x_i = c\}$ have positive measure. Consider, for $i = 1, 2, \cdots, N$, a collection of points $0 = c_{i,0} < c_{i,1} < \cdots < c_{i,f(i)} = 1$ such that for each j,

(a) $\rho_i(c_{i,j}, c_{i,j+1}) < 1/N$, and
(b) $\lambda\{x \in I^\omega: x_i = c_{i,j}\} = 0$.

Each of the sets of the form

$$A = A_{j_1, \cdots, j_N} = \{x \in I^\omega: c_{i,j_i} \leqq x_i \leqq c_{i,j_i+1} \quad \text{for} \quad i = 1, 2, \cdots, N,\ 0 \leqq j_i \leqq f(i) - 1\}$$

is closed, of diameter less than $1/N$, and intersects any other set $\hat{A} = A_{\hat{j}_1, \cdots, \hat{j}_N}$ in a set of measure 0. It is straightforward to verify that

$$X = \cup\{\overline{X \cap A^0}: A = A_{j_1, \cdots, j_N} \quad \text{and} \quad \lambda(X \cap A^0) > 0\}$$

so that these sets form a λ-null partition of X, of mesh less than $1/N$. A partition such as this clearly admits refinement by any similarly constructed partition, and an inductive argument provides the desired sequence $\mathcal{P}_1, \mathcal{P}_2, \cdots$.

Partitions of K. Any Cantor set is homeomorphic to the Cantor Discontinuum, or "middle thirds set" K; and any Borel measure on that Cantor set induces on K a Borel measure with respect to which the homeomorphism is measure-preserving. We may therefore assume without loss of generality that the Cantor set K of Theorem 2 is the "middle thirds" set. We note also that the condition $\operatorname{Supp}(\mu) = K$ causes no loss of generality. For μ is nonatomic, so $\operatorname{Supp}(\mu)$ is perfect, and itself a Cantor set; and any closed subset of K is a retract of K ([3], p. 165).

We now develop the apparatus for partitioning K inductively.

Notation. Let K be the "middle thirds" set. We will denote the set

$$\{x \in K : a \leqq x \leqq b\} \quad \text{by} \quad [a, b]_K.$$

PROPOSITION 2.2. *Let $A = [a, b]_K$ be an open-closed subset of K, $\mu(A) = \alpha > 0$. Suppose that $\beta > 0$ and $\{\beta_i : i = 1, 2, \cdots, m\}$ are positive numbers such that $\beta = \Sigma_{i=1}^m \beta_i$, and $\gamma = \max(\alpha/\beta, \beta/\alpha)$. Then for any $\gamma^* > \gamma$, there is a partition of A into m disjoint open-closed sets $A_i = [a_i, b_i]_K$ such that $\max_{1 \leqq i \leqq m} (\alpha_i/\beta_i, \beta_i/\alpha_i) < \gamma^*$, where $\alpha_i = \mu(A_i)$, for each i.*

Proof. Setting

$$\hat{\alpha}_i = \beta_i + \frac{\beta_i}{\beta}(\alpha - \beta) \quad \text{for} \qquad i = 1, 2, \cdots, m.$$

we have that $\Sigma \hat{\alpha}_i = \alpha$ and $\max(\hat{\alpha}_i/\beta_i, \beta_i/\hat{\alpha}_i) = \max(\alpha/\beta, \beta/\alpha) = \gamma$ for each i. Now for each i there is a $\delta_i > 0$ such that

$$|x - \hat{\alpha}_i| < \delta_i \Rightarrow \max\left(\frac{x}{\beta_i}, \frac{\beta_i}{x}\right) < \gamma^*. \tag{2.8}$$

Set

$$\delta = \frac{1}{m-1} \min_{1 \leqq i \leqq m} (\delta_i), \tag{2.9}$$

and $a_1 = a$. Note that $\{x : [a_1, x]_K$ is open-closed$\}$ is dense in A, and recall that μ is nonatomic. Thus there is a point $b_1 \in A$, $b_1 < b$, such that $A_1 = [a_1, b_1]_K$ is open-closed, and $|\mu(A_1) - \hat{\alpha}_i| < \delta$. Now $A \backslash A_1$ is open-closed, and is itself of the form $[a_2, b]_K$ $(a_2 = \inf(x \in K : x > b_1))$. We may proceed inductively to define open-closed sets $A_i = [a_i, b_i]_K$ for $1 \leqq i \leqq m-1$ such that

$$i \neq j \Rightarrow A_i \cap A_j = \emptyset,$$

$$[a, b_{m-1}]_K = \bigcup_{i=1}^{m-1} A_i,$$

and

$$|\mu(A_i) - \hat{\alpha}_i| < \delta \quad \text{for} \qquad i = 1, 2, \cdots, m-1. \tag{2.10}$$

Thus $A_m = A \backslash \bigcup_{i=1}^{m-1} A_i$ is of the form $[a_m, b]_K$ and is open-closed. Set

$\alpha_i = \mu(A_i)$ for $i = 1, 2, \cdots, m$. By (2.10), $|\alpha_i - \hat{\alpha}_i| < \delta$ for each $i < m$, and

$$|\alpha_m - \hat{\alpha}_m| = |\mu(A_m) - \hat{\alpha}_m| = \left| \left[\alpha - \sum_{i=1}^{m-1} \alpha_i\right] - \left[\alpha - \sum_{i=1}^{m-1} \hat{\alpha}_i\right] \right|$$
$$\leqq \sum_{i=1}^{m-1} |\alpha_i - \hat{\alpha}_i| < (m-1)\delta < \delta_m.$$

The proposition follows from (2.9) and (2.8).

Proof of Theorem 2.

Step I. Construction of f_ϵ.

Let $\epsilon > 0$. Apply Proposition 2.2 to obtain a sequence $\{\mathcal{P}_n : n = 1, 2, \cdots\}$ of λ-null partitions of X, where each $\mathcal{P}_n$ refines its predecessor, and $\Delta(\mathcal{P}_n) < 2^{-n}$. Let $1 < \gamma_0 < \gamma_1 < \gamma_2 < \cdots$ be a sequence of real numbers such that $\sup_n \gamma_n < \min(1 + \epsilon, 2)$.

We begin with the inductive construction of a sequence of partitions $\{\hat{\mathcal{P}}_n\}$ of K and a correspondence between members of $\mathcal{P}_n$ and $\hat{\mathcal{P}}_n$. For $n = 0$: Let $\hat{\mathcal{P}}_0 = \{K\}$. For $n = k$: suppose we have that

$\mathcal{P}_k = \{P_{k,1}, \cdots, P_{k,m}\}$, and
$\hat{\mathcal{P}}_k = \{\hat{P}_{k,1}, \cdots, \hat{P}_{k,m}\}$ has been constructed as a partition of K, where

(2.11.a) Each $\hat{P}_{k,j}$ is an open-closed subset of K, of the form $[a, b]_K$.

(2.11.b) $$\max_j \left(\frac{\mu(\hat{P}_{k,j})}{\lambda(P_{k,j})}, \frac{\lambda(P_{k,j})}{\mu(\hat{P}_{k,j})} \right) < \gamma_k.$$

We proceed as follows for $n = k + 1$: Let $P \in \mathcal{P}_k$, and $\lambda(P) = \beta$. We have that $\mathcal{P}_{k+1}$ partitions P into (one or more) closed sets, say $P_1, P_2, \cdots, P_m$, each of positive measure. Say $\lambda(P_i) = \beta_i$; then $\Sigma \beta_i = \beta$. Corresponding to P we have $\hat{P} \in \hat{\mathcal{P}}_k$, where $\hat{P}$ is an open-closed subset of K, of the form $[a, b]_K$. If $\mu(\hat{P}) = \alpha$, we have $\max(\alpha/\beta, \beta/\alpha) < \gamma_k$. Thus we may apply Proposition 2.2 and obtain a partition of $\hat{P}$ into m disjoint open-closed sets of the form $\hat{P}_i = [a_i, b_i]_K$ such that if $\alpha_i = \mu(\hat{P}_i)$, $\max_i (\alpha_i/\beta_i, \beta_i/\alpha_i) < \gamma_{k+1}$.

Obtaining in this manner a partitioning of each $\hat{P} \in \hat{\mathcal{P}}_k$ provides the desired partition of $\hat{\mathcal{P}}_k$, and the correspondence from the elements of $\hat{\mathcal{P}}_{k+1}$ to those of $\mathcal{P}_{k+1}$. This completes the inductive construction.

Let n be fixed. Each $\hat{P} \in \hat{\mathcal{P}}_n$ is a Cantor set, and the corresponding $P \in \mathcal{P}_n$ a nonempty compact metric space. Where $\hat{P}$ and P correspond, let $f_n|_P : \hat{P} \to P$ be a continuous surjection. Since distinct

$\hat{P}$ in $\hat{\mathscr{P}}_n$ are disjoint, the map f_n defined piecewise in this fashion is a continuous surjection from K to X.

Now let $x \in K$. Say $x \in \hat{P}_n \in \hat{\mathscr{P}}_n$, and $x \in \hat{P}_{n+1} \in \hat{\mathscr{P}}_{n+1}$. The P_{n+1} corresponding to $\hat{P}_{n+1}$ is contained in the P_n corresponding to $\hat{P}_n$, so that $|f_n(x) - f_{n+1}(x)| \leqq D(P_n) \leqq \Delta(\mathscr{P}_n) < 2^{-n}$. Thus the sequence f_n converges uniformly to a limit function $f = f_\epsilon$, which is easily seen to be a continuous surjection from K to X. We note for future use:

(2.12) If $\hat{P} \in \hat{\mathscr{P}}_n$ and $f(\hat{P}) = P \in \mathscr{P}_n$, $\max(\lambda(P)/\mu(\hat{P}), \mu(\hat{P})/\lambda(P)) \leqq \gamma_n < \min(1+\epsilon, 2)$.

Step II. f is ϵ-almost measure-preserving.

We first prove:

(2.13) Let n be fixed. If $P, Q \in \mathscr{P}_n$ and $P \neq Q$, $\mu(f^{-1}(P \cap Q)) = 0$. Let $\eta > 0$.

We have by the construction that $\lambda(P \cap Q) = 0$. Since λ is a finite measure on the compact measure space X, λ is regular: and there is an open set $V \subset X$ such that $(P \cap Q) \subset V$ and $\lambda(V) < \eta/2$. Since X is compact, there is a $\delta > 0$ such that for each $x \in X$, $\rho(x, P \cap Q) < \delta \Rightarrow x \in V$. Choose $N > n$ such that $\Delta(\mathscr{P}_n) < \delta$. Let $\hat{R} \in \hat{\mathscr{P}}_N$, and $f(\hat{R}) = R \in \mathscr{P}_N$. Suppose $\exists \hat{x} \in \hat{R}$, $f(\hat{x}) \in P \cap Q$. For any $\hat{y} \in \hat{R}$, $\rho(f(\hat{x}), f(\hat{y})) \leqq D(R) \leqq \Delta(\mathscr{P}_N) < \delta$, so that $R = f(\hat{R}) \subset V$. Thus $f^{-1}(P \cap Q) \subset \cup\{R \subset \mathscr{P}_N : f(R) \subset V\}$. As a consequence of (2.12) and Proposition 2.4 (c), we have

$$\begin{aligned}\mu(f^{-1}(P \cap Q)) &\leqq \Sigma\{\mu(\hat{R}) : \hat{R} \in \hat{\mathscr{P}}_N \text{ and } f(\hat{R}) \subset V\} \\ &< 2\Sigma\{\lambda(R) : R \in \mathscr{P}_N \text{ and } R \subset V\} < 2\lambda(V) < \eta.\end{aligned}$$

This proves (2.13).

Now let n be fixed, $R_1, R_2, \cdots, R_m \in \mathscr{P}_n$, and, for each i, $\hat{R}_i$ the element of $\hat{\mathscr{P}}_n$ such that $f(\hat{R}_i) = R_i$. Note that

$$\bigcup_{i=1}^m \hat{R}_i \subset f^{-1}\left(\bigcup_{i=1}^m R_i\right) \subset \bigcup_{i=1}^m \hat{R}_i \cup \left\{\cup\left\{f^{-1}\left(P \cap Q\right) : P, Q \in \mathscr{P}_n\right\}\right\}.$$

Recalling (2.13), (2.12), and the fact that

$$i \neq j \Rightarrow \mu(\hat{R}_i \cap \hat{R}_j) = 0 = \lambda(R_i \cap R_j),$$

we have

$$(2.14)\ (1+\epsilon)^{-1}\mu\left(f^{-1}\left(\bigcup_{i=1}^m R_i\right)\right) < \lambda\left(\bigcup_{i=1}^m R_i\right) < (1+\epsilon)\mu\left(f^{-1}\left(\bigcup_{i=1}^m R_i\right)\right).$$

If V is any open subset of X, one sees easily that

$$V = \bigcup_{k=1}^{\infty} \{P \in \mathcal{P}_k : P \subset V\};$$

and thus

$$\lambda(V) = \lim_{n\to\infty} \lambda\left(\bigcup_{k=1}^{n} \{P \in \mathcal{P}_k : P \subset V\}\right) = \lim_{n\to\infty} \lambda\left(\bigcup \{P \in \mathcal{P}_n : P \subset V\}\right).$$

Similarly,

$$\mu(f^{-1}(V)) = \lim_{n\to\infty} \mu(f^{-1}(\bigcup \{P \in \mathcal{P}_n : P \subset V\})).$$

A simple limit argument employing the preceding and (2.14) yields

$$(1+\epsilon)^{-1}\mu(f^{-1}(V)) \leqq \lambda(V) \leqq (1+\epsilon)\mu(f^{-1}(V)), \quad \text{for each open } V \text{ in } X. \tag{2.15}$$

Now let A be any measurable subset of X, and $\eta > 0$. λ is a finite measure on the compact metric space X, and is therefore regular. There are open sets V and $\hat{V}$ in X such that $A \subset V$, $\lambda(V \backslash A) < \eta/(1+\epsilon)^2$, $(V\backslash A) \subset \hat{V}$, and $\lambda(\hat{V}) < \eta/(1+\epsilon)^2$. Thus

$$\begin{aligned}(1+\epsilon)^{-1}\mu(f^{-1}(A)) \leqq (1+\epsilon)^{-1}\mu(f^{-1}(V)) \leqq \lambda(V) &= \lambda(A) + \lambda(V\backslash A) \\ &< \lambda(A) + \eta;\end{aligned}$$

and

$$\begin{aligned}\lambda(A) \leqq \lambda(V) &\leqq (1+\epsilon)\mu(f^{-1}(V)) \leqq (1+\epsilon)[\mu(f^{-1}(A)) + \mu(f^{-1}(\hat{V}))] \\ &\leqq (1+\epsilon)\mu(f^{-1}(A)) + (1+\epsilon)^2\lambda(\hat{V}) < (1+\epsilon)\mu(f^{-1}(A)) + \eta.\end{aligned}$$

As η is arbitrary,

$$(1+\epsilon)^{-1}\mu(f^{-1}(A)) \leqq \lambda(A) \leqq (1+\epsilon)\mu(f^{-1}(A)),$$

and our proof of Theorem 2 is complete.

COROLLARY. *Let μ be a nonatomic Borel measure on a Cantor set K with* $\operatorname{supp}(\mu) = K$ *and $\mu(K) < \infty$; λ a Borel measure on the compact metric space X, with* $\operatorname{supp}(\lambda) = X$, *and $\lambda(X) < \infty$. There is a continuous measure-positive surjection $f: K \to X$.*

Proof. Normalize μ and λ to total mass 1, and apply Theorem 2. Any almost measure-preserving map is measure-positive.

3. Continous measure-preserving maps onto Peano spaces. For the balance of this paper μ will denote Lebesgue measure on $I = [0, 1]$. X will be a Peano space with fixed metric ρ, and λ a Borel measure on X, with $\operatorname{supp}(\lambda) = X$ and $\lambda(X) = 1$. Our main objective is to prove

THEOREM 1. *There is a continuous measure-preserving surjection from $\langle I, \mu\rangle$ to $\langle X, \lambda\rangle$.*

This will follow easily from

THEOREM 1′. *There is a continuous measure-positive surjection from $\langle I, \mu\rangle$ to $\langle X, \lambda\rangle$.*

In order to prove this, we need two rather technical propositions. They will allow us to construct a uniformly convergent sequence of continuous surjections from I to X, each of which is measure-positive on an increasingly large part of I, and whose limit is measure-positive. Note that if $X = \{x\}$, Theorem 1 is trivial; we will assume that X contains more than one point.

PROPOSITION 3.1. (a) *Let $\epsilon > 0$. There is a $\gamma = \gamma(\epsilon) > 0$ such that if $x, y \in X$ and $\rho(x, y) < \gamma$, there is a Peano subspace $S = S(x, y) \subset X$ such that $x, y \in S$, $\lambda(S) > 0$, and $D(S) < \epsilon$.*

(b) *If $[r, s] \subset I$, there is a continuous surjection $f\colon [r, s] \to S(x, y)$ with $f(r) = x$ and $f(s) = y$.*

Proof. X is uniformly locally arcwise connected ([2], p. 130), so

(3.1) There is a $\delta = \delta(\epsilon/3)$ such that for each pair $a, b \in X$ with $\rho(a, b) < \delta$, there is an arc $A = A(a, b)$ from a to b with $D(A(a, b)) < \epsilon/3$.

Let $0 < \gamma < \delta$ and suppose we have $x, y \in X$ with $\rho(x, y) < \gamma$. Let $f\colon I \to X$ be any continuous surjection such that $f(0) \neq y \neq f(1)$. Set

$$U = \{z \in X\colon \rho(z, y) < \min(\gamma, \epsilon/3, \rho(f(0), y), \rho(f(1), y))\}.$$

U is open in X, so $\lambda(U) > 0$; $f^{-1}(U) \subset (0, 1)$ is a disjoint union of open intervals, at least one of which, say (p, q), must have $\lambda(f(p, q)) > 0$. Note $D(f([p, q])) \leqq \epsilon/3$ and $\rho(y, f(p)) \leqq \gamma < \delta$. By (3.1), there are arcs $A(x,y)$ and $A(y,f(p))$ from x to y to $f(p)$ respectively, each of diameter less than $\epsilon/3$. The subspace

$$S = S(x,y) = A(x,y) \cup A(y,f(p)) \cup f([p,q])$$

clearly exhibits the desired properties.

Note that Proposition 3.1 provides us with a measure-theoretic replacement for uniform local arcwise connectedness: if any two points of X are "close", we can connect them with a Peano subspace of X, of small diameter and positive measure (instead of an arc of small diameter).

PROPOSITION 3.2. *Let S be a Peano subspace of X with $\lambda(S) > 0$, and $x, y \in X$. Let $[r,s] \subset I$ and $\gamma > 0$. There is a Cantor set $K \subset [r,s]$ and a continuous function $f\colon K \to S$ which satisfy the following properties:*

(3.2) $\qquad r, s \in K;\ f(r) = x \quad$ *and* $\quad f(s) = y.$

(3.3) $\qquad \mu(K) > 2/3(r-s) = 2/3(\mu([r,s])).$

(3.4) $\qquad f\colon K \to S$ *is a measure-positive surjection.*

(3.5) $\qquad$ *If (r', s') is a component of $[r,s]\backslash K$, $\rho(f(r), f(s)) < \gamma$.*

Proof. A familiar construction provides a "fat Cantor set" $\hat{K} \subset (r,s)$ such that $\mu(\hat{K}) > 2/3(r-s)$ and $\operatorname{supp}(\mu\,|_{\hat{K}}) = \hat{K}$. Let $\hat{S} = \operatorname{supp}(\lambda\,|_S)$. The pairs $\langle \hat{K}, \mu\,|_{\hat{K}}\rangle$ and $\langle \hat{S}, \lambda\,|_{\hat{S}}\rangle$ satisfy the conditions of the corollary to Theorem 2, so there is a continuous measure-positive surjection $\hat{f}\colon \hat{K} \to \hat{S}$. Note that $\lambda(S\backslash\hat{S}) = 0$. If we have a Cantor set K and function f such that $K \supset \hat{K}$, $\mu(K\backslash\hat{K}) = 0$, and $f\colon K \to S$ is an extension of $\hat{f}\colon \hat{K} \to \hat{S}$, then f is measure-positive (proof trivial). Since $\hat{K} \subset (r,s)$, we may choose disjoint Cantor sets $\hat{K}_1, \hat{K}_2, \hat{K}_3$ such that $r \in \hat{K}_1$, $s \in \hat{K}_2$, $\hat{K}_i \cap \hat{K} = \varnothing$ and $\mu(\hat{K}_i) = 0$ for $i = 1,2,3$. Let $f\colon \hat{K}_3 \to S$ be any continuous surjection. Set

$$K^* = \hat{K} \cup \hat{K}_1 \cup \hat{K}_2 \cup \hat{K}_3,$$

and define $f^*\colon K^* \to S$ by

$$f^*(k) = \begin{cases} \hat{f}(k), & k \in \hat{K} \\ x, & k \in \hat{K}_1 \\ y, & k \in \hat{K}_2 \\ f(k), & k \in \hat{K}_3. \end{cases}$$

The pair K^*, f^* now satisfies (3.2) through (3.4). Since K^* is compact, f^* is uniformly continuous. There is a $\delta > 0$ such that $|k_1 - k_2| < \delta$ implies $\rho(f^*(k_1), f^*(k_2)) < \gamma$; and thus there are at most finitely many components of $[r, s]\backslash K^*$, say $\{(r_i, s_i)\colon i = 1, 2, \cdots, n\}$, such that $\rho(f^*(r_i), f^*(s_i)) > \gamma$.

Now S is a compact connected metric space, so for each i we can find a chain of points $x_{i,1}, x_{i,2}, \cdots, x_{i,m(i)}$ from $x_{i,0} = f^*(r_i)$ to $x_{i,m(i)+1} = f^*(s_i)$ such that

$$\max\{\rho(x_{i,j}, x_{i,j+1})\colon \quad j = 0, 1, 2, \cdots, m(i)\} < \gamma. \tag{3.6}$$

For each i, $1 \leqq i \leqq n$, let $K_{i,1}, K_{i,2}, \cdots, K_{i,m(i)}$ be a collection of Cantor subsets of (r_i, s_i), each of μ-measure zero, such that $\sup(k \in K_{i,j}) < \inf(k \in K_{i,j+1})$ for $j = 1, 2, \cdots, (m(i) - 1)$. (That is, each $K_{i,j}$ "sits to the left" of $K_{i,j+1}$.) Now set

$$K = \cup\{K_{i,j(i)}\colon i = 1, 2, \cdots, n; j(i) = 1, 2, \cdots, m(i)\}$$

and define $f\colon K \to S$ by

$$f(k) = \begin{cases} f^*(k), & k \in K^* \\ x_{i,j'}, & k \in K_{i,j}. \end{cases}$$

We note that f and K inherit (2.2) and (2.3) from f^* and K^*, and (3.4) as well, since $\mu(K\backslash K^*) = 0$. To examine (3.5), suppose that (r', s') is a component of $[r, s]\backslash K$. If (r', s') is also a component of $[r, s]\backslash K^*$, we have that $\rho(f(r'), f(s')) = \rho(f^*(r'), f^*(s')) < \gamma$. If (r', s') is not a component of $[r, s]\backslash K^*$, there are an i and j such that $f(r') = x_{i,j}$ and $f(s')$ is either $x_{i,j}$ or $x_{i,j+1}$. By (3.6), $\rho(f(r'), f(s')) < \gamma$, and (3.5) holds. This proves Proposition 3.2, and provides us with the inductive mechanism for proving Theorem 1′.

THEOREM 1′. *There is a continuous measure-positive surjection* $F\colon \langle I, \mu\rangle \to \langle X, \lambda\rangle$.

Proof. We wish to construct a uniformly convergent sequence of continuous surjections $F_n\colon I \to X$ such that $F = \lim F_n$ is measure-positive. Begin by applying Proposition 3.1 to obtain $\gamma_n = \gamma(1/2^n)$ for each $n = 1, 2, \cdots$. For two points $x, y \in X$ with $\rho(x, y) < \gamma_n$, we can "connect" x to y with a Peano subspace of X, of positive measure and diameter less than 2^{-n}.

Step 1. *Construction of* F_1: Apply Proposition 3.2 to obtain a Cantor set $K_1 \subset I$ and a continuous surjection $f_1\colon K_1 \to X$ such that

(3.7.a) $\mu(K_1) > 1/2$; $\{0, 1\} \subset K_1$.

(3.7.b) $f_1\colon K_1 \to X$ is measure-positive.

(3.7.c) If (r, s) is a component of $I \backslash K_1$, $\rho(f_1(r), f_1(s)) < \gamma_1$.

Recall the note following Proposition 3.2. In view of (3.7.c), we may minic the proof of the Hahn-Maxurkiewitz theorem and extend $f_1\colon K_1 \to X$ to a continuous surjection $F_1\colon I \to X$ such that

(3.7.d) If (r, s) is a component of $I \backslash K_1$, $F_1([r, s])$ is a Peano subspace of positive measure and diameter less than 1/2.

Step 2. *Inductive Construction.* Suppose we have Cantor sets $K_1 \subset K_2 \subset \cdots \subset K_m$ and continuous surjections $F_n\colon I \to X$ $(i = 1, 2, \cdots, m)$ such that

(3.8.a) For each n, $\mu(K_n) > 1 - 2^{-n}$.

(3.8.b) If $n_1 > n_2$, $F_{n_1}|_{K_{n_2}} = F_{n_2}|_{K_{n_2}}$.

(3.8.c) $F_n|_{K_n} = f_n\colon K_n \to X$ is measure-positive.

(3.8.d) If (r, s) is a component of $I \setminus K_n$, $F_n([r, s])$ is a Peano subspace of X, of positive measure and diameter less than $\gamma_n = \gamma(2^{-n}) \leqq 2^{-n}$.

(3.8.e) $$\operatorname*{Sup}_{k \in I} |F_n(k) - F_{n-1}(k)| \leqq 2^{-n+2}.$$

We now construct K_{m+1} and $F_{m+1}\colon I \to X$. First note that F_m is uniformly continuous, so that there is an $\eta > 0$ such that

$$|r - s| < \eta \Rightarrow \rho(F_m(r), F_m(s)) < \gamma_{m+1},$$

and there are at most finitely many (if indeed any) components of $I \backslash K_m$, say $\{(r_i, s_i)\colon i = 1, 2, \cdots, t\}$ such that $\rho(F_m(r_i), F_m(s_i)) \geqq \gamma_{m+1}$. Add to this collection if need be finitely many other components of $I \backslash K_m$, say $\{(r_i, s_i))\colon i = t + 1, t + 2, \cdots, T\}$ so that

(3.9) $$\mu\left(\bigcup_{i=1}^{T} (r_i, s_i)\right) > \frac{3}{4}\mu(I \backslash K_m).$$

For each $i = 1, 2, \cdots, T$, consider the Peano subspace $F_m([r_i, s_i])$. From (3.8.d) we have that $\lambda(F_m([r_i, s_i])) > 0$, and we may thus apply Proposition 3.2 to obtain a Cantor subset $K_{m+1,i} \subset [r_i, s_i]$ and a continuous map $f_{m+1,i}: K_{m+1,i} \to F_m([r_i, s_i])$ such that

(3.10) $r_i, s_i \in K_{m+1,i}$; $f_{m+1,i}(r_i) = F_m(r_i)$ and $f_{m+1,i}(s_i) = F_m(s_i)$.

(3.11($\mu(K_{m+1,i}) > \frac{2}{3}(r_i - s_i)$.

(3.12) $f_{m+1,i}: K_{m+1,i} \to F_m([r_i, s_i])$ is a measure-positive surjection.

(3.13) If (r, s) is a component of $[r_i, s_i] \backslash K_{m+1,i}$ then

$$\rho(f_{m+1,i}(r), f_{m+1,i}(s)) < \gamma_{m+1} \leqq 2^{-(m+1)}.$$

In view of these, we may extend $f_{m+1,i}: K_{m+1,i} \to F_m([r_i, s_i])$ to a continuous map $F_{m+1,i}: [r_i, s_i] \to X$ such that

(3.14) For each component (r, s) of $[r_i, s_i] \backslash K_{m+1,i}$, $F_{m+1}([r, s])$ is of diameter less than $2^{-(m+1)}$ and is of positive measure. Now set

(3.15) $K_{m+1} = K_1 \cup \left(\bigcup_{i=1}^{T} K_{m+1,i} \right)$, and define $F_{m+1}: I \to X$ by

(3.16)
$$F_{m+1}(k) = \begin{cases} F_m(k), & k \in I \backslash \bigcup_{i=1}^{T} [r_i, s_i] \\ F_{m+1,i}(k), & k \in [r_i, s_i]. \end{cases}$$

As an easy consequence of (3.10), $F_{m+1}: I \to X$ is continuous; as an extension of $F_1|_{K_1}$, F_{m+1} is onto. We now consider (3.8.a) through (3.8.e) for $n = m + 1$. We have from (3.15), (3.12), and (3.9) that

$$\mu(K_{m+1}) > \mu(K_m) + \frac{2}{3}\left[\left(\frac{3}{4}\right)(1 - \mu(K_m))\right]$$
$$= \frac{1}{2} + \frac{1}{2}\mu(K_m) > \frac{1}{2} + \frac{1}{2}(1 - 2^{-m}) = 1 - 2^{-(m+1)},$$

proving (3.8.a). (3.8.b) holds trivially, for $F_{m+1}|_{K_m} = F_m|_{K_m}$ by definition. Hence $F_{m+1}|_{K_m}$ is measure-positive, and each $F_{m+1}|_{K_{m+1,i}} = f_{m+1,i}$ is measure-positive by construction; (3.8.c) follows.

Now suppose (r, s) is a component of $I \backslash K_{m+1}$. Recall that the collection $\{(r_i, s_i): i = 1, 2, \cdots, T\}$ includes all those components (r, s) of $I \backslash K_m$ such that $D(F_m([r, s])) \geqq \gamma_{m+1}$. If $(r, s) \not\subset \cup_{i=1}^{T} (r_i, s_i)$, we have that

$F_{m+1}([r, s]) = F_m([r, s])$ and thus that $D(F_{m+1}([r, s])) < \gamma_{m+1}$. If, on the other hand, $(r, s) \subset (r_i, s_i)$ for some i, we may employ (3.14). This proves (3.8.d).

Prior to verifying (3.8.e) for $n = m + 1$, we show that

$$D(f_{m+1}([r_i, s_i])) \leqq 2^{-m+1} \quad \text{for each} \quad i = 1, 2, \cdots, T. \tag{3.17}$$

Now if $x \in [r_i, s_i]$, either $x \in K_{m+1,i}$ or $x \in (r, s)$, a component of $[r_i, s_i] \backslash K_{m+1,i}$; as a consequence of (3.13), $\rho(F_{m+1}(x), F_{m+1}(K_{m+1,i})) < 2^{-m-1}$. By (3.12) and (3.8.d),

$$D(F_{m+1}\ (K_{m+1,i})) = D(f_{m+1,i}(K_{m+1,i})) = D(F_m([r_i, s_i])) \leqq 2^{-m}.$$

(3.17) follows from the triangle inequality.

Now let $x \in I$, and examine $\rho(F_{m+1}(x), F_m(x))$. If $x \in I \backslash \cup_{i=1}^{T} (r_i, s_i)$, $F_{m+1}(x) = F_m(x)$; if $x \in (r_i, s_i)$ for some i, $F_m(x) \in F_m([r_i, s_i]) \subset F_{m+1}([r_i, s_i])$, and by (3.17), $\rho(F_m(x), F_{m+1}(x)) \leqq 2^{-m+1}$. Thus (3.8.e) holds for $n = m + 1$, and our inductive construction is complete.

From (3.8.e) we have that the inductively defined sequence $\{F_n : I \to X\}$ of continuous surjections converges uniformly to a continuous surjection $F: I \to X$. F is also measure-positive.

(a) Let $A \subset X$ with $\lambda(A) > 0$. Since $F|_{K_1} = F_1|_{K_1}$ is measure-positive, we have $\mu(F^{-1}(A)) \geqq \mu(K_1 \cap F^{-1}(A)) > 0$.

(b) From (3.8.a), $\mu(\cup_{i=1}^{\infty} K_i) = 1$.

If $B \subset I$ and $\mu(B) > 0$, then $\exists K_i$, such that $\mu(B \cap K_i) > 0$. Since $F|_{K_i}$ is measure-positive, $F(B \cap K_i)$ contains a set of positive λ-measure. This completes the proof of Theorem 1′. We now obtain

THEOREM 1. *There is a continuous measure-preserving surjection* $F: I \to X$.

Proof. Use Theorem 1′ to obtain a continuous measure-positive surjection $F: I \to X$, and consider the measure M induced on X by F:

$$M(B) = \mu(F^{-1}(B)) \text{ for each Borel set } B \subseteq X. \tag{3.18}$$

We note that

(3.19) $\displaystyle\int_B (g(dM)) = \int_{f^{-1}(B)} ((g \cdot f) d\mu)$, for every Borel set B in X and every extended real-valued Borel measureable function g in X (in the sense that if either integral exists, then the other does and the two are equal).

An easy consequence of (3.17) and the fact that F is measure-positive, we have that the measures M and λ are mutually absolutely continuous. As $\lambda \ll M$, we may apply the Radon-Nikodym theorem to obtain a non-negative Borel measureable function $u: X \to R$ such that

$$\lambda(B) = \int_B (u(dM)) \text{ for each Borel set } B \text{ in } X. \tag{3.20}$$

As $M \ll \lambda$, we have that u is positive everywhere (M), and, since the null sets of λ and M coincide, u is positive a.e. (λ). We may therefore assume

$$u(x) > 0 \quad \text{for} \quad \text{all} \quad x \in X. \tag{3.21}$$

Define the measure L on I by setting

$$L(E) = \int_E ((u \cdot F)d\mu), \text{for each Borel set } E \subset I. \tag{3.22}$$

As a consequence of (3.19) and (3.20), we have

$$L(F^{-1}(B)) = \int_{F^{-1}(B)} (u \cdot F)d\mu = \int_B (u(dM)) = \lambda(B) \tag{3.23}$$

for each Borel set B in X.

Thus the map $F: \langle I, L\rangle \to \langle X, \lambda\rangle$ is a continuous measure-preserving surjection.

It is a simple consequence of (3.21) and (3.22) that the measures L and μ are mutually absolutely continuous. Since $F^{-1}(X) = I$, we have from (3.23) that $L(I) = \lambda(X) = 1$. Thus the distribution function of L,

$$D(x) = L([0, x]),$$

is a strictly monotonic continuous surjection from I to I. It is straightforward to verify that the homeomorphism $D: \langle I, L\rangle \to \langle I, \mu\rangle$ is measure-preserving, so that the map $D^{-1}: \langle I, \mu\rangle \to \langle I, L\rangle$ is a measure-preserving homeomorphism.

Now set

$$f = FD^{-1}: \langle I, \mu\rangle \to \langle X, \lambda\rangle.$$

Being the composition of two measure-preserving surjections, f is itself measure-preserving and onto. This completes the proof.

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