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THE INVARIANCE PRINCIPLE FOR WAVE OPERATORS

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The invariance principle for wave operators is proved. It is shown that the existence of wave operators $W_\pm(B, A)$ does not imply the existence of $W_\pm(g(B), g(A))$, in general.

1. Introduction. Let A and B be two selfadjoint operators on a separable Hilbert space $\mathscr{H}$ and let P_A and P_B be the orthogonal projections on the spaces of absolute continuity for A and B, respectively. The wave operators $W_\pm(B, A)$ are defined by the strong limits

$$W_\pm(B, A) \equiv \underset{t\to\pm\infty}{s\text{-lim}}\, e^{itB}e^{-itA}P_A \tag{1.1}$$

when they exist (cf. [2, Chapter X]). The invariance principle of M. S. Birman and T. Kato says: If the wave operators $W_\pm(B, A)$ and $W_\pm(g(B), g(A))$ exist and $g(\lambda)$ is real-valued and piecewise monotone increasing, with a certain mild smoothness, then

$$W_\pm(g(B), g(A)) = W_\pm(B, A)\ . \tag{1.2}$$

As stated by T. Kato and S. T. Kuroda in [3]: "It would be nice if the existence of $W_\pm(B, A)$ implied the existence of $W_\pm(g(B), g(A))$ and the invariance principle.

However, this has not been shown in general".

For example, the existence of $W_\pm(g(B), g(A))$ and the invariance principle have been proved under the condition that $B - A$ or $(B - \xi)^{-1} - (A - \xi)^{-1}$($\xi$ a nonreal number) is a trace-class operator (see for instance [2, Chapter X]).

The aim of this paper is

1. the proof of the invariance principle for wave operators,

2. the proof, that the existence of $W_\pm(B, A)$ does not imply the existence of $W_\pm(g(B), g(A))$, in general.

In the present work we restrict our considerations to real-valued functions $g(\lambda)$ on $(-\infty, \infty)$ with the following properties (cf. [2, p. 543]): The whole interval $(-\infty, \infty)$ can be divided into a countable number of subintervals Δ_n with lengths l_n in such a way that min $l_n > 0$ and in each open subinterval $g(\lambda)$ is differentiable with $g'(\lambda)$

continuous, locally of bounded variation, and positive. A function with these properties is called an allowable function. Furthermore, we shall consider only the wave operators W_+ because the theorems and proofs for the wave operators W_- are entirely similar. In §2 we prove

THEOREM 1 (invariance principle). *Let A and B be two selfadjoint operators on a separable Hilbert space $\mathscr{H}$ and let $g(\lambda)$ be an allowable function. If $W_+(B, A)$ and $W_+(g(B), g(A))$ exist and if $W_+(B, A)$ is complete, then $W_+(g(B), g(A)) = W_+(B, A)$.*

From Theorem 1 we also see that the existence of $W_+(B, A)$, $W_+(g(B), g(A))$ and the completeness of $W_+(B, A)$ imply the completeness of $W_+((g, B), g(A))$. The next two theorems concern the existence of the wave operator $W_+(g(B), g(A))$. They will be proved in §§2 and 3, respectively.

THEOREM 2. *Let A and B be two selfadjoint operators on a separable Hilbert space $\mathscr{H}$ with the absolutely continous spectrum Λ and let $g(\lambda)$ be an allowable function. If the wave operator $W_+(B, A)$ exists, is complete and if $g(\lambda)$ is piecewise linear on Λ, then $W_+(g(B), g(A))$ exists.*

THEOREM 3. *Let A be a selfadjoint operator on a separable Hilbert space $\mathscr{H}$ with the absolutely continuous spectrum $\Lambda \neq 0$. Let $g(\lambda)$ be an allowable function for which a finite interval $\Delta \subset (-\infty, \infty)$ with $|\Delta \cap \Lambda| \neq 0$* (Lebesgue measure) *exists such that on Δ $g'(\lambda)$ exists and is a continuous strictly monotone function. Then there is a selfadjoint operator B such that $W_+(B, A)$ exists, is complete, however, $W_+(g(B), g(A))$ does not exist.*

It is easily seen, for instance, that all allowable functions $g(\lambda)$ which are piecewise twice continuously differentiable satisfy the assumptions either of Theorem 2 or of Theorem 3 for fixed Λ.

For the proofs of the theorems we use the following result of H. Baumgärtel [1, Theorem 3]:

(NS) Let W be a partial isometry with

$$W^* W = P_A,\ WW^* = P_B,\ WAP_A W^* \supseteqq BP_B\ .$$

Then $W_+(B, A)$ exists and $W = W_+(B, A)$ if and only if

$$W = P_A + C\ , \tag{1.3}$$

$$\underset{t\to\infty}{s\text{-}\lim}\, e^{itA}Ce^{-itA}P_A = 0\ . \tag{1.4}$$

By Theorem 1 we obtain from (NS) that the existence and completeness of $W_+(B, A)$ imply the existence of $W_+(g(B), g(A))$ if and only if for the operator C defined by (1.3) the strong limit

$$s\text{-}\lim_{t\to\infty} e^{itg(A)}Ce^{-itg(A)}P_A = 0 \tag{1.5}$$

exists. Here it was used that $P_{g(A)} = P_A$ for allowable functions $g(\lambda)$ (see §2).

Hence we know that the proof of Theorem 3 leads to the construction of an operator C for which $s\text{-}\lim_{t\to\infty} e^{itA}Ce^{-itA}P_A = 0$ and $s\text{-}\lim_{t\to\infty} e^{itg(A)}Ce^{-itg(A)}P_A$ does not exist for the function $g(\lambda)$ defined by Theorem 3. To prove Theorem 2 we shall show that the equation

$$s\text{-}\lim_{t\to\infty} e^{itA}Ce^{-itA}P_A = 0$$

implies $s\text{-}\lim_{t\to\infty} e^{itg(A)}Ce^{-itg(A)}P_A = 0$ for piecewise linear functions $g(\lambda)$. The invariance principle will be proved by means of

LEMMA 1. *Let T be a nonnegative bounded selfadjoint operator and $g(\lambda)$ an allowable function. If the strong limits*

$$s\text{-}\lim_{t\to\infty} e^{itA}Te^{-itA}P_A = 0,\ s\text{-}\lim_{t\to\infty} e^{itg(A)}Te^{-itg(A)}P_A$$

exist, then they are equal.

In §5 we prove Lemma 1 and formulate and prove two other lemmas which concern the behavior of the function $e^{-itg(\lambda)}$ for large t.

2. *Proof of Theorem* 1. First we introduce several notations and simple relations which are needed for the proof. As in §1 let H be a selfadjoint operator on a separable Hilbert space $\mathscr{H}$ and P_H be the orthogonal projection on the space of absolute continuity. We note that for every allowable function $g(\lambda)$

$$P_{g(H)} = P_H\ . \tag{2.1}$$

(2.1) has been proved in [2, p. 545] for a class of functions slightly more restrictive than the allowable functions. The proof can easily be generalized for all allowable functions. Furthermore, we introduce the notations $\{H\}'$ for the commutant of H and

$$V_H^+(X) \equiv s\text{-}\lim_{t\to\infty} e^{itH}Xe^{-itH}P_H\ , \tag{2.2}$$

whenever for the bounded operator X the strong limit exists. If $V_H^+(X)$ for the bounded operator X exists, then we have the unambiguous decomposition (cf. [1])

$$X = X_1 + X_2 , \tag{2.3}$$

where

$$X_1 = P_H X_1 = X_1 P_H \in \{H\}' , \tag{2.4}$$

$$V_H^+(X_2) = 0 . \tag{2.5}$$

For continuous functions $f(\lambda)$ and a selfadjoint operator X one easily verifies that

$$f(V_H^+(X)) = V_H^+(f(X)) . \tag{2.6}$$

Now we prove Theorem 1. By (NS), we find that $W_+(B, A) = P_A + C$ with $V_A^+(C) = 0$. Further we also have

$$V_A^+(C^*) = 0 , \tag{2.7}$$

since

$$\begin{aligned} V_A^+(C^*) &= V_A^+(C^* + P_A) - P_A = V_A^+(W_+^*(B, A)) - P_A \\ &= s\text{-}\lim_{t\to\infty} e^{itA} W_+^*(B, A) e^{-itA} P_A - P_A \\ &= s\text{-}\lim_{t\to\infty} W_+^*(B, A) e^{itB} e^{-itA} P_A - P_A \\ &= W_+^*(B, A) W_+(B, A) - P_A = 0 \end{aligned}$$

with the intertwining relation $W_+(B, A)e^{itA} = e^{itB} W_+(B, A)$. We define

$$W_1 \equiv W_+^*(B, A) W_+(g(B), g(A)) . \tag{2.8}$$

From this definition we obtain that $V_{g(A)}^+(W_+^*(B, A))$ exists and

$$V_{g(A)}^+(W_+^*(B, A)) = W_1 , \tag{2.9}$$

since

$$\begin{aligned} V_{g(A)}^+(W_+^*(B, A)) &= s\text{-}\lim_{t\to\infty} e^{itg(A)} W_+^*(B, A) e^{-itg(A)} P_A \\ &= s\text{-}\lim_{t\to\infty} W_+^*(B, A) e^{itg(B)} e^{-itg(A)} P_A = W_1 . \end{aligned}$$

By (2.3) and (2.5) then we have

$$W_+^*(B, A) = W_1 + C_1 \tag{2.10}$$

with

$$V_{g(A)}^+(C_1) = 0 \tag{2.11}$$

$$W_1 = P_A W_1 = W_1 P_A \in \{g(A)\}' . \tag{2.12}$$

From the Definition (2.8) and the completeness of $W_+(B, A)$ one easily verifies that W_1 is *a* partially isometrie with

$$W_1^* W_1 = P_A,\ W_1 W_1^* = P_1 \leqq P_A\ . \tag{2.13}$$

Further we have

$$V^+_{g(A)}(C_1^* P_1) = 0\ . \tag{2.14}$$

This follows from

$$\begin{aligned} V^+_{g(A)}(C_1^* P_1) = 0 \longleftrightarrow\ & V^+_{g(A)}(W_+^*(B, A)C_1{}^* P_1) \\ &= V^+_{g(A)}(W_+^*(B, A)C_1^* P_1 + C_1 W_1^* + P_1) - P_1 \\ &= V^+_{g(A)}(W_+^*(B, A)(C_1^* + W_1^*)P_1) - P_1 \\ &= V^+_{g(A)}(W_+^*(B, A) W_+(B, A)P_1) - P_1 = 0 \end{aligned}$$

with $V^+_{g(A)}(C_1 W_1^* + P_1) = P_1$ by $W_1^*,\ P_1 \in \{g(A)\}'$. Combining (2.10) with $W_+^*(B, A) = P_A + C^*$ and (2.13) we obtain

$$\begin{aligned} CP_1C^* &= (W_1 + C_1 - P_A)^* P_1 (W_1 + C_1 - P_A) \\ &= (W_1 - P_1)^*(W_1 - P_1) + (W_1 - P_1)^* C_1 + C_1^* P_1 (W_1 - P_1) \\ &\quad + C_1^* P_1 C_1 = (W_1 - P_1)^*(W_1 - P_1) + C_2\ . \end{aligned}$$

By (2.11), (2.14) and $(W_1 - P_1),\ (W_1 - P_1)^* \in \{g(A)\}'$ we have $V^+_{g(A)}(C_2) = 0$ and therefore,

$$V^+_{g(A)}(CP_1C^*) = (W_1 - P_1)^*(W_1 - P_1)\ . \tag{2.15}$$

Furthermore, it follows from (2.7) that

$$V_A^+(CP_1C^*) = 0\ . \tag{2.16}$$

The operator CP_1C^* satisfies the assumptions of Lemma 1. Hence, we have $(W_1 - P_1)^*(W_1 - P_1) = 0$ and also $(W_1 - P_1) = 0$. With (2.13) and (2.8), we finally obtain $W_1 = P_A$ and

$$W_+(g(B), g(A)) = W_+(B, A)\ .$$

3. *Proof of Theorem* 2. We shall use the same notations as in §2. By Theorem 1 and (NS) it is necessary and sufficient for the existence of $W_+(g(B), g(A))$ that

$$V^+_{g(A)}(C) = 0\ . \tag{3.1}$$

Let $\varphi \in P_A\mathscr{H}$, $\varepsilon > 0$ and $P_A(\varDelta)$ be the spectral measure of A. Then there is a finite interval $\varDelta'$ such that

$$\|\varphi - P_A(\varDelta')\varphi\| \leqq \frac{\varepsilon}{3\|C\|}\ . \tag{3.2}$$

From the definition of the functions $g(\lambda)$ in Theorem 2 we see that there exists a finite number of disjoint intervals $\Delta_n \subset \Delta'(n = 1, 2, \cdots, N)$ such that

$$g(A)P_A P_A(\Delta_n) = q_n A P_A P_A(\Delta_n) , \tag{3.3}$$

$$\left\| P_A(\Delta')\varphi - \sum_{n=1}^{N} P_A(\Delta_n)\varphi \right\| \leqq \frac{\varepsilon}{3\|C\|} \tag{3.4}$$

with $0 < q_n < \infty$. By (3.2), (3.3), and (3.4) we find

$$\| Ce^{-itg(A)} P_A \varphi \| \leqq \frac{2}{3}\varepsilon + \sum_{n=1}^{N} \| Ce^{-itq_n A} P_A(\Delta_n)\varphi \| . \tag{3.5}$$

C satisfies the relation (see (NS))

$$\| Ce^{-itA} P_A \psi \| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty$$

for every $\psi \in \mathscr{H}$. Hence, for the functions $\varphi_n = P_A(\Delta_n)\varphi$ there are numbers T_n such that

$$\| Ce^{-itq_n A} \varphi_n \| \leqq \frac{\varepsilon}{3N} \quad \text{for all} \quad t > T_n . \tag{3.6}$$

By (3.5), (3.9) we obtain

$$\| Ce^{-itg(A)} P_A \varphi \| \leqq \varepsilon \quad \text{for} \quad t > T = \max_n T_n$$

and (3.1) is proved.

4. *Proof of Theorem* 3. For simplicity we shall assume that $\Delta \subset \Lambda$ and $\Delta = [0, 2\pi]$. Let $u \in P_A(\Delta)P_A\mathscr{H}$ with $P_A(\Delta')u \neq 0$ for all $\Delta' \subset \Delta$, $|\Delta'| \neq 0$. A restricted on the subspace $\mathscr{H}_1 = \overline{\mathrm{sp}}\,\{P_A(\Delta')u, \Delta' \subset \Delta$ Borel set$\}$ is an operator with simple absolutely continuous spectrum Δ. Hence we may identify $\mathscr{H}_1$ with $\mathscr{L}^2(\Delta)$ and a restricted on $\mathscr{H}_1$ with the multiplication operator by λ on $\mathscr{L}^2(\Delta)$ denoted by H.

Therefore it follows that for the proof of Theorem 3 it is sufficient to show that for H such an operator B defined by Theorem 3 exists. At first we construct a projector P such that $V_H^+(P) = 0$ and $V_{g(H)}^+(P)$ do not exist. We consider the function

$$\frac{1}{\sqrt{2\pi}} e^{-itg(\lambda)} = \sum_m \psi_t^m \frac{1}{\sqrt{2\pi}} e^{-i\lambda m} \in \mathscr{L}^2(\Delta) .$$

By Lemma 2 for $\varepsilon = 1/2$ we can find sequences of natural numbers S_n, N_n with $S_n, N_n \longrightarrow \infty$ as $n \longrightarrow \infty$ such that

$$\sum_{m=N_n}^{N_{n+1}-1} |\psi_{s_n}^m|^2 = a_n \geqq 1 - \varepsilon = \frac{1}{2} . \tag{4.1}$$

Now we define P by

$$P = \sum_{n=1}^{\infty} f_n(\cdot, f_n) , \tag{4.2}$$

$$f_n(\lambda) = \sum_{m=N_n}^{N_{n+1}-1} \frac{1}{\sqrt{a_n}} \psi^m_{S_n} \frac{1}{\sqrt{2\pi}} e^{-i\lambda m} . \tag{4.3}$$

Next we prove $V^+_H(P) = 0$, i.e.,

$$\lim_{t\to\infty} \| Pe^{-itH}\psi \| = 0 \quad \text{for every} \quad \psi \in \mathscr{L}^2(\Delta) . \tag{4.4}$$

As is easily shown, for the proof of (4.4) it is sufficient to consider the sequence $\| Pe^{-inH}\psi_0 \|$ with $n \to \infty$ (n a natural number) and $\psi_0 = 1/\sqrt{2\pi}$.

We have

$$\begin{aligned} \| Pe^{-ipH}\psi_0 \|^2 &= \sum_{n=1}^{\infty} |(f_n, e^{-ipH}\psi_0)|^2 \\ &= \sum_{n=1}^{\infty} \left| \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f_n(\lambda) e^{ip\lambda} d\lambda \right|^2 \\ &= \sum_{n=1}^{\infty} \left| \sum_{m=N_n}^{N_{n+1}-1} \psi^m_{S_n} \frac{1}{\sqrt{a_n}} \delta_{m;p} \right|^2 \\ &= \frac{\Lambda}{a_{x(p)}} | \psi^p_{Sx(p)} |^2 , \end{aligned} \tag{4.5}$$

where $x(p) = r$ if $p \in (N_r, N_r + 1, \cdots, N_{r+1} - 1)$.

It is clear that $x(p) \to \infty$ as $p \to \infty$. Since $g(\lambda)$ satisfies the assumptions of Lemma 3 we find $|\psi^p_{Sx(p)}|^2 \to 0$ as $p \to \infty$ and also $\| Pe^{-ipH}\psi_0 \|^2 \to 0$ as $p \to \infty$. This proves (4.4). To prove that $V^+_{g(H)}(P)$ does not exist by Lemma 1 it is sufficient to show that there are a $\psi \in \mathscr{L}^2(\Delta)$, a sequence of real numbers $t_n \to \infty$ as $n \to \infty$ and an $X > 0$ such that

$$\| Pe^{-it_n \cdot g(H)}\psi \| > X \text{ for all } t_n . \tag{4.6}$$

We set $\psi = \psi_0 = 1/\sqrt{2\pi}$ and $t_n = S_n$ (see (4.1)). Then by (4.1), (4.2)

$$\begin{aligned} \| Pe^{-iS_n \cdot g(H)}\psi_0 \|^2 &= \sum_{q=1}^{\infty} |(f_q, e^{-iS_n \cdot g(H)}\psi_0)|^2 \\ &\geqq |(f_n, e^{-iS_n \cdot g(H)}\psi_0)|^2 = \left| \frac{1}{\sqrt{a_n}} \sum_{m=N_n}^{N_{n+1}-1} \psi^m_{S_n} \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\lambda e^{iS_n \cdot g(\lambda)} e^{-im\lambda} \right|^2 \\ &= \left| \frac{1}{\sqrt{a_n}} \sum_{m=N_n}^{N_{n+1}-1} |\psi^m_{S_n}|^2 \right|^2 = a_n \geqq \frac{1}{2} \text{ for all } S_n . \end{aligned}$$

This proves that $U^+_{g(H)}(P)$ does not exist.

Now we define by $U \equiv 1 - 2P$ a unitary operator and we set

$B = UHU$. From the definitions of U, B, and (NS) it immediately follows that $W_+(B, H)$ exists and $U = W_+(B, H)$. Since, however, $V^+_{g(H)}(-2P)$ does not exist, it follows from Theorem 1 and (NS) that also $W_+(g(B), g(H))$ does not exist.

5. *Proofs of the Lemmas.* Proof of Lemma 1: We shall prove Lemma 1 indirectly. Thus we suppose that for a nonnegative bounded selfadjoint operator T and an allowable function $g(\lambda)$ the strong limits

$$V^+_A(T) = s\text{-}\lim_{t\to\infty} e^{itA} T e^{-itA} P_A = 0 \,, \tag{5.1}$$

$$V^+_{g(A)}(T) = s\text{-}\lim_{t\to\infty} e^{itg(A)} T e^{-itg(A)} P_A = S \tag{5.2}$$

exist with $S \neq 0$, and from these assumptions we construct a contradiction. It is obvious that S is also a nonnegative bounded selfadjoint operator with $S = SP_A = P_A S \in \{g(A)\}'$ by (2.3) to (2.5). By $S \neq 0$ it exists a $u \in P_A\mathscr{H}$ with $Su \neq 0$. From the definition of the allowable functions $g(\lambda)$ it follows that there is a finite interval $\Delta \subset (-\infty, \infty)$ such that $P_A(\Delta)u \neq 0$, $SP_A(\Delta)u \neq 0$ and $g'(\lambda)$ is continuous, positive and of bounded variation on Δ. For a nonnegative operator S it follows from $Sv \neq 0$ that also $(v, Sv) \neq 0$. Hence we have $(P_A(\Delta)u, SP_A(\Delta)u) \neq 0$ and then $QSQ \neq 0$ where Q is the orthogonal projection on the subspace $\mathscr{H}_1 = \overline{\mathrm{sp}}\,\{P_A(\Delta')u,\ \Delta' \subset \Delta\}$. It is $Q \in \{A\}'$ and therefore $Q \in \{g(A)\}'$. By $S \in \{g(A)\}'$ we obtain $QSQ \in \{g(A)\}'$. Since $g(\lambda)$ is strictly increasing on Δ it is clear that $\{QAQ\}' = \{Qg(A)Q\}'$. From this identity and $QSQ \in \{g(A)\}'$ we finally obtain $QSQ \in \{A\}'$. Furthermore, we have $E(\Delta) \in \{A\}'$, where $E(\Delta)$ is the spectral measure of QSQ. We choose a $\alpha > 0$ such that $E(0, \alpha) < Q$. With $R \equiv (Q - E(0, \alpha)) \in \{A\}'$ and (5.1), (5.2) we find

$$V^+_A(RTR) = 0,\ V^+_{g(A)}(RTR) = RSR \neq 0 \,. \tag{5.3}$$

RSR is a nonnegative selfadjoint operator with the spectrum $\delta \in 0 \cup [a, b] (0 < a < b < \infty)$. Now we consider continuous functions $f(\lambda)$ which are 1 on $[a, b]$ and 0 in a neighborhood of 0.

By (2.6) and (5.3) we find

$$V^+_A(f(RTR)) = 0,\ V^+_{g(A)}(f(RTR)) = f(RSR) = R \,. \tag{5.4}$$

From the independence of the right sides of these $f(\lambda)$ it can easily be shown that (5.4) is also true for the step-function

$$f(\lambda) = \begin{cases} 1 & \text{on } [a_1, b_1]\ (0 < a_1 < a < b < b_1 < \infty) \\ 0 & \lambda \notin [a_1, b_1] \,. \end{cases}$$

Hence we have

$$V^+_A(P) = 0,\ V^+_{g(A)}(P) = R \tag{5.5}$$

where $P = f(RTR)$ is an orthogonal projection with $P < R$. $\mathscr{H}_1$ reduces A, P, R and P, R are distinct from 0 only on $\mathscr{H}_1$. Thus it is sufficient to consider (5.5) in $\mathscr{H}_1$. A restricted on $\mathscr{H}_1$ is an operator with a simple absolutely continuous spectrum $\sigma \subset \varDelta$. Then we may identify $\mathscr{H}_1$ with $\mathscr{L}^2(\sigma)$ and a restricted on $\mathscr{H}_1$ with the multiplication operator by λ and regard $\mathscr{H}_1 \cong \mathscr{L}^2(\sigma)$ as a subspace of the large Hilbert space $\mathscr{L}^2(\varDelta)$. In $\mathscr{L}^2(\varDelta)$ we may identify R with the multiplication operator by $\chi_{\bar{\rho}}(\lambda)$, where $\chi_{\bar{\rho}}(\lambda)$ is the characteristic function on $\bar{\rho} \subset \sigma$ with $|\bar{\rho}| \neq 0$. H denotes the multiplication operator by λ in $\mathscr{L}^2(\varDelta)$. Then we obtain from (5.5)

$$\lim_{t\to\infty} \| Pe^{-iHt}\psi \| = 0 \quad \text{for every} \quad \psi \in \mathscr{L}^2(\varDelta)\,, \tag{5.6}$$

$$\lim_{t\to\infty} \| Pe^{-ig(H)t}\psi \| = \| \chi_{\bar{\rho}}\psi \| \,. \tag{5.7}$$

For the sake of simplicity, we shall assume that $\varDelta = [0, 2\pi]$. We can write $g(\lambda) = \alpha \cdot \bar{g}(\lambda) + \beta$, where $\bar{g}(0) = 0$, $\bar{g}(2\pi) = 2\pi$, and α, β are real numbers with $\alpha > 0$. Then we put

$$\varphi_n(\lambda) \equiv \frac{1}{\sqrt{2\pi}} e^{-i\lambda n}, \; \psi_n(\lambda) \equiv \frac{1}{\sqrt{2\pi}} \sqrt{\bar{g}'(\lambda)}\, e^{-in\bar{g}(\lambda)} \,.$$

It is easy to verify that both φ_n and ψ_n form a complete orthonormal family in $\mathscr{L}^2(\varDelta)$. Furthermore, we have $\|\chi_{\bar{\rho}}\psi_n\|^2 = C > 0$. With these notations we easily obtain from (5, 6), (5.7)

$$\| P\varphi_n \|^2 = \varepsilon_n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \tag{5.8}$$

$$\| P\psi_n \|^2 - C = \alpha_n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \,. \tag{5.9}$$

We set $\psi_s(\lambda) = \sum_m a_s^m \varphi_m(\lambda)$. Now we consider the functions $\psi_s(\lambda)$ for which

$$\left\| \psi_s - \sum_{n=1}^{N} a_s^m \varphi_m \right\|^2 \leqq \frac{C}{2} \tag{5.10}$$

with fixed $N > 0$. For the functions $\bar{\psi}_s(\lambda) = (\sqrt{\bar{g}^1(\lambda)})^{-1}\psi_s(\lambda)$ by Lemma 2 we have

$$\left\| \bar{\psi}_s - \sum_{m=[sq_1]-p}^{[sq_2]-p} \bar{a}_s^m \varphi_m \right\| \leqq \varepsilon$$

where q_1, q_2 are positive real numbers independent of s, ε and p is a natural number independent of s. It is clear that then also

$$\left\| \psi_s - \sum_{m=[sq_1]-p'}^{[sq_2]+p'} a_s^m \varphi_m \right\|^2 \leqq \frac{C}{2} \,, \tag{5.11}$$

with an appropriate p' independent of s. An elementary computation shows that for all ψ_s with

$$s \in \left(\left[\frac{1+p'}{q_1}\right], \left[\frac{1+p'}{q_1}\right]+1, \cdots, \left[\frac{N-p'}{q_2}\right]\right)$$

the inequality (5.10) is true.

Hence there are natural numbers N_1, s_1 and an $\alpha > 0$ such that for every fixed N with $N > N_1$ and $s \in (s_1, s_1+1, \cdots, [\alpha N]+s_1)$ ψ_s satisfies (5.10). Now we consider the sum

$$S_N = \sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N]+s_1} |(\varphi_n, \psi_s)|^2$$

and introduce the orthogonal projection $\bar{P} = 1 - P$. Then

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N]+s_1} |(p\varphi_n, \psi_s) + (\varphi_n, \bar{p}\psi_s)|^2 \\ &\leqq \sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N]+s_1} \{|(p\varphi_n, \psi_s)|^2 + |(\varphi_n, \bar{p}\psi_s)|^2 + 2|(p\varphi_n, \psi_s)|\cdot|(\varphi_n, \bar{p}\psi_s)|\} \\ &\leqq \sum_{n=1}^{N} \sum_{s=s_1}^{[\beta N]+s_1} \{|(p\varphi_n, \psi_s)|^2 + |(\varphi_n, \bar{p}\psi_s)|^2\} \\ &\quad + 2\sqrt{\left(\sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N]+s_1} |(p\varphi_{n_1}\psi_s)|^2\right)\left(\sum_{n=1}^{N} \sum_{s=s_1}^{[\alpha N]+s_1} |(\varphi_{n_1}\bar{p}\psi_s)|^2\right)} \\ &\leqq \sum_{n=1}^{N} \|P\varphi_n\|^2 + \sum_{s=s_1}^{[\alpha N]+s_1} \|\bar{P}\psi_s\|^2 \\ &\quad + 2\sqrt{\left(\sum_{n=1}^{N} \|P\varphi_n\|^2\right)\left(\sum_{s=s_1}^{[\alpha N]+s_1} \|\bar{P}\psi_s\|^2\right)}\,. \end{aligned} \tag{5.12}$$

By (5.9) we have $\|\bar{P}\psi_s\|^2 = 1 - C - \alpha_s$ and with (5.8), (5.12)

$$\begin{aligned} S_N &\leqq \sum_{n=1}^{N} \varepsilon_n + [\alpha N](1-C) - \sum_{s=s_1}^{[\alpha N]+s_1} \alpha_s \\ &\quad + 2\sqrt{\left(\sum_{n=1}^{N} \varepsilon_n\right)\left(1 - C - \sum_{s=s_1}^{[\alpha N]+s_1} \alpha_s\right)}\,. \end{aligned} \tag{5.13}$$

On the other hand, by (5.10) we find

$$S_N \geqq [\alpha N]\left(1 - \frac{C}{2}\right)\,. \tag{5.14}$$

Combining (5.13), (5.14) we obtain

$$\frac{C}{2} \leqq \frac{1}{[\alpha N]}\left\{\sum_{n=1}^{N} \varepsilon_n - \sum_{s=s_1}^{[\alpha N]+s_1} \alpha_s\right\} + 2\sqrt{\frac{1}{[\alpha N]}\sum_{n=1}^{N} \varepsilon_n}\cdot\sqrt{1 - C - \frac{1}{[\alpha N]}\sum_{s=s_1}^{[\alpha N]+s_1} \alpha_s}\,.$$

Since ε_n, α_n are zero sequences, also

$$\gamma_N = \frac{1}{[\alpha N]}\sum_{n=1}^{N} \varepsilon_n\,, \quad \delta_N = \frac{1}{[\alpha N]}\sum_{s=s_1}^{[\alpha N]+s_1} \alpha_s$$

are zero sequences. Hence for sufficiently large N the last inequality which we have got from the assumption $S \neq 0$ is not true, which proves Lemma 1.

LEMMA 2. *Let $g(\lambda)$ be a real-valued function and $g'(\lambda)$ a continuous positive function on $\Delta = [0, 2\pi]$. Then the functions $\bar{\psi}_s(\lambda) \in \mathscr{L}^2(\Delta)$ defined by*

$$\bar{\psi}_s(\lambda) \equiv \frac{1}{\sqrt{2\pi}} e^{-isg(\lambda)} = \sum_{m=-\infty}^{\infty} \bar{a}_s^m \frac{1}{\sqrt{2\pi}} e^{-i\lambda m} \tag{5.15}$$

possess the following properties: For every ε with $0 < \varepsilon < 1$ and every natural nnmber $s > 0$ there exist two real positive numbers q_1, q_2 independent of s, ε and a natural number p independent of s such that

$$\sum_{m=[sq_1]-p}^{[sq_2]+p} |\bar{a}_s^m|^2 \geqq 1 - \varepsilon \, . \tag{5.16}$$

Proof. Let $a_1 = \min_{\lambda \in \Delta} g'(\lambda)$, $a_2 = \max_{\lambda \in \Delta} g'(\lambda)$. Let s be a fixed natural number. We consider integral numbers m with $m > s \cdot a_2$ or $s \cdot a_1 > m$. For these m we have $|s \cdot g'(\lambda) - m| > 0$ and we can write

$$\bar{a}_s^m = \frac{1}{2\pi} \int_0^{2\pi} d\lambda e^{-is\cdot g(\lambda)} e^{im\lambda} = \frac{1}{2\pi} \int_0^{2\pi} d\lambda \frac{1}{-i(sg'(\lambda) - m)} \left(\frac{d}{d\lambda} e^{-i(sg(\lambda) - m\cdot\lambda)} \right)$$

Integrating by parts and an elementary computation shows that

$$\begin{aligned} |\bar{a}_s^m| &\leqq \frac{1}{2\pi} \left| \left[\frac{1}{-i(sg'(\lambda) - m)} e^{-i(sg(\lambda) - m\lambda)} \right]_0^{2\pi} \right. \\ &\qquad \left. + \int_0^{2\pi} e^{-i(sg(\lambda) - m\lambda)} d\left(\frac{1}{i(g'(\lambda)s - m)} \right) \right| \\ &\leqq \frac{1}{2\pi} \left\{ \frac{2}{|s\alpha - m|} + \int_0^{2\pi} \left| d\left(\frac{1}{sg'(\lambda) - m} \right) \right| \right\} \\ &\leqq \frac{1}{2\pi} \left\{ \frac{2}{|s\alpha - m|} + \frac{M\cdot s}{|s\alpha - m|^2} \right\} \\ &= \frac{1}{2\pi |s\alpha - m|} \left\{ 2 + \frac{M}{\left| \alpha - \dfrac{m}{s} \right|} \right\} , \end{aligned} \tag{5.17}$$

where M is the total variation of $g'(\lambda)$ on Δ and $\alpha = a_1$ if $m < s \cdot a_1$ or $\alpha = a_2$ if $m > s \cdot a_2$. Let p' be a positive integral number, then by (5.17) we have

$$\begin{aligned}\sum_{m=[s2a_2]+p'+1}^{\infty}|\bar{a}_s^m|^2 &\leqq \frac{1}{(2\pi)^2}\left(2+\frac{M}{a_2}\right)^2\sum_{m=[s2a_2]+p'+1}^{\infty}\frac{1}{|m-[sa_2]-1|^2}\\ &< \frac{1}{(2\pi)^2}\left(2+\frac{M}{a_2}\right)^2\sum_{n=p'}^{\infty}\frac{1}{n^2}\end{aligned}\tag{5.18}$$

and entirely analogous

$$\begin{aligned}\sum_{m=[s(1/2)a_1]-p'-1}^{-\infty}|\bar{a}_s^m|^2 &\leqq \frac{1}{(2\pi)^2}\left(2+\frac{2M}{a_1}\right)^2\sum_{m=[s(1/2)a_1]-p'-1}^{-\infty}\frac{1}{|sa_1-m|^2}\\ &< \frac{1}{(2\pi)^2}\left(2+\frac{2M}{a_1}\right)^2\sum_{n=-p'}^{\infty}\frac{1}{n^2}\,,\end{aligned}\tag{5.19}$$

where $[a]$ is the smallest integer $r > a - 1$.

For sufficiently large p' from (5.18), (5.19) we find

$$\sum_{m=[s2a_2]+p'+1}^{\infty}|\bar{a}_s^m|^2 + \sum_{m=[s(1/2)a_1]-p'-1}^{-\infty}|\bar{a}_s^m|^2 < \varepsilon$$

for all positive integral numbers s and every $\varepsilon > 0$. With $q_1 = (1/2)a_1$, $q_2 = 2\cdot a_2$ and by $|\psi_s|^2 = \sum_m |\bar{a}_s^m|^2 = 1$ we finally obtain (5.16).

LEMMA 3. *Let $g(\lambda)$, $\bar{\psi}_s(\lambda)$ be defined as in Lemma 2 and let $g'(\lambda)$ be continuous, strictly monotone on Δ. Then the functions $\bar{\psi}_s(\lambda)$ possess the following properties: For every ε with $0 < \varepsilon < 1$ there exists an N such that for all integral numbers m, s with $s > N$*

$$|\bar{a}_s^m| < \varepsilon\,.\tag{5.20}$$

Proof. From the continuity and strict monotony of the positive function $g'(\lambda)$ on Δ it follows that for every real number x and $\varepsilon > 0$ there is an interval $\Delta_x \subseteqq \Delta$ of the length $l_x \leqq \varepsilon\cdot\pi$ such that

$$\alpha(\varepsilon) \equiv \min_{x\in R_1}\left(\min_{\lambda\in\Delta-\Delta_x}|g'(\lambda)+x|\right)\tag{5.21}$$

exists and $\alpha(\varepsilon) > 0$. Hence with $x = -m/s$ we have

$$\begin{aligned}|\bar{a}_s^m| &= \left|\frac{1}{2\pi}\int_0^{2\pi}d\lambda e^{-isg(\lambda)}e^{im\lambda}\right|\\ &\leqq \frac{\varepsilon}{2} + \frac{1}{2\pi}\left|\int_{\Delta-\Delta_x}d\lambda\, e^{-is(g(\lambda)-x\lambda)}\right|\,.\end{aligned}\tag{5.22}$$

The domain of integration $(\Delta - \Delta_x)$ consists of one or two intervals in dependence on x and ε. Let $\Delta' = [a, b] \subseteqq \Delta$ be such an interval. Then

$$\left|\int_a^b d\lambda e^{-is(g(\lambda)-x\lambda)}\right| = \left|\int_a^b\frac{d\lambda}{-is(g'(\lambda)-x)}\left(\frac{d}{d\lambda}e^{-is(g(\lambda)-x\lambda)}\right)\right|$$

$$
\begin{aligned}
&\leqq \left| \left[\frac{1}{-is(g'(\lambda)-x)} e^{-is(g(\lambda)-x\lambda)} \right]_a^b + \int_a^b e^{-is(g(\lambda)-x\lambda)} d\left(\frac{1}{is(g'(\lambda)-x)} \right) \right| \\
(5.23) \quad &\leqq \frac{2}{s\cdot\alpha} + \int_a^b \left| d\left(\frac{1}{s(g'(\lambda)-x)} \right) \right| \\
&\leqq \frac{2}{s\cdot\alpha} + \frac{M}{s\cdot\alpha}
\end{aligned}
$$

where α is defined by (5.21) and $M = |g'(b) - g'(a)|$. From (5.22) and (5.23) we have

$$|a_s^m| \leqq \frac{\varepsilon}{2} + \frac{1}{\pi s} \cdot \frac{2+M}{\alpha} .$$

If we put $N = (2/\varepsilon\cdot\pi)\cdot\dfrac{2+M}{\alpha}$, then this implies (5.20).

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