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ON RESTRICTING IRREDUCIBLE CHARACTERS TO NORMAL SUBGROUPS

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This paper is about the situation where χ is an irreducible character of a finite group G and K is a normal subgroup. A construction of Serre's relating the characters of G with those of G/K is used to give a new proof of a well-known lemma concerning the case that $\chi|_K$ is irreducible and to generalize this lemma. It is seen that the irreducibility of $\chi|_K$ is equivalent to the property that $(1/|K|)\sum_{x\in hK}|\chi(x)|^2 = 1$ for each coset of G modulo K and also to the property that χ is not a component of $\lambda\chi$ for any irreducible character λ of G/K except for $\lambda = 1$. The subgroup $J_1 = J_1(\chi)$ is defined as the intersection of the kernels of the irreducible characters λ of G/K for which χ is a component of $\lambda\chi$. It is seen that an irreducible component σ of the restriction of χ to K will extend to J_1, $e_{J_1}(\chi) = e_K(\chi)$ and J_1 is the maximal normal subgroup with these two properties.

Preliminary remarks. $\hat{G}$ denotes the set of irreducible complex characters of G. 1 will often be used for the one-character of the appropriate group (according to context). $\langle\chi, \varphi\rangle_G = (1/|G|)\sum_{g\in G}\chi(g)\bar{\varphi}(g)$, the usual inner product.

We include here a couple of well-known theorems to be referred to later.

THEOREM A. (Clifford) *If $K \lhd G$, $\chi \in \hat{G}$, $\sigma \in \hat{K}$ and σ a component of $\chi|_K$ then $\chi|_K = e_K(\chi)\sum_{i=1}^{m}\sigma^{g_i}$ where $e_K(\chi)$ is a positive integer called the ramification index, $m = [G: I(\sigma)]$ with $I(\sigma)$ being the inertial group for σ and $\{g_1, \cdots, g_m\}$ are a set of coset representatives for G modulo $I(\sigma)$.* (See for example [1, Theorem 9.10].)

THEOREM B. *Let $K \lhd G$ and χ an irreducible character of G which remains irreducible when restricted to K. Then the characters $\lambda\chi$ are distinct and irreducible as λ varies over the characters of G/K. Further if θ is an irreducible character of G such that $\chi|_K$ is a component of $\theta|_K$, then θ is of the form $\lambda\chi$ as above.* (See [3, Lemma 3.1].)

1. In this section we review a construction due to Serre which bears some resemblance to the familiar process for inducing characters from a subgroup. Theorem 1.1(b) is analogous to the Frobenius reciprocity theorem and was stated by Serre without proof in [8, p. 106].

By a class function on a group G is meant any function from G to the complex numbers which is constant on conjugacy classes. Let K be a normal subgroup of the finite group G. If φ is any class function of G/K, let φ^* denote the corresponding class function on G obtained in the usual way by $\varphi^*(g) = \varphi(gK)$. [Note that it is usually the custom to write φ instead of φ^* and this will be done in the latter part of this paper but here it is useful to make the distinction.] If ψ is a class function on G let ψ_* denote the function on G/K defined by $\psi_*(hK) = (1/|K|) \sum_{x \in hK} \psi(x)$.

THEOREM 1.1. (Serre) (a) *ψ_* is a class function on G/K.*
(b) *$\langle \varphi^*, \psi \rangle_G = \langle \varphi, \psi_* \rangle_{G/K}$ where φ is any class function on G/K.*

Proof. (a) If hK and h_1K are conjugate in G/K then $h_1K = g^{-1}hKg$ for some $g \in G$. Hence

$$\begin{aligned}\psi_*(h_1K) &= \frac{1}{|K|} \sum_{x \in h_1K} \psi(x) = \frac{1}{|K|} \sum_{x \in g^{-1}hKg} \psi(x) \\ &= \frac{1}{|K|} \sum_{y \in hK} \psi(g^{-1}yg) = \frac{1}{|K|} \sum_{y \in hK} \psi(y) = \psi_*(hK) .\end{aligned}$$

(b)
$$\begin{aligned}\langle \varphi, \psi_* \rangle_{G/K} &= \frac{1}{|G/K|} \sum_{hK \in G/K} \varphi(hK)\overline{\psi_*(hK)} \\ &= \frac{|K|}{|G|} \sum_{hK \in G/K} \left[\varphi(hK) \cdot \frac{1}{|K|} \sum_{x \in hK} \overline{\psi(x)}\right] \\ &= \frac{1}{|G|} \sum_{x \in G} \varphi^*(x)\overline{\psi(x)} = \langle \varphi^*, \psi \rangle_G .\end{aligned}$$

The following corollary shows that the construction appears not as promising as Frobenius' induction; nevertheless it has some use as will be seen shortly.

COROLLARY 1.2. *Let $\psi \in \hat{G}$.*
(a) *If $K \subseteq \operatorname{Ker} \psi$ and hence ψ may be regarded also as an element of $\widehat{G/K}$, then $\psi_* = \psi$* (under the latter identification).
(b) *If $K \nsubseteq \operatorname{Ker} \psi$ then $\psi_* \equiv 0$.*

Proof. If $\varphi \in \widehat{G/K}$ then $\langle \varphi, \psi_* \rangle_{G/K} = \langle \varphi^*, \psi \rangle_G = 1$ or 0 depending on whether $\psi = \varphi^*$ or not. Case (b) means that $\psi \neq \varphi^*$ for any $\varphi \in \widehat{G/K}$ and since $\widehat{G/K}$ forms a basis for the class functions on G/K we get that $\psi_* \equiv 0$. If (a) holds, then $\psi = \varphi^*$ for exactly one φ and so $\psi_* = \varphi$.

COROLLARY 1.3. *If ψ is any class function on G write $\psi =$*

$\sum a_i\chi_i + \sum b_j\psi_j$ *where* $\chi_i, \psi_j \in \hat{G}$, $K \subseteq \operatorname{Ker}\chi_i$ *each* i *but* $K \not\subseteq \operatorname{Ker}\psi_j$ *each* j. *Then* $(\psi_*)^* = \sum a_i\chi_i$. *Further if* ψ *is a character then* $(\psi_*)^*$ *is also a character or the zero function.*

In what follows, we omit the upper star and identify characters of G/K with characters of G.

2. We now use the Serre construction to give a proof of a theorem which generalizes both [2, Lemma, p. 178] of Gallagher and [5, Lemma 4.2] of Iwahori and Matsumoto (see corollaries which follow).

THEOREM 2.1. *Let* $\chi \in \hat{G}$. *Let* $S(\chi)$ *denote the set of irreducible characters* λ *of* G *such that* $\lambda\chi$ *contains* χ *as a component,* i.e., $\langle\lambda\chi, \chi\rangle_G = n_\lambda > 0$. *Then* $(\chi\bar{\chi})_* = \sum_{\lambda\in S(\chi)\cap\widehat{G/K}} n_\lambda\lambda$ i.e., $(1/|K|)\sum_{x\in hK}|\chi(x)|^2 = \sum_{\lambda\in S(\chi)\cap\widehat{G/K}} n_\lambda\lambda(hK)$.

Proof. $n_\lambda = \langle\lambda\chi, \chi\rangle_G = \langle\lambda, \chi\bar{\chi}\rangle_G$ so that $\chi\bar{\chi} = \sum n_\lambda\lambda$ summed over $\lambda \in S(\chi)$. By Corollary 1.3, $(\chi\bar{\chi})_* = \sum_{\lambda\in S(\chi)\cap\widehat{G/K}} n_\lambda\lambda$.

COROLLARY 2.2. *The one-character always occurs with multiplicity one in* $(\chi\bar{\chi})_*$.

COROLLARY 2.3. (Iwahori-Matsumoto [5, Lemma 4.2]) *If* G/K *is abelian and* $H(\chi)$ *is the group of* (linear) *characters* $\lambda \in \widehat{G/K}$ *such that* $\lambda\chi = \chi$ *then* $(\chi\bar{\chi})_* = \sum_{\lambda\in H(\chi)}\lambda$.

Proof. In this case $S(\chi) \cap \widehat{G/K} = H(\chi)$ since if λ is linear and $\langle\lambda\chi, \chi\rangle = n_\lambda > 0$ then $\lambda\chi = \chi$ and $n_\lambda = 1$.

COROLLARY 2.4. (Gallagher [2, Lemma, p. 178]; also Isaacs [4, Lemma 3.4]) *If* $\chi|_K$ *is irreducible then* $(\chi\bar{\chi})_* = 1$.

It is instructive to give two different short proofs.

Proof 1. By Theorem B in the preliminary remarks the characters $\{\lambda\chi: \lambda \in \widehat{G/N}\}$ are all distinct and irreducible. Thus $S(\chi) \cap \widehat{G/N} = \{1\}$.

Proof 2. $\chi|_K$ irreducible means that

$$1 = \langle\chi, \chi\rangle_K = \frac{1}{|K|}\sum \chi(g)\overline{\chi(g)} = (\chi\bar{\chi})_*(K) .$$

Hence $(\chi\bar{\chi})_*$ is a character (Corollary 1.3) of degree 1. By Corollary

2.2, we have $(\chi\bar{\chi})_* = 1$.

COROLLARY 2.5 (the converse to Corollary 2.4). *If* $(\chi\bar{\chi})_* = 1$ *then* $\chi|_K$ *is irreducible.*

Proof. As in Proof 2 of Corollary 2.4 above, note that $\langle\chi, \chi\rangle_K = (\chi\bar{\chi})_*(K) = 1$.

As a summary it is convenient to make a list of equivalent statements.

THEOREM 2.5. *Let* $\chi \in \hat{G}$, $K \lhd G$. *The following conditions are equivalent*:

(a) $\chi|_K$ *is irreducible.*

(b) $(\chi\bar{\chi})_* = 1$.

(c) *If* $\lambda \in \widehat{G/K}$ *and* $\langle\lambda\chi, \chi\rangle_G \neq 0$ *then* $\lambda = 1$.

(d) *The characters in the set* $\{\lambda\chi : \lambda \in \widehat{G/K}\}$ *are distinct and irreducible.*

Proof. (a) $\Leftrightarrow$ (b) by Corollaries 2.4 and 2.5. (b) $\Leftrightarrow$ (c) by Theorem 2.1. So (a), (b) and (c) are equivalent. Clearly (d) $\Rightarrow$ (c). (a) $\Rightarrow$ (d) is by Theorem B of the preliminary remarks.

3. In [6] the author considered the effect of the characters $\widehat{G/K}$ on an irreducible character of G in the case that G/K is abelian (see also [5] for a similar treatment). In particular the irreducible characters H(χ) that "fix" χ (i.e., $\lambda\chi = \chi$) were studied and the intersection of their kernels was singled out as the "dual inertial group" $J(\chi)$. If G/K is non-abelian then its irreducible characters need not be linear, and there are several ways to generalize the above concept. In [7] we called $H(\chi)$ the set of irreducible characters λ such that $\lambda\chi = (\deg\lambda)\chi$. Some properties of $J(\chi)$ were dealt with there where $J(\chi)$ is the intersection of the kernels of set of characters $H(\chi)$. An alternative approach which we look at briefly here is to examine instead $H_1(\chi) = S(\chi) \cap \widehat{G/K}$ = the irreducible characters λ of G/K such that $\lambda\chi$ contains χ as a component. Then let $J_1(\chi) = \bigcap\{\mathrm{Ker}\,\lambda : \lambda \in H_1(\chi)\}$. It is seen below that $J_1 = J_1(\chi)$ has at least some of the properties of the "dual inertial group" of [6], namely that if, (1) σ is a component of $\chi|_K$ then σ may be extended to $J_1(\chi)$ and (2) $e_{J_1}(\chi) = e_K(\chi)$. Further it is shown (Theorem 3.5) that $J_1(\chi)$ might be characterized as the (unique) maximal normal subgroup between G and K having these two properties. (This latter fact is new even for the case of G/K abelian treated in [6].)

THEOREM 3.1. *Let* $\chi \in \hat{G}$, $K \lhd G$, *and* $J_1 = J_1(\chi)$ *be defined as above. Let* ψ *be an irreducible component of* $\chi|_{J_1}$. *Then* $(\psi\bar{\psi})_* = 1$ *on* J_1/K *and hence* $\psi|_K$ *is irreducible.*

Proof. $(\chi\bar{\chi})_* = \sum_{\lambda \in H_1(\chi)} n_\lambda \lambda$. So $(\chi\bar{\chi})_*$ restricted to J_1/K consists of a multiple of the one-character. Since ψ is a component of $\chi|_{J_1}$, $\chi\bar{\chi}|_{J_1} = \psi\bar{\psi} + \tau$ where τ is another character of J_1, and the restriction of $(\chi\bar{\chi})_*$ to J_1/K equals $(\psi\bar{\psi})_* + \tau_*$. Hence $(\psi\bar{\psi})_*$ is a multiple of the one-character, and hence is the one-character by Corollary 2.2.

COROLLARY 3.2. *Let* $K \lhd G$, $\chi \in \hat{G}$ *and let* σ *be a component of* $\chi|_K$. *Then* σ *may be extended to a character* ψ *of* J_1 *and* $I(\sigma) \supseteqq J_1(\chi)$ *where* $I(\sigma)$ *denotes the inertial group of* σ.

Proof. Let ψ be a component of $\chi|_K$. By Theorem 3.1 $\psi|_K = \tau$ is an irreducible component of $\chi|_K$. For some $g \in G$, $\sigma = \tau^g$ (by Theorem A in the preliminary remarks) and ψ^g is an extension of σ to J_1. For simplicity of notation we may assume henceforth that $\psi|_K = \sigma$. Thus if $h \in J_1$ then $\psi^h = \psi$ so $\sigma^h = \sigma$ and hence $h \in I(\sigma)$.

The following notation which was used in [7] will be helpful in proving Theorem 3.3. If $\rho, \chi \in G$ then $\rho^*\chi$ is the set of irreducible components of $\rho\chi$. If $T \subseteqq \hat{G}$ then $\rho^*T = \bigcup \{\rho^*\gamma \mid \gamma \in T\} = T^*\rho$. Associativity holds: $(\rho^*\gamma)^*\tau = \rho^*(\gamma^*\tau) =$ the irreducible components of $\rho\gamma\tau$. In this notation, $S(\chi) = \chi^*\bar{\chi}$ since $\langle \lambda\chi, \chi \rangle = \langle \lambda, \chi\bar{\chi} \rangle$. Theorem 2.5 of [7] states that if $K \lhd G$. $\chi \in \hat{G}$ and ψ an irreducible component of $\chi|_K$ then the set of irreducible components of ψ^G equals $\bigcup \{\chi^*\rho \mid \rho \in \widehat{G/K}\} = \chi^*\widehat{G/K}$.

THEOREM 3.3. $e_K(\chi) = e_{J_1}(\chi)$.

Proof. Let ψ be an irreducible component of $\chi|_{J_1}$ and $\psi|_K = \sigma$. then

$$\chi|_K = e_K(\chi) \sum_{i=1}^{m} \sigma^{g_i} \qquad m = [G\colon I(\sigma)]$$

and

$$\chi|_{J_1} = e_{J_1}(\psi) \sum_{i=1}^{n} \psi^{h_i} \qquad n = [G\colon I(\psi)]$$

by Clifford's theorem ("Theorem A"). Since $\psi|_K = \sigma$ it is clear that $I(\psi) \subseteqq I(\sigma)$. It suffices to prove that $I(\psi) = I(\sigma)$ for then $m = n$ and since $\deg \sigma = \deg \psi$ the above equations show that $e_K(\chi) = e_{J_1}(\chi)$.

Hence let $g \in I(\sigma)$. We must show that $\psi^g = \psi$. Clearly $\psi^g|_K = \psi|_K = \sigma$ hence by Theorem B, $\psi^g = \lambda\psi$ for some $\lambda \in \widehat{J_1/K}$. And $(\lambda\psi)^G = (\psi^g)^G = \psi^G$. Let γ be an irreducible component of λ^G. Then $\gamma|_{J_1} = e_{J_1}(\gamma)\sum \lambda^g$ and $K \subseteqq \operatorname{Ker}\gamma$. Thus any irreducible component of $\psi^G = (\lambda\psi)^G$ must be included among the irreducible components of $\gamma\psi^G = (\gamma|_{J_1}\psi)^G$. In particular, $\langle\chi, \psi^G\rangle > 0$ and hence $\langle\chi, \gamma\psi^G\rangle > 0$. By Theorem 2.5 of [7], cited earlier, there exists $\tau \in \widehat{G/J_1}$ such that $\chi \in \gamma^*(\tau^*\chi) = (\gamma^*\tau)^*\chi$. Hence there exists $\delta \in \gamma^*\tau$ such that $\chi \in \delta^*\chi$. Hence $\delta \in S(\chi)$.

Now $\gamma \in \widehat{G/K}$, $\tau \in \widehat{G/J_1} \subseteqq G/K$ so $\delta \in \gamma^*\tau \subseteqq \widehat{G/K}$; i.e., $\delta \in S(\chi) \cap \widehat{G/K} = H_1(\chi)$. Thus $J_1(\chi) \subseteqq \operatorname{Ker}\delta$; i.e., $\delta \in \widehat{G/J_1}$. But $\langle\delta, \gamma\tau\rangle > 0$ means that $\langle\delta\bar{\tau}, \gamma\rangle > 0$ so $\gamma \in \delta^*\bar{\tau} \subseteqq \widehat{G/J_1}$ so that $J_1 \subseteqq \operatorname{Ker}\gamma$ and hence λ is trivial. Thus $\psi^g = \lambda\psi = \psi$.

COROLLARY 3.4. $I(\psi) = I(\sigma)$.

THEOREM 3.5. *Let G, K, χ, σ be as in the previous theorems. Let N be a normal subgroup of G containing K such that*

(1) $e_N(\chi) = e_K(\chi)$ and

(2) σ extends to an irreducible character θ of N. Then $N \subseteqq J_1(\chi)$. Hence $J_1(\chi)$ is the unique normal subgroup which is maximal with respect to having properties (1) *and* (2).

Proof. Since $\theta|_K = \sigma$ is irreducible, $\theta^h|_K = \sigma^h$ is irreducible for each $h \in G$. Using Theorem A and writing $e = e_K(\chi) = e_N(\chi)$ we have:

$$\chi_K = e\sum \sigma^h \qquad \{\sigma^h\} = \text{set of distinct conjugates of } \sigma\,,$$
$$\chi|_N = e\sum \theta^g \qquad \{\theta^g\} = \text{set of distinct conjugates of } \theta\,.$$

Since $\chi|_K = (\chi|_N)|_K$ we see that different conjugates θ^h of θ must restrict to different conjugates of σ. Hence if $\theta \neq \theta^h$ then $\sigma = \theta|_K \neq \theta^h|_K = \sigma^h$.

Now let $\gamma \in H_1(\chi)$. We will show that $N \subseteqq \operatorname{Ker}\gamma$ and hence $N \subseteqq J_1(\chi) = \bigcap\{\operatorname{Ker}\gamma\colon \gamma \in H_1(\chi)\}$.

$\gamma \in H_1(\chi)$ means $\chi \in \gamma^*\chi$ and hence $\gamma|_N\chi|_N$ contains θ as a component. Hence there exists λ an irreducible component of $\gamma|_N$, and θ^g a conjugate of θ such that $\theta \in \lambda^*\theta^g$. So $(\lambda\theta^g)|_N$ contains σ as a component and yet $(\lambda\theta^g)|_N = (\deg\lambda)\sigma^g$. Thus $\sigma = \sigma^g$ and by the initial discussion this means that $\theta = \theta^g$ and that $\theta \in \lambda^*\theta$. By Theorem B, λ must be trivial. Since any component of $\gamma|_N$ is a conjugate of λ, we have that $N \subseteqq \operatorname{Ker}\gamma$.

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