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CHARACTERIZING LOCAL CONNECTEDNESS IN INVERSE LIMITS

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Let X denote the limit of an inverse system $\underset{\sim}{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$ of locally connected Hausdorff continua. The main purpose of this paper is to define a notion of local connectedness for inverse systems, and to prove that if $\underset{\sim}{X}$ is locally connected, then so is the limit X. If the bonding maps $p_{\alpha\alpha'}$ are surjections, then X is locally connected if and only if $\underset{\sim}{X}$ is. The following corollaries are obtained. (1) If $\underset{\sim}{X}$ is σ-directed and surjective, then X is locally connected. (2) If $\underset{\sim}{X}$ is well-ordered, surjective, and weight $(X_\alpha) \leqq \lambda$ for each α in A, then either weight $(X) \leqq \lambda$, or X is locally connected. (3) If $\underset{\sim}{X}$ is σ-directed and the factor spaces X_α are trees (generalized arcs), then X is a tree (generalized arc). (4) If $\underset{\sim}{X}$ is well-ordered and the factor spaces X_α are dendrites (arcs), then either X is metrizable, or X is a tree (generalized arc).

1. Introduction. By a continuum we mean a compact connected Hausdorff space. Let X denote the limit of an inverse system $\underset{\sim}{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$ where the factor spaces X_α are locally connected continua, and A is an arbitrary directed set. It is well-known that every continuum X can be obtained as the limit of such a system where the factor spaces are polyhedra (see Theorem 10.1, p. 284, [2]). Hence local connectedness of the factor spaces X_α does not imply local connectedness of the limit X. It is the main purpose of this paper to introduce a notion of *local connectedness* for inverse systems, and to prove that for such systems $\underset{\sim}{X}$ the limit space X is locally connected (see Theorem 1). The converse holds if $\underset{\sim}{X}$ is a surjective system, i.e., if the bonding maps $p_{\alpha\alpha'}$ are surjections. An immediate corollary is the known result that if $\underset{\sim}{X}$ is a monotone inverse system, then X is locally connected [1].

In §3 the main theorem is applied to well-ordered and σ-directed inverse systems, i.e., systems in which every countable subset of the index set is bounded above. The following somewhat surprising results are obtained. (1) If the inverse system $\underset{\sim}{X}$ is σ-directed and surjective, then the limit X is locally connected. (2) If $\underset{\sim}{X}$ is well-ordered, surjective, and weight $(X_\alpha) \leqq \lambda$ for each α in A, then weight $(X) \leqq \lambda$ or X is locally connected.

Section 4 contains similar results about well-ordered and σ-directed inverse systems of trees (i.e., locally connected, hereditarily unicoherent continua [9]) and generalized arcs (i.e., ordered continua).

For example, the limit of a σ-directed inverse system of trees (generalized arcs) is a tree (generalized arc).[1]

The problem of characterizing locally connected inverse limits has been studied from a different point of view in [3].

The reader is referred to [1] for basic results concerning inverse limits of compact Hausdorff spaces.

2. Locally connected inverse systems. A continuum X has *property* S if given any open cover $\mathcal{U}$ of X, there exists a finite cover $\mathcal{C}$ of X which refines $\mathcal{U}$ and consists of connected subsets of X. A continuum is locally connected if and only if it has property S (e.g., Chapter IV, Theorem 3.7, p. 106, [11]).

DEFINITION Let $f: X \to Y$ be a mapping of locally connected continua, and let $F \subset U \subset Y$ where F is closed and U is open. We define the *splitting number* $s(f, U, F)$ of the triple (f, U, F) to be the number of components of $f^{-1}(U)$ which meet $f^{-1}(F)$.

LEMMA 1. *The splitting number $s(f, U, F)$ is finite.*

Proof. Since X is locally connected, the components of $f^{-1}(U)$ are open sets. By compactness, only finitely many components of $f^{-1}(U)$ can meet the closed set $f^{-1}(F)$.

DEFINITION. Let $\underset{\sim}{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$ be an inverse system of continua over an arbitrary directed set A. We say that the system $\underset{\sim}{X}$ is *locally connected* if (1) the factor spaces X_α are locally connected; and (2) whenever $F_\alpha \subset U_\alpha \subset X_\alpha$, where F_α is closed and U_α is open, there exists an $\alpha' \geqq \alpha$ in A such that the splitting number $s(p_{\alpha\alpha'}, U_\alpha, F_\alpha)$ agrees with $s(p_{\alpha\alpha''}, U_\alpha, F_\alpha)$ for every $\alpha'' \geqq \alpha'$.

THEOREM. 1. *The limit of a locally connected inverse system is locally connected.*

Proof. Let $\underset{\sim}{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$ be a locally connected inverse system with limit X and projections $p_\alpha: X \to X_\alpha$. We shall prove that X has property S. Let $\mathcal{U}$ be any open cover of X. There exists an $\alpha \in A$ and a finite open cover $\mathcal{U}_\alpha = (U_1, \cdots, U_n)$ of X_α such that $\{p_\alpha^{-1}(U_i)\}_{i=1}^n$ refines $\mathcal{U}$ (e.g., Lemma 3.7, p. 263, [2]). Choose open covers $\mathcal{U}'_\alpha = (U'_1, \cdots, U'_n)$ and $\mathcal{U}''_\alpha = (U''_1, \cdots, U''_n)$ of X_α such that $U''_i \subset \mathrm{cl}(U''_i) \subset U'_i \subset \mathrm{cl}(U'_i) \subset U_i$. Let $F_i = \mathrm{cl}(U''_i)$ and consider the pairs (U'_i, F_i). Since the system $\underset{\sim}{X}$ is locally connected, there exists an $\alpha' \in A$ such that for $\alpha'' \geqq \alpha'$ we have $s(p_{\alpha\alpha'}, U'_i, F_i) = s(p_{\alpha\alpha''}, U'_i, F_i)$ for $1 \leqq i \leqq n$. Let s_i denote the splitting number $s(p_{\alpha\alpha'}, U'_i, F_i)$. For $\alpha' \in A$ as above, let

[1] M. Smith has announced results similar to Corollary 5 and Theorem 6 at the Topology Conference held at the University of North Carolina at Charlotte, March, 1974.

$\{V^i_{\alpha'j}\}_{j=1}^{s_i}$ denote the collection of components of $p_{\alpha\alpha'}^{-1}(U'_i)$ which intersect $p_{\alpha\alpha'}^{-1}(F_i)$. For $\alpha'' \geqq \alpha'$ there are also s_i components of $p_{\alpha\alpha'}^{-1}(U'_i)$ which intersect $p_{\alpha\alpha'}^{-1}(F_i)$. Denote these components by $\{V^i_{\alpha''j}\}_{j=1}^{s_i}$, and assume that they are labelled so that $p_{\alpha'\alpha''}(V^i_{\alpha''j}) \subset V^i_{\alpha'j}$. Define $C^i_{\alpha''j} = \mathrm{cl}(V^i_{\alpha''j})$ for all $\alpha'' \geqq \alpha'$, and let

$$C^i_j = \mathrm{inv}\lim\{C^i_{\alpha''j}\,;\ \alpha'' \geqq \alpha'\}.$$

Since $\{F_i\}$ covers X_α, it follows that $\{C^i_{\alpha''j}\}$ covers $X_{\alpha''}$ for each $\alpha'' \geqq \alpha'$. To every $x \in X$ one can assign a pair (i,j) such that $p_{\alpha''}(x) \in C^i_{\alpha''j}$. Since i and j vary through a finite set, some pair (i,j) occurs cofinally often; and consequently $x \in C^i_j$. Consequently, $\{C^i_j\}_{i,j}$ covers X and refines $\{p_\alpha^{-1}(U_i)\}_{i=1}^n$ which refines $\mathcal{U}$. Since each C^i_j is a subcontinuum of X, it follows that X has property S.

The next theorem provides a converse to Theorem 1 for inverse systems with surjective bounding maps.

THEOREM 2. *Let $X = \mathrm{inv}\lim \underset{\sim}{X}$ where $\underset{\sim}{X}$ is a surjective inverse system of continua. If X is locally connected, then the system $\underset{\sim}{X}$ is locally connected.*

The proof of Theorem 2 depends on two simple lemmas.

LEMMA 2. *Let X_1, X_2 and Y be locally connected continua and suppose that $f_i\colon X_i \to Y$ $(i = 1,2)$ and $g\colon X_2 \to X_1$ are continuous surjections such that $f_2 = f_1 g$. Let $F \subset U \subset Y$ where F is closed and U is open. Then $s(f_1, U, f) \leqq s(f_2, U, F)$.*

Proof. Let $s_1 = s(f_1, U, F)$, and let $V_1, \cdots, V_{s_1}$ denoted the components of $f_1^{-1}(U)$ which meet $f_1^{-1}(F)$. For each $i \leqq s_1$, at least one component of $g^{-1}(V_i)$ meets $g^{-1}(f_1^{-1}(F)) = f_2^{-1}(F)$. Since each component of $g^{-1}(V_i)$ is a component of $f_2^{-1}(U)$, at least s_1 components of $f_2^{-1}(U)$ meet $f_2^{-1}(F)$. Thus $s_1 \leqq s(f_2, U, F)$.

LEMMA 3. *Let A be a directed set and N the set of natural numbers. If $\pi\colon A \to N$ is an order preserving bounded function, then π is eventually constant.*

Proof. Let $m = \max \pi(A)$, and choose $\alpha \in A$ such that $\pi(\alpha) = m$. Thus for $\alpha' \geqq \alpha$, $\pi(\alpha') = m$.

Proof of Theorem 2. Let $\underset{\sim}{X} = \{X_\alpha\,;\ p_{\alpha\alpha'}\,;\ A\}$ be a surjective system of continua with locally connected limit X and projections $p_\alpha\colon X \to X_\alpha$. Since the projections p_α are surjections (e.g., Theorem 2.6, **[1]**), each factor space X_α is the image of a locally connected continuum; hence each X_α is locally connected (e.g., Theorem 3–22, p. 126, [5]). Given $\alpha \in A$, let $A(\alpha) = \{\alpha' \in A \mid \alpha' \geqq \alpha\}$, and let $F_\alpha \subset U_\alpha \subset X_\alpha$ where F_α is closed and U_α is open. Define $\pi\colon A(\alpha) \to N$ by $\pi(\alpha') = s(p_{\alpha\alpha'}, U_\alpha, F_\alpha)$. Lemma 2 implies that π is order preserving and bounded by $s(p_\alpha, U_\alpha, F_\alpha)$. By Lemma 3, there exists $\alpha' \in A(\alpha)$ such that for all $\alpha'' \geqq \alpha'$, $\pi(\alpha') = \pi(\alpha'')$; i.e., $s(p_{\alpha\alpha'}, U_\alpha, F_\alpha) = s(p_{\alpha\alpha''}, U_\alpha, F_\alpha)$.

COROLLARY 1. *Let $\underline{X}$ be a surjective inverse system of locally connected continua with limit X. Then X is locally connected if and only if $\underline{X}$ is locally connected.*

A surjective continuous function $f\colon X \to Y$ between continua is *monotone* if $f^{-1}(y)$ is a continuum for each $y \in Y$. An inverse system of continua is *monotone* if each bonding map is monotone.

COROLLARY 2. (Capel [1]). *The limit of a monotone inverse system of locally connected continua is locally connected.*

Proof. Let $\{X_\alpha;\, p_{\alpha\alpha'};\, A\}$ be a monotone inverse system of locally connected continua. Let $F_\alpha \subset U_\alpha \subset X_\alpha$ where F_α is closed and U_α is open in X_α. If $\alpha' \geqq \alpha$, then since $p_{\alpha\alpha'}$ is monotone, the splitting number $s(p_{\alpha\alpha'}, U_\alpha, F_\alpha)$ is precisely the number of components of U_α which meet F_α. Thus, for $\alpha' \geqq \alpha$ the splitting number $s(p_{\alpha\alpha'}, U_\alpha, F_\alpha)$ is independent of α', and so the inverse system is locally connected. By Theorem 1, the limit of the system is locally connected.

3. Well-ordered and σ-directed inverse systems of locally connected continua.

We say that a quasi-ordered set A is *σ-directed* (directed) if every countable (finite) subset of A is bounded above. Thus every bounded quasi-ordered set is σ-directed. Clearly, an unbounded well-ordered set is σ-directed if and only if it contains no cofinal sequence. Another example of a σ-directed set is the collection of all countable subsets of a given set, ordered by inclusion. An inverse system is said to be *σ-directed* (*well-ordered*) if its index set is σ-directed (well-ordered).

LEMMA 4. *Let A be a σ-directed set and let N denote the set of natural numbers. If $\pi\colon A \to N$ is an order preserving function, then π is eventually constant.*

Proof. If π is not eventually constant, then there exists an increasing sequence $\{\alpha_i\}_{i=1}$ in A such that $\{\pi(\alpha_i)\}_{i=1}$ is cofinal in N.

Since A is σ-directed, there exists $\alpha \in A$ such that $\alpha_i \leqq \alpha$ for every $i \in N$. Thus $\pi(\alpha_i) \leqq \pi(\alpha)$ for every i, which is a contradiction.

THEOREM 3. *The limit of a σ-directed surjective inverse system of locally connected continua is locally connected.*

Proof. Let $\underset{\sim}{X} = \{X_\alpha; p_{\alpha\alpha'}; A\}$ be a σ-directed surjective inverse system of locally connected continua. According to Theorem 1, it suffices to show that $\underset{\sim}{X}$ is a locally connected system. Let $F_\alpha \subset U_\alpha \subset X_\alpha$ where F_α is closed and U_α is open. Let $A(\alpha) = \{\alpha' \in A \mid \alpha' \geqq \alpha\}$ and note that $A(\alpha)$ is a σ-directed set. We define a function $\pi: A(\alpha) \to N$ by $\pi(\alpha') = s(p_{\alpha\alpha'}, U_\alpha, F_\alpha)$. By Lemma 2, π is an increasing function. Thus, by Lemma 4, π is eventually constant, and there exists $\alpha' \in A(\alpha)$ such that $\pi(\alpha') = \pi(\alpha'')$ whenever $\alpha' \leqq \alpha''$. Thus for $\alpha' \leqq \alpha''$ we have $s(p_{\alpha\alpha'}, U_\alpha, F_\alpha) = s(p_{\alpha\alpha''}, U_\alpha, F_\alpha)$, and $\underset{\sim}{X}$ is locally connected.

COROLLARY 3. *If X is the limit of a σ-directed inverse system of hereditarily locally connected continua, then X is hereditarily locally connected.*

Proof. Let $X = \operatorname{inv\,lim}\{X_\alpha; p_{\alpha\alpha'}; A\}$ where A is σ-directed and the factor spaces X_α are hereditarily locally connected continua. Let Y be any subcontinuum of X. Then $\{p_\alpha(Y); p_{\alpha\alpha'} | p_{\alpha'}(Y); A\}$ is a σ-directed surjective inverse system of locally connected continua with limit Y (see [1]). By Theorem 3, Y is locally connected.

The *weight* of a topological space X, denoted $w(X)$, is the smallest cardinal number λ such that X admits a basis for its topology of cardinality λ.

THEOREM 4. *Let X be the limit of a well-ordered surjective inverse system $\underset{\sim}{X}$ of locally connected continua X_α such that $w(X_\alpha) \leqq \lambda$ for each X_α. Then, either $w(X) \leqq \lambda$, or X is locally connected. In particular, if the factor spaces X_α are metrizable, then either X is metrizable, or X is locally connected.*

Proof. Let A denote the well-ordered index set for the system $\underset{\sim}{X}$. If A contains a cofinal sequence, then X is the limit of an inverse sequence of continua X_n such that $w(X_n) \leqq \lambda$; hence $w(X) \leqq \lambda$. Otherwise, A is σ-directed and X is locally connected by Theorem 3.

REMARK. Suppose that the nonmetrizable continuum X is the limit of a well-ordered surjective inverse system of metric continua X_α. If X is non-locally connected, then by Theorem 4 the factor spaces X_α are eventually nonlocally connected as well. This remark applies to all continua of weight $\aleph_1$, since such continua are known to be limits of well-ordered surjective inverse systems of metric continua [7].

COROLLARY 4. *Let X be the limit of a well-ordered inverse system $\underline{X}$ of hereditarily locally connected continua X_α such that $w(X_\alpha) \leqq \lambda$ for each $\alpha \in A$. Then either $w(X) \leqq \lambda$, or X is hereditarily locally connected.*

4. Well-ordered and σ-directed inverse systems of trees and generalized arcs. A continuum X is a *tree* [9] if each pair of points is separated by a third point. A continuum X with precisely two nonseparating points is called a *generalized arc* (or an *ordered continuum*). According to [9], a continuum X is a tree if and only if X is locally connected and hereditarily unicoherent. Clearly every subcontinuum of a tree X is a tree, and consequently X is hereditarily locally connected. It follows immediately from Theorem 4.1(3) of [4] that a tree is a generalized arc if and only if it is atriodic.

It is known that the limit of a monotone inverse system of trees is a tree (see the proof of Theorem 4.2 in [4]); and that the limit of a monotone inverse system of generalized arcs is a generalized arc (Lemma 4.7 of [1], or [8]). We shall obtain the same conclusions for σ-directed inverse systems of trees and generalized arcs without any assumptions about the bonding maps.

LEMMA 5. *Suppose that X is the limit of an arbitrary inverse system of trees* (generalized arcs). *If X is locally connected, then X is a tree* (generalized arc).

Proof. Since the factor spaces are hereditarily unicoherent, X is also hereditarily unicoherent by a routine application of ((2.9), p. 235, [1]). Consequently, X is a tree. If the factor spaces are generalized arcs, then X is chainable (e.g., [6]). Since chainable continua are atriodic, X is an atriodic tree; i.e., a generalized arc.

REMARK. The proof of Lemma 5 can be modified to show that a locally connected tree-like (arc-like, i.e., chainable) continuum is a tree (generalized arc). If X is tree-like, then X is hereditarily unicoherent. Consequently, if X is locally connected, then X is a tree. If, in addition, X is arc-like, then X is atriodic; hence X is a generalized arc (see [8] for a different proof).

THEOREM 5. *If X is the limit of a σ-directed inverse system of trees* (generalized arcs), *then X is a tree* (generalized arc).

Proof. Apply Corollary 3 and Lemma 5.

THEOREM 6. *Let X be the limit of a well-ordered inverse system of trees* (generalized arcs) *X_α such that $w(X_\alpha) \leqq \lambda$ for each X_α. Then, either $w(X) \leqq \lambda$, or X is a tree* (generalized arc).

Proof. Apply Corollary 4 and Lemma 5.

COROLLARY 5. *Let X be the limit of a well-ordered inverse system of dendrites* (arcs). *Then, either X is metrizable, or X is a tree* (generalized arc).

Proof. A dendrite (arc) is a metrizable tree (generalized arc) (see (1.1), p. 88 and Theorem (6.2), p. 54 of **[10]**). Thus the desired conclusion follows from Theorem 6.

REMARK. The limit of a well-ordered inverse system of arcs need not be metrizable. For example, the long line (p. 55, **[5]**) is the limit of a well-ordered monotone inverse system of arcs.

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