PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 2, 1975

BIRNBAUM-ORLICZ SPACES OF FUNCTIONS ON GROUPS

IRACEMA M. BUND

It is natural to ask how far the theory of closed invariant subspaces for $\mathfrak{L}_p(G)$ can be extended to Birnbaum-Orlicz spaces $\mathfrak{L}_A(G)$. If G is a compact group and A satisfies the Δ_2-condition for $u \geqq u_0 \geqq 0$, the class of all closed invariant subspaces of $\mathfrak{L}_A(G)$ is exactly the family $\{(\mathfrak{L}_A)_P : P \subset \Sigma\}$ where Σ is the dual object of G. Distinct subsets of Σ engender distinct subspaces.

The generalization of the classical $\mathfrak{L}_p$-spaces foreshadowed by Z. W. Birnbaum in 1930 [1] was the subject of a long article by Z. W. Birnbaum and W. Orlicz [2]. In the next four decades their theory has been extended by many writers, among them G. Weiss [9] and W. Luxemburg who invented convenient new definitions. More recently M. Jodeit and A. Torchinsky [7] introduced a generalization of the concept of Young's function which we adopt here.

The essential introductory definitions and theorems are stated in §1; proofs may be found in [3], [8] and [9]. In §2 we show that if G is a locally compact group, the Birnbaum-Orlicz space $\mathfrak{L}_A(G)$ is a left Banach $\mathfrak{L}_1$-module and a right Banach $(\mathfrak{L}_1 \cap \mathfrak{L}_1^*)$-module. Finally in §3 we establish the result stated in the synopsis. Our notation is as in [4], [5] and [6].

1. Preliminaries.

(1.1) A function A on $[0,\infty[$ into $[0,\infty]$ will be called a generalized Young's function if it is left continuous on $]0,\infty[$, $A(u)/u$ is nondecreasing for $u > 0$, and $A(0) = 0$. It easily follows that

$$\text{(i)} \quad A(\alpha u) \leqq \alpha A(u) \quad \text{for} \quad 0 \leqq \alpha \leqq 1 \quad \text{and} \quad 0 \leqq u < \infty.$$

The zero function and the function $A(u) = \infty . \xi_{]0,\infty[}(u)$ are trivial generalized Young's functions. Throughout the remaining of this work the letter A will denote a nontrivial generalized Young's function. We also fix $a = \sup\{u : A(u) = 0\}$.

A Young's function A_0 is associated to A by the equality $A_0(u) = \int_0^u A(t)/t \, dt$.

(1.2) Let $(X, \mathcal{M}, \mu)$ be an arbitrary measure space. The set $\mathfrak{L}_A(X, \mathcal{M}, \mu)$ of all complex-valued, $\mathcal{M}$-measurable functions defined μ-a.e. on X, such that $\int_X A(\alpha |f|) \, d\mu < \infty$ for some positive number α is called a Birnbaum-Orlicz space. Where no confusion seems possible, we will write $\mathfrak{L}_A(X)$ for $\mathfrak{L}_A(X,\mathcal{M},\mu)$.

The equality

(i) $p_A(f) = \inf\{k \in]0,\infty[: \int_X A(|f|/k)\, d\mu \leqq 1\}$

defines a nonnegative finite-valued function on $\mathfrak{L}_A(X)$ which is a norm in case A is convex. This suggests that we define a norm on $\mathfrak{L}_A(X)$ by the equality $\|f\|_A = p_{A_0}(f)$. With this norm, $\mathfrak{L}_A(X)$ is a Banach space.

If $f \in \mathfrak{L}_A(G)$ the following hold:

(ii) $\|f\|_A \leqq p_A(f) \leqq 2\|f\|_A$;

(iii) $\int_X A(|f|/p_A(f))\, d\mu \leqq 1$, provided that $p_A(f) > 0$.

Denoting the Young's complement of A by $\bar{A}$, for f in $\mathfrak{L}_A(X)$ and g in $\mathfrak{L}_{\bar{A}}(X)$ we obtain

(iv) $\int_X |fg|\, d\mu \leqq 2p_A(f)p_{\bar{A}}(g)$.

If $\mu(X)$ is finite, $\mathfrak{L}_A(X)$ is contained in $\mathfrak{L}_1(X)$ and for $f \in \mathfrak{L}_A(X)$ we have

(v) $\|f\|_1 \leqq [4/(\bar{A})^{-1}(1/\mu(X))]\|f\|_A$,

where $(\bar{A})^{-1}$ denotes the right inverse of $\bar{A}$.

(1.3) THEOREM. *Let f be a complex-valued measurable function vanishing outside of a σ-finite set. Suppose that*

$$N_A(f) = \sup\left\{\int_X |fg|\, d\mu : g \in \mathfrak{L}_{\bar{A}}(X), p_{\bar{A}}(g) \leqq 1\right\} < \infty.$$

Then $f \in \mathfrak{L}_A(X)$ and we have $\|f\|_A \leqq N_A(f)$.

(1.4) THEOREM. *Let X be a locally compact Hausdorff space. Let μ be a measure obtained from a nonnegative linear functional on $\mathfrak{C}_{00}(X)$, and let $\mathcal{M}$ be the σ-algebra of all μ-measurable subsets of X. Then each function f in $\mathfrak{L}_A(X)$ can be written as $f_1 + f_2$, where $f_1 = f\,\xi_F$ for some σ-compact set F, and $|f_2| \leqq ap_A(f)$ μ-*a.e. *on X. In particular, if $a = 0$, then f vanishes μ-* a.e. *outside of a σ-con pact set.*

2. Birnbaum-Orlicz spaces of functions on groups. From here on we consider spaces $\mathfrak{L}_A(G,\mathcal{M},\lambda)$, where G is a locally compact group, λ is a left Haar measure on G, and $\mathcal{M}$ is the σ-algebra of λ-measurable subsets of G. We will often write $\int_G f d\lambda$ as $\int_G f(x)dx$.

Our first theorem follows easily from (20.2) in [4], and the fact that $\mathfrak{L}_1(G,\mathcal{M},\max\{1,1/\Delta\}\lambda)$ is complete.

(2.1) THEOREM. *A complex-valued measurable function f belongs to* $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^\star(G)$ *if and only if* $\max\{1, 1/\Delta\} f \in \mathfrak{L}_1(G)$. *The equalities*

(i) $\|f\| = \|f\|_1 + \|(1/\Delta)f\|_1$,

and

(ii) $|||\, f \,||| = \|\max\{1, 1/\Delta\} f\|_1$

define equivalent norms on the linear space $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^\star(G)$. *Precisely, we have*

(iii) $|||\, f \,||| \leqq \|f\| \leqq 2\, |||\, f \,|||$ *for all* $f \in \mathfrak{L}_1(G) \cap \mathfrak{L}_1^\star(G)$.

With either of these two norms, $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^\star(G)$ *is a Banach space.*

(2.2) THEOREM. *Let f be a function in* $\mathfrak{L}_A(G)$ *and let s be an arbitrary element of G. Then the functions* ${}_sf$ *and* f_s *belong to* $\mathfrak{L}_A(G)$ *and we have:*

(i) $p_A({}_sf) = p_A(f)$;

(ii) $p_A(f_s) \leqq \max\{1, \Delta(s^{-1})\} p_A(f)$.

Proof. It is clear that ${}_sf$ and f_s are λ-measurable. Relations (i) and (ii) trivially become equalities if $p_A(f) = 0$. Suppose that $p_A(f) > 0$.

Theorem (20.1.i) in **[4]**, and (1.2.iii) yield the inequality $p_A({}_sf) \leqq p_A(f)$, from which (i) easily follows. Using (20.1.ii) in **[4]**, and once again (1.2.iii) we write

$$\int_G A(|f_s|/p_A(f))\, d\lambda \leqq \Delta(s^{-1}), \tag{1}$$

which establishes (ii) in case $\Delta(s^{-1}) \leqq 1$. For $\Delta(s^{-1}) > 1$, use (1) and (1.1.i).

The following result is part of (20.7) in the Russian edition of Hewitt and Ross "Abstract Harmonic Analysis", to be published.

(2.3) LEMMA. *Let f be a* λ*-measurable function on G. The following functions are* $\lambda \times \lambda$*-measurable on* $G \times G$:

$$(x, y) \to f(xy^{-1}), \qquad (x, y) \to f(y^{-1}x), \qquad (x, y) \to f(x),$$

$$(x, y) \to f(x^{-1}), \qquad (x, y) \to f(y), \qquad (x, y) \to f(y^{-1}).$$

(2.4) THEOREM *Let f be a function in* $\mathfrak{L}_A(G)$ *vanishing outside of a* σ*-compact set F and let g be a function in* $\mathfrak{L}_1(G)$. *The integral*

(i) $g * f(x) = \int_G f(y^{-1}x) g(y) dy$

exists and is finite for almost all x in G. The function $g*f$ *is in* $\mathfrak{L}_A(G)$ *and we have*

(ii) $\|g * f\|_A \leqq 4\|f\|_A \|g\|_1$.
If $g \in \mathfrak{L}_1(G) \cap \mathfrak{L}_1^\star(G)$, *the integral*
(iii) $f * g(x) = \int_G \Delta(y^{-1}) f(xy^{-1}) g(y)\, dy$
exists and is finite for λ*-almost all* x *in* G. *The function* $f * g$ *is in* $\mathfrak{L}_A(G)$ *and we have*
(iv) $\|f * g\|_A \leqq 4\|f\|_A \|g\|$,
where $\|\cdot\|$ *is as in* (2.1.i).

Proof. We may suppose that g vanishes outside of a σ-compact set E. Thus the function $(x,y) \to f(y^{-1}x)g(y)$ vanishes outside of the σ-compact set $(EF) \times E$.

Let v be an arbitrary function in $\mathfrak{L}_{\bar{A}}(G)$. From (2.3) we know that the mapping $(x,y) \to v(x)f(y^{-1}x)g(y)$ is $\lambda \times \lambda$-measurable. Plainly this function vanishes outside of $(EF) \times E$.

Recalling (1.2.iv) and (2.2.i), we obtain

$$\begin{aligned} &\int_G \int_G |v(x)f(y^{-1}x)g(y)|\, dxdy \\ (1) \qquad & \leqq 2p_A(f)\, p_{\bar{A}}(v)\, \|g\|_1. \end{aligned}$$

Thus we may apply (13.10) of **[4]** to conclude that

$$\begin{aligned} &\int_G \int_G |v(x)f(y^{-1}x)g(y)|\, dydx \\ (2) \qquad & = \int_G \int_G |v(x)f(y^{-1}x)g(y)|\, dxdy. \end{aligned}$$

From (13.10) and (13.8) in **[4]**, we see that the integral $\int_G v(x)f(y^{-1}x)g(y)dy$ exists and is finite for λ-almost all x in G, and that

$$(3) \qquad x \to v(x) \int_G f(y^{-1}\dot{x})g(y)dy.$$

is a function in $\mathfrak{L}_1(G)$; in particular it is a λ-measurable function.

We define $g * f(x)$ by the equality (i), provided the integral exists, and put $g * f(x) = 0$, otherwise. It is easy to see that $g * f(x)$ is finite λ-a.e. on G.

In (3) we may take v to be any function in $\mathfrak{C}_{00}(G)$. Recalling (11.42) in **[4]**, we see that $g * f$ is λ-measurable.

Consider v in $\mathfrak{L}_{\bar{A}}(G)$ with $p_{\bar{A}}(v) \leqq 1$. Taking account of (1) and (2), we obtain

$$\oint_G |v(x)(g * f)(x)|\,dx \leqq \int_G \int_G |v(x)f(y^{-1}x)g(y)|\,dydx$$

$$= \int_G \int_G |v(x)f(y^{-1}x)g(y)|\,dxdy \leqq 2p_A(f)\|g\|_1.$$

This implies that

$$N_A(g * f) \leqq 2p_A(f)\|g\|_1. \tag{4}$$

Now we observe that $g * f(x) = 0$ for x outside of the σ-compact set EF. Thus from (4) and (1.3), we conclude that $g * f \in \mathfrak{L}_A(G)$ and that $\|g * f\|_A \leqq 2p_A(f)\|g\|_1$. Applying (1.2.ii) to this last inequality, we obtain (ii).

Next suppose that $g \in \mathfrak{L}_1(G) \cap \mathfrak{L}_1^*(G)$. Consider the function

$$(x, y) \rightarrow v(x)f(xy^{-1})g(y)\Delta(y^{-1}), \tag{5}$$

where v is an arbitrary function in $\mathfrak{L}_{\bar{A}}(G)$. As in the previous case, we see that the function (5) is $\lambda \times \lambda$-measurable and vanishes outside of the σ-compact set $(FE) \times E$. From (1.2.iv) and (2.2.ii) we obtain

$$\int_G |v(x)f(xy^{-1})|\,dx \leqq 2\max\{1, \Delta(y)\}p_A(f)p_{\bar{A}}(v).$$

Thus we have

$$\int_G \int_G |v(x)f(xy^{-1})g(y)\Delta(y^{-1})|\,dxdy$$

$$\leqq 2p_A(f)p_{\bar{A}}(v)\int_G \max\{1, \Delta(y^{-1})\}|g(y)|\,dy$$

$$= 2p_A(f)p_{\bar{A}}(v)\|\max\{1, 1/\Delta\}g\|_1$$

$$\leqq 2p_A(f)p_{\bar{A}}(v)\|g\|,$$

the last inequality being a consequence of (2.1.ii) and (2.1.iii).

From this point on the proof is completely analogous to that presented above for $g * f$ and we omit it.

Theorem (2.4) serves as a lemma for the following general result.

(2.5) THEOREM. *Suppose that* $f \in \mathfrak{L}_A(G)$ *and* $g \in \mathfrak{L}_1(G)$. *Then the integral*

(i) $g * f(x) = \int_G f(y^{-1}x)g(y)dy$

exists and is finite for λ*-almost all* x *in* G. *The function* $g * f$ *is in* $\mathfrak{L}_A(G)$ *and we have*

(ii) $\|g * f\|_A \leqq k\|f\|_A\|g\|_1$,

where $k = 4$ *if* $a = 0$ *or if* G *is* σ*-compact, and* $k = 6$ *otherwise.*

If $g \in \mathfrak{L}_1(G) \cap \mathfrak{L}_1^{\star}(G)$, *the integral*

(iii) $f * g(x) = \int_G \Delta(y^{-1})g(y)dy$

exists and is finite for λ*-almost all* x *in* G. *The function* $f * g$ *is in* $\mathfrak{L}_A(G)$ *and we have*

(iv) $\|f * g\|_A \leqq k\|f\|_A\|g\|$,

where k *is as above and* $\|\cdot\|$ *is as in* (2.1.i).

Proof. If G is σ- compact, the assertion follows immediately from (2.4). If $a = 0$, it follows from (1.4) and (2.4). Thus we may suppose that $a > 0$ and that G fails to be σ- compact.

Using (1.4), we may write $f = f_1 + f_2$, where $f_1 = f\xi_F$ for some σ- compact set F, and $|f_2| \leqq ap_A(f)$. It follows that

$$\int_G |f_2(y^{-1}x)g(y)|dy \leqq ap_A(f)\|g\|_1 \tag{1}$$

for all x in G, and hence that $g * f_2(x)$ exists and is finite for all x in G. A short computation, in which we use (1), gives us

$$g * f_2(x)\|g * f_2\|_A \leqq p_A(g * f_2) \leqq a^{-1}\|g * f_2\|_\infty \leqq 2\|f\|_A\|g\|_1.$$

Applying (2.4.i) to f_1, we conclude that

$$\int_G f_1(y^{-1}x)g(y)dy + \int_G f_2(y^{-1}x)g(y)dy$$

exists and is finite for λ- almost all x in G. Hence the same is true of $g * f(x)$.

Inequality (ii) follows from (2) and (2.4.ii) applied to f_1. The remaining assertions are similarly established.

(2.6) THEOREM. *The space* $\mathfrak{L}_1(G) \cap \mathfrak{L}_1^{\star}(G)$ *is a Banach algebra.*

Proof. For f and g in $\mathfrak{L}_1(G)\cap\mathfrak{L}_1^*(G)$ we obtain

$$(1)\qquad ((1/\Delta)g)*((1/\Delta)f)=(1/\Delta)(g*f).$$

Thus (2.1) and (2.5.i) tell us that $g*f\in\mathfrak{L}_1(G)\cap\mathfrak{L}_1^*(G)$. We use (1) to prove that $\mathfrak{L}_1(G)\cap\mathfrak{L}_1^*(G)$, with the norm $\|\cdot\|$ defined in (2.1.i), is a normed algebra:

$$\|g*f\|\leqq\|g\|_1\|f\|_1+\|(1/\Delta)g\|_1\|(1/\Delta)f\|_1\leqq\|g\|\,\|f\|.$$

(2.7) THEOREM. *The space $\mathfrak{L}_A(G)$ is a left Banach $\mathfrak{L}_1$-module and a right Banach $(\mathfrak{L}_1\cap\mathfrak{L}_1^*)$-module.*

Proof. For g in $\mathfrak{L}_1(G)$ and f in $\mathfrak{L}_A(G)$, (2.5.ii) tells us that there is a positive number k such that $\|g*f\|_A\leqq k\|f\|_A\|g\|_1$.

Next we show that, for f as above, and g_1 and g_2 in $\mathfrak{L}_1(G)$, we have $g_1*(g_2*f)=(g_1*g_2)*f$. Using (20.1) of **[4]**, we obtain the equality

$$\int_G f(v^{-1}y^{-1}x)g_2(v)dv=\int_G f(v^{-1}x)g_2(y^{-1}v)\,dv,$$

which implies that

$$(1)\qquad g_1*(g_2*f)(x)=\int_G\int_G f(v^{-1}x)g_2(y^{-1}v)g_1(y)dvdy.$$

By (2.5.i), $g_1*(g_2*f)$ is in $\mathfrak{L}_A(G)$, and hence the integral in (1) exists and is finite λ-almost everywhere in G. From (1.4) we know that g_1 and g_2 vanish outside of σ-compact sets E_1 and E_2, respectively. Thus the function $(v,y)\to f(v^{-1}x)g_2(y^{-1}v)g_1(y)$ vanishes outside of the σ-compact set $(E_1E_2)\times E_1$. By (2.3) this function is $\lambda\times\lambda$-measurable.

We apply (13.10) in **[4]** to conclude that for λ-almost all x in G we have

$$g_1*(g_2*f)(x)=\int_G\int_G f(v^{-1}x)g_2(y^{-1}v)g_1(y)dydv$$
$$=\int_G f(v^{-1}x)(g_1*g_2)(v)dv=(g_1*g_2)*f(x).$$

It is now clear that $\mathfrak{L}_A(G)$ is a left Banach $\mathfrak{L}_1$-module. The proof that $\mathfrak{L}_A(G)$ is a right Banach $(\mathfrak{L}_1\cap L_1^*)$-module is similar and we omit it.

3. Closed ideals in $\mathfrak{L}_A(G)$ for G a compact group. Throughout this section we suppose that G is compact and that $\lambda(G)=1$.

(3.1) THEOREM. *If f and g are in $\mathfrak{L}_A(G)$ the equality $g * f(x) = \int_G f(y^{-1}x)g(y)dy$ defines a function in $\mathfrak{L}_A(G)$. We have*

(i) $$\|g * f\|_A \leqq (16/(\bar{A})^{-1}(1))\|f\|_A \|g\|_A.$$

Proof. Follows from (2.5.i), (1.2.v) and (2.5.ii).

(3.2) THEOREM. *The Birnbaum-Orlicz space $\mathfrak{L}_A(G)$ is a Banach algebra under a norm which is a positive constant times $\|\cdot\|_A$.*

Proof. Define $n_A(f) = (16/(\bar{A})^{-1}(1))\|f\|_A$ and use (3.1).

(3.3) THEOREM. *Suppose that A satisfies the Δ_2-condition for $u \geqq u_0 \geqq 0$. Then the space $\mathfrak{T}(G)$ of trigonometric polynomials on G is $\|\cdot\|_A$-dense in $L_A(G)$.*

Proof. Our hypothesis imply that $\mathfrak{C}(G)$ is $\|\cdot\|_A$-dense in $\mathfrak{L}_A(G)$: see **[3]** or **[8]**. Theorem (27.39.ii) of **[5]** tells us that $\mathfrak{T}(G)$ is uniformly dense in $\mathfrak{C}(G)$, and it is easy to see that $\mathfrak{T}(G)$ is also $\|\cdot\|_A$-dense in $\mathfrak{C}(G)$.

(3.4) THEOREM. *Let A be as in* (3.3). *Suppose that S is a closed linear subspace of $\mathfrak{L}_A(G)$. Then S is a left* [right] *ideal in $\mathfrak{L}_A(g)$ if and only if S is closed under the formation of left* [right] *translates.*

Proof. Since G is unimodular, it follows from (2.1) and (2.7) that $\mathfrak{L}_A(G)$ is a Banach $\mathfrak{L}_1$-module with respect to convolution. From (3.2) we know that $\mathfrak{L}_A(G)$ is a subalgebra of $\mathfrak{L}_1(G)$ which is a Banach algebra with the norm n_A. Taking (3.3) into account, we see that $L_A(G)$ has the properties stated in (38.6.a) in **[5]**. Thus the theorem follows immediately from (38.22.b) of **[5]**.

(3.5) THEOREM. *Let A be as in* (3.3). *Then the class of all closed two-sided ideals in $\mathfrak{L}_A(G)$ is exactly the family $\{(\mathfrak{L}_A)_P: P \subset \Sigma\}$. Distinct subsets of Σ engender distinct ideals.*

Proofs. This is a direct application of (38.7) in **[5]**.

REFERENCES

1. Z. W. Birnbaum, *Uber Approximation im Mittel*, Nachrichten von der Gesellschaft der Wissenschaften zu Gottingen, Fachgruppe I, Nr. **15**, (1930), 338–343.

2. Z. W. Birnbaum und W. Orlicz, *Uber die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen*, Studia Math., 3 (1931), 1–67.

3. I. M. Bund, *Fourier Analysis on Birnbaum-Orlicz spaces, Dissertation*, University of Washington, 1973, unpublished.
4. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. I, Heidelberg, Springer-Verlag, Grundlehren der Math. Wiss., Band 115, 1963.
5. ———, *Abstract Harmonic Analysis*, Vol. II, Heidelberg, Springer-Verlag Grundlehren der Math. Wiss, Band 152, 1970.
6. E. Hewitt and K. R. Stromberg, *Real and Abstract Analysis*, Heidelberg, Springer-Verlag, 1965.
7. M. Jodeit Jr. and A. Torchinsky, *Inequalities for Fourier transforms*, Studia Math., T. **XXXVII** (1971), 245–276.
8. M. A. Krasnosel'skii and Ja. B. Rutickii, *Convex functions and Orlicz spaces*, Groningen (The Netherlands), transl. from Russian, 1961.
9. G. Weiss, *A note on Orlicz spaces*, Portugaliae Math., **15** (1956), 35–47.

Received February 22, 1974. This research was supported by the Fundacão de Amparo à Pesquisa do Estado de São Paulo. This work is part of a doctoral dissertation written at the University of Washington. The author is grateful for the invaluable guidance of her research adviser, Professor Edwin Hewitt.

UNIVERSITY OF SÃO PAULO.