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ON GROSS DIFFERENTIATION ON BANACH SPACES

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Let $p_t(x, \cdot)$ denote the Wiener measure in an abstract Wiener space (H, B) with variance parameter $t > 0$ and mean x in B. It is shown that if $f \in L^2 p_t(x, \cdot), t > 0$ and x are fixed, then the function $p_t f$ defined by $p_t f(x + h) = \int_B f(y) p_t(x + h, dy)$ for h in H is infinitely Gross differentiable at x. The first two derivatives are given by $(Dp_t f(x), h) = t^{-1} \int_B f(y)(h, y - x) p_t(x, dy)$ and $(D^2 p_t f(x)k, h) = t^{-1} \int_B f(y) \{t^{-1}(h, y - x)(k, y - x) - (h, k)\} p_t(x, dy)$, where h and k are in H. Moreover, $D^2 p_t f(x)$ is a Hilbert-Schmidt operator and $\| D^2 p_t f(x) \|_2 \leqq \sqrt{2}\, t^{-1} \left\{ \int_B |f(y)|^2 p_t(x, dy) \right\}^{1/2}$. An application to Uhlenbeck-Ornstein process is also given.

1. Introduction. It is well-known that in a general Banach space B the Frechet differentiable functions on B do not form a very large class of functions. The works [1; 7], among others, show that for many separable Banach spaces the bounded continuously Frechet differentiable functions are not dense in the space of bounded uniformly continuous functions. However, by regarding a real separable Banach space B as an abstract Wiener space [3], Goodman [2] is able to show that the set of bounded continuously quasi-differentiable functions on B is dense in the space of bounded uniformly continuous functions on B. Regarding B as an abstract Wiener space has a more important advantage, namely, we can talk about whether the second derivative is a Hilbert-Schmidt or trace class operator.

Let (H, B) be an abstract Wiener space. In [4], a real valued function u defined in an open subset V of B is said to be H-differentiable at $x \in V$ if there is an element y in H such that $|u(x + h) - u(x) - (y, h)| = o(|h|)$ for h in H, where $|\cdot|$ and $(,)$ are the norm and the inner product of H, respectively. y is easily seen to be unique and is denoted by $Du(x)$. Higher derivatives are defined similarly. Let $p_t(x, \cdot)$ denote the Wiener measure in (H, B) with variance parameter $t > 0$ and mean $x \in B$. $p_t(0, \cdot)$ will be written as $p_t(\cdot)$. If f is a bounded measurable complex valued function on B, we define $p_t f(x) = \int_B f(x + y) p_t(dy)$.

THEOREM (Gross [4]). *Let f be a bounded measurable function on B. Then $p_t f$ is infinitely H-differentiable on B with the first* *and second derivatives given by*

$$(Dp_tf(x), h) = t^{-1}\int_B f(x+y)(h, y)p_t(dy)\ ,$$

$$(D^2p_tf(x)k, h) = t^{-1}\int_B f(x+y)\{t^{-1}(h, y)(k, y) - (h, k)\}p_t(dy)\ ,$$

where h and k are in H. If T is a test operator then

$$\text{trace}\,[TD^2p_tf(x)] = t^{-1}\int_B f(x+y)\{t^{-1}\langle Ty, y\rangle - \text{trace}\ T\}p_t(dy)\ ,$$

where $\langle\,,\rangle$ is the natural pairing between B^* and B. Moreover, $D^2p_tf(x)$ is a Hilbert-Schmidt operator and

$$\|D^2p_tf(x)\|_2 \leqq \sqrt{2}\,t^{-1}\left\{\int_B |f(x+y)|^2p_t(dy)\right\}^{1/2}.$$

The assumption that f is bounded makes the above theorem uneasy to apply. For instance, it is desirable to differentiate the function r_tf defined by $r_tf(x) = \int_B f(y)r_t(x, dy)$, where $f \in L^2(p_1)$ and $r_t(x, \cdot) = p_{1-e^{-2t}}(e^{-t}x, \cdot)$ are the transition probabilities of Uhlenbeck-Ornstein process. Since f may not be bounded, the above theorem is not applicable. To overcome this difficulty Piech considers an orthonormal basis of $L^2(p_1)$ consisting of Hermite cylinder functions and proves the following theorem.

THEOREM (Piech [6]). *Let $g \in L^2(p_1)$. Then for p_1-a.e. x and for $t > 0$, r_tg is twice H-differentiable at x with derivatives given by*

$$(Dr_tg(x), h) = -[e^t(1-e^{-2t})]^{-1}\int_B g(y)(e^{-t}x - y, h)r_t(x, dy)\ ,$$

$$(D^2r_tg(x)h, k) = (e^{2t}-1)^{-1}\int_B g(y)\{(1-e^{-2t})^{-1}(e^{-t}x-y, h)(e^{-t}x-y, k) - (h, k)\}r_t(x, dy)\ .$$

Moreover, $|Dr_tg|$ and $\|D^2r_tg\|_2$ are in $L^2(p_1)$.

In this paper, we will reexamine how the H-derivatives are defined by Gross in [4] and prove his formulas in a rather general situation which we believe will be quite useful in $L^2(p_1)$-theory associated with Uhlenbeck-Ornstein process as well as other processes in B. We will see that our theorem yields Piech's in a beautiful way. The novelties in our approach are Lemma 1, Theorem 2 and the way we rewrite the expressions in Gross' theorem. For instance, the first derivative will be rewritten as $t^{-1}\int_B f(y)(h, y-x)p_t(x, dy)$, which exist when $f \in L^2(p_t(x, \cdot))$ since $(h, \cdot - x) \in L^2(p_t(x, \cdot))$.

DEFINITION 1. Let V be a subset of B and x in V. V is called *H-coset open at x* if there exists $\delta = \delta(x) > 0$ such that $x + h \in V$ for all $|h| < \delta$ in H. V is called *H-coset open if V is* H-coset open at every point of V.

Obviously, if G is an open subset of H then $z + G$ is H-coset open for any z in B. An open subset of B is also H-coset open because B-norm is weaker then H-norm. But there are other subsets which are H-coset open. For instance, let (H, B_0) be another abstract Wiener space with $B_0 \subset B$ then any open subset of B_0 is also H-coset open. Perhaps this is the reason why our theorem is easier to apply because whenever a function is defined on an H-coset open set then we can talk about Gross differentiation of f defined below.

DEFINITION 2. Let f be a function defined on an H-coset open subset V of B. Let $x \in V$. If the function $v(h) = f(x + h)$ defined on some H-neighborhood of 0 is kth$(k \geqq 1)$ Frechet differentiable at 0, then we say that f is kth *Gross differentiable* at x. The jth *Gross derivative* $D^j f(x)$ of f at x is defined to be the jth Frechet derivative of v at 0. (Hence $D^j f(x)$ is a j-linear form on H, $1 \leqq j \leqq k$).

From now on, $t > 0$ and $x \in B$ will be fixed through §4. Suppose $f \in L^2(p_t(x, \cdot))$, it will follow from Lemma 1 that $f \in L^1(p_t(x + h, \cdot))$ for all h in H. Therefore, $p_t f(x + h) = \int_B f(y) p_t(x + h, dy)$ is a function defined on the H-coset open set $x + H$.

THEOREM 1. *Let $f \in L^2(p_t(x, \cdot))$. Then the function $p_t f$ defined on the H-coset open set $x + H$ is infinitely Gross differentiable at x. The first and second Gross derivatives are given by*

$$(Dp_t f(x), h) = t^{-1} \int_B f(y)(h, y - x) p_t(x, dy) ,$$

$$(D^2 p_t f(x) k, h) = t^{-1} \int_B f(y)\{t^{-1}(h, y - x)(k, y - x) - (h, k)\} p_t(x, dy) .$$

If T is a test operator then

$$\text{trace } TD^2 p_t f(x) = t^{-1} \int_B f(y)\{t^{-1}\langle T(y - x), y - x\rangle - \text{trace } T\} p_t(x, dy) .$$

Moreover, $D^2 p_t f(x)$ is a Hilbert-Schmidt operator of H and

$$\| D^2 p_t f(x) \|_2 \leqq \sqrt{2}\, t^{-1} \left\{ \int_B |f(y)|^2 p_t(x, dy) \right\}^{1/2} .$$

The proof of this theorem will be given in §4. In §2 and §3 we will state some integral formulas and prove some lemmas necessary for the proof. We remark that the idea of the proof comes essentially from [5, Proposition 1]. In §5 we will apply our theorem to Uhlenbeck-Ornstein process and derive Piech's theorem.

2. Some integral formulas. In the following h and k are in H and $J_t(h, z) = \exp\{[-|h|^2 + 2(h, z)]/2t\}$, $h \in H$ and $z \in B$.

F 1). $$\int_B (h, z)^2 p_t(dz) = t|h|^2 .$$

F 2). $$\int_B (h, z)^4 p_t(dz) = 3t^2|h|^4 .$$

F 3). $$\int_B (h, z)^8 p_t(dz) = 105\, t^4|h|^8 .$$

F 4). $$\int_B J_t(h, z)^2 p_t(dz) = e^{|h|^2/t}$$

F 5). $$\int_B J_t(h, z)^4 p_t(dz) = e^{6|h|^2/t}$$

F 6). Let $\theta_t(r) = e^{6r^2/t} - 4e^{3r^2/t} + 6e^{r^2/t} - 3,\ r \geqq 0$.

Then $\int_B [J_t(h, z) - 1]^4 p_t(dz) = \theta_t(|h|)$.

3. Some lemmas. Recall that $t > 0$ and $x \in B$ are fixed. Note that for h in H the distribution function of $(h, y - x)$ with respect to $p_t(x, dy)$ is the same as that of (h, z) with respect to $p_t(dz)$. Hence if ϕ is any complex valued Borel measurable function on R then $\int_B \phi((h, y - x))p_t(x, dy) = \int_B \phi((h, z))p_t(dz)$. This remark will play an important role in the following computations.

LEMMA 1. *If $f \in L^2(p_t(x, \cdot))$ then $f \in L^1(p_t(x + h, \cdot))$ for all h in H. In fact,*

$$\int_B |f(y)|\, p_t(x + h, dy) \leqq e^{|h|^2/2t}\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2} .$$

Proof. By the translation formula for the Wiener measures,

$$p_t(x + h, dy) = J_t(h, y - x)p_t(x, dy) .$$

Hence

$$\begin{aligned}&\int_B |f(y)|\, p_t(x+h, dy)\\ &\quad \leqq \left\{\int_B J_t(h, y-x)^2 p_t(x, dy)\right\}^{1/2}\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\\ &\quad = \left\{\int_B J_t(h, z)^2 p_t(dz)\right\}^{1/2}\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\\ &\quad = e^{|h|^2/2t}\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}.\end{aligned}$$

Here we have used the integral formula F4).

LEMMA 2. *If $f \in L^2(p_t(x, \cdot))$ then for all h and k in H, $f(\cdot)(h, \cdot - x)$ and $f(\cdot)\{t^{-1}(h, \cdot - x)(k, \cdot - x) - (h, k)\}$ are in $L^1(p_t(x, \cdot))$. In fact,*

$$\int_B |f(y)(h, y-x)|\, p_t(x, dy) \leqq \sqrt{t}\,|h|\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}.$$

$$\begin{aligned}&\int_B |f(y)\{t^{-1}(h, y-x)(k, y-x) - (h, k)\}|\, p_t(x, dy)\\ &\quad \leqq (1+\sqrt{3})\,|h|\,|k|\left\{\int_B |f(y)|^2 p_t(x, dy)\right)^{1/2}.\end{aligned}$$

Proof. Use F1), F2) and the Schwarz inequality.

LEMMA 3. *If $f \in L^2(p_t(x, \cdot))$ then $f(\cdot)(h, \cdot - x) \in L^1(p_t(x+k, \cdot))$ for all h and k in H. In fact,*

$$\begin{aligned}&\int_B |f(y)(h, y-x)|\, p_t(x+k, dy)\\ &\qquad \leqq \sqrt[4]{3}\sqrt{t}\,|h|\, e^{3|k|^2/2t}\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\end{aligned}$$

Proof.

$$\begin{aligned}&\int_B |f(y)(h, y-x)|\, p_t(x+k, dy)\\ &\quad = \int_B |f(y)|\,|(h, y-x)|\, J_t(k, y-x) p_t(x, dy)\\ &\quad \leqq \left\{\int_B (h, y-x)^4 p_t(x, dy)\right\}^{1/4}\left\{\int_B J_t(k, y-x)^4 p_t(x, dy)\right\}^{1/4}\\ &\qquad \times \left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\\ &\quad = \left\{\int_B (h, z)^4 p_t(dz)\right\}^{1/4}\left\{\int_B J_t(k, z)^4 p_t(dz)\right\}^{1/4}\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\\ &\quad = \sqrt[4]{3}\sqrt{t}\,|h|\, e^{3|k|^2/2t}\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}.\end{aligned}$$

Here we have used integral formulas F2) and F5).

4. **Proof of Theorem 1.** By Lemma 1 the function $p_t f(x+h)$ is defined on the H-coset open subset $x+H$ of B. We divide the proof into several steps.

Step 1. To show that $p_t f$ is Gross differentiable and $(Dp_t f(x), h) = t^{-1}\int_B f(y)(h, y-x)p_t(x, dy)$. (This integral exists by Lemma 2).

Define $\phi(h) = p_t f(x+h) - p_t f(x) - t^{-1}\int_B f(y)(h, y-x)p_t(x, dy)$. We need to show that $|\phi(h)| = o(|h|)$ for small $|h|$. Let

$$\phi_1(h) = t^{-1}\int f(y)(h, y-x)\int_0^1 [J_t(sh, y-x) - 1]ds\,p_t(x, dy)\,,$$

$$\phi_2(h) = t^{-1}\int_B f(y)\int_0^1 s|h|^2 J_t(sh, y-x)ds\,p_t(x, dy)\,.$$

Then by [4, p. 153] we have $\phi(h) = \phi_1(h) - \phi_2(h)$. But now we have to make better estimates.

$$\begin{aligned}
&t|\phi_1(h)| \\
&\leqq \left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\left\{\int_B (h, z)^4 p_t(dz)\right\}^{1/4} \\
&\quad\times\left\{\int_0^1\int_B [J_t(sh, z) - 1]^4 p_t(dz)ds\right\}^{1/4} \\
&\leqq \left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\{3t^2|h|^4\}^{1/4}\left\{\int_0^1 \theta_t(s|h|)ds\right\}^{1/4} \\
&= \sqrt[4]{3}\sqrt{t}\,|h|\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\left\{\int_0^1 \theta_t(s|h|)ds\right\}^{1/4}.
\end{aligned}$$

Here we have used the integral formulas F2) and F6). It is easy to see that $\int_0^1 \theta_t(s|h|)ds \to 0$ as $|h| \to 0$. Hence $|\phi_1(h)| = o(|h|)$ for small $|h|$. On the other hand,

$$\begin{aligned}
t|\phi_2(h)| &\leqq |h|^2\int_0^1 s\left|\int_B f(y)J_t(sh, y-x)p_t(x, dy)\right|ds \\
&= |h|^2\int_0^1 s\left|\int_B f(y)p_t(x+sh, dy)\right|ds \\
&\leqq |h|^2\int_0^1 s e^{s^2|h|^2/2t}\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2} ds \\
&= t(e^{|h|^2/2t} - 1)\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}.
\end{aligned}$$

Hence $|\phi_2(h)| = o(|h|)$. Therefore, we have $|\phi(h)| = o(|h|)$.

Step 2. To show that $p_t f$ is twice Gross differentiable and

$$(D^2 p_t f(x)k, h) = t^{-1}\int_B f(y)\{t^{-1}(h, y-x)(k, y-x) - (h, k)\} p_t(x, dy) .$$

It is easy to see that

$$\begin{aligned}(Dp_t f(x+k), h) = {} & t^{-1}\int_B f(y)(h, y-x)J_t(k, y-x)p_t(x, dy) \\ & - t^{-1}\int_B f(y)(h, k)J_t(k, y-x)p_t(x, dy) .\end{aligned}$$

Hence we have

$$\begin{aligned}(Dp_t f(x+k), h) - (Dp_t f(x), h) = {} & t^{-1}\int_B f(y)(h, y-x) \\ & \times [J_t(k, y-x) - 1]p_t(x, dy) - t^{-1} \\ & \times \int_B f(y)(h, k)J_t(k, y-x)p_t(x, dy) .\end{aligned}$$

Note that

$$J_t(k, y-x) - 1 = t^{-1}\int_0^1 [(k, y-x) - s|k|^2]J_t(sk, y-x)ds .$$

Therefore,

$$\begin{aligned}(Dp_t f(x+k), h) - (Dp_t f(x), h) = {} & t^{-2}\int_B f(y)(h, y-x)(k, y-x) \\ & \times \int_0^1 J_t(sk, y-x)ds p_t(x, dy) - t^{-2}\int_B f(y)(h, y-x) \\ & \times \int_0^1 s|k|^2 J_t(sk, y-x)ds p_t(x, dy) - t^{-1}(h, k) \\ & \times \int_B f(y)J_t(k, y-x)p_t(x, dy) .\end{aligned}$$

Now, define

$$\begin{aligned}\psi(h, k) = {} & (Dp_t f(x+k), h) - (Dp_t f(x), h) \\ & - t^{-1}\int_B f(y)\{t^{-1}(h, y-x)(k, y-x) - (h, k)\}p_t(x, dy) ,\end{aligned}$$

$$\psi_1(h, k) = t^{-2}\int_B f(y)(h, y-x)(k, y-x)\int_0^1 [J_t(sk, y-x) - 1]ds p_t(x, dy) ,$$

$$\psi_2(h, k) = t^{-2}\int_B f(y)(h, y-x)\int_0^1 s|k|^2 J(sk, y-x)ds p_t(x, dy) ,$$

$$\psi_3(h, k) = t^{-1}(h, k)\int_B f(y)[J_t(k, y-x) - 1]p_t(x, dy) .$$

Then by the above computation, we have $\psi(h, k) = \psi_1(h, k) + \psi_2(h, k) + \psi_3(h, k)$. We will show that $\psi_j(h, k) = 0(|h|)o(|k|)$ and our assertion in this step follows immediately.

$$\begin{aligned} t^2|\psi_1(h, k)| &\leqq \left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\left\{\int_B (h, y-x)^8 p_t(x, dy)\right\}^{1/8} \\ &\quad \times\left\{\int_B (k, y-x)^8 p_t(x, dy)\right\}^{1/8}\left\{\int_0^1\int_B [J_t(sk, y-x)-1]^4 p_t(x, dy)ds\right\}^{1/4} \\ &= \sqrt[4]{105}\, t\,|h|\,|k|\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\left\{\int_0^1 \theta_t(s|k|)ds\right\}^{1/4} \\ &= o(|h|)o(|k|)\,. \end{aligned}$$

$$\begin{aligned} t^2|\psi_2(h, k)| &= |k|^2\left|\int_0^1 s\int_B f(y)(h, y-x)J_t(sk, y-x)p_t(x, dy)ds\right| \\ &= |k|^2\left|\int_0^1 s\int_B f(y)(h, y-x)p_t(x+sk, dy)ds\right| \\ &\leqq |k|^2\int_0^1 s\int_B |f(y)(h, y-x)|\,p_t(x+sk, dy)ds \\ &\leqq \sqrt[4]{3}\sqrt{t}\,|h|\,|k|^2\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\left\{\int_0^1 s e^{3s^2|k|^2/2t}ds\right\} \\ &= \frac{1}{\sqrt[4]{27}}t^{3/2}|h|\,[e^{3|k|^2/2t}-1]\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2} \\ &= O(|h|)o(|k|). \end{aligned}$$

In the third step of estimating $\psi_2(h, k)$ we have used Lemma 3. Finally,

$$\begin{aligned} t|\psi_3(h, k)| &\leqq |h|\,|k|\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\left\{\int_B [J_t(k, y-x)-1]^4 p_t(x, dy)\right\}^{1/4} \\ &= |h|\,|k|\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\{\theta_t(|k|)\}^{1/4} \\ &= O(|h|)o(|k|). \end{aligned}$$

Step 3. Let T be a test operator whose restriction to H is symmetric. Then T can be expressed as follows:

$$Tx = \sum_{j=1}^n \lambda_j\langle e_j, x\rangle e_j\,,$$

where $\langle\,,\rangle$ is the natural pairing between B^* and B, and $e_j \in B^*$, $j = 1, 2, \cdots, n$, are orthonormal. Then

$$\begin{aligned} \operatorname{trace} TD^2p_tf(x) &= \sum_{j=1}^n (TD^2p_tf(x)e_j, e_j) \\ &= \sum_{j=1}^n \lambda_j(D^2p_tf(x)e_j, e_j) \\ &= \sum_{j=1}^n t^{-1}\int_B f(y)\{t^{-1}\lambda_j\langle e_j, y-x\rangle^2 - \lambda_j\}p_t(x, dy) \\ &= t^{-1}\int_B f(y)\{t^{-1}\langle T(y-x), y-x\rangle - \operatorname{trace} T\}p_t(x, dy)\,. \end{aligned}$$

Hence, for any symmetric test operator T,

$$\begin{aligned}&\text{trace } TD^2p_tf(x)\\&\quad = t^{-1}\int_B f(y)\{t^{-1}\langle T(y-x),\, y-x\rangle - \text{trace } T\}p_t(x, dy)\,.\end{aligned}$$

Observe that $D^2p_tf(x)$ is symmetric and that both sides of the above equality depend only on the symmetric part of T. Hence the above equality holds for all test operators. Moreover, this equality implies that

$$\begin{aligned}&|\text{trace } TD^2p_tf(x)|\\&\quad \leqq t^{-1}\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\left\{\int_B [t^{-1}\langle T(y-x),\, y-x\rangle\right.\\&\quad \left. - \text{trace } T]^2 p_t(x, dy)\right\}^{1/2}\\&\quad = t^{-1}\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\left\{\int_B [t^{-1}\langle Tz, z\rangle - \text{trace } T]^2 p_t(dz)\right\}^{1/2}\\&\quad = t^{-1}\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\sqrt{2}\,\|T\|_2\,.\end{aligned}$$

Hence, by the same argument as in [4, pp. 155–156], $D^2p_tf(x)$ is a Hilbert-Schmidt operator and

$$\|D^2p_tf(x)\|_2 \leqq \sqrt{2}\,t^{-1}\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2}\,.$$

Step 4. To see the existence of the higher Gross derivatives of p_tf, observe that we need only the integrability with respect to $p_t(dz)$ of all polynomials in (h, z) and in $J_t(h, z)$ for h in H. It is easy to see that the method used in the previous computation yields the following estimate

$$\begin{aligned}&|D^np_tf(x+k)(h_1, \cdots, h_n) - D^np_tf(x)(h_1, \cdots, h_n)\\&\quad - D^{n+1}p_tf(x)(h_1, \cdots, h_n, k)| = O(|h_1|\cdot|h_2|\cdots|h_n|)o(|k|)\,,\end{aligned}$$

where $D^np_tf(x)$ is the nth Gross derivative of p_tf at x, $n \geqq 3$. Of course, we have to note that $D^np_tf(x+k)$ is also a continuous n-linear form on H for all k in H.

5. An application to Uhlenbeck-Ornstein process. Let $r_t(x, dy) = p_{1-e^{-2t}}(e^{-t}x, dy)$, $t > 0$ and $x \in B$. $\{r_t(x, \cdot); t > 0, x \in B\}$ generate a Markov process with continuous sample paths known as Uhlenbeck-Ornstein process. It is easy to check that p_1 is an invariant measure for $r_t(x, \cdot)$ for all $t > 0$. That is, for all Borel subsets A of B,

$$\int_B r_t(x, A)p_1(dx) = p_1(A),\ t > 0 .$$

THEOREM 2. *Let $g \in L^2(p_1)$. Then for each $t > 0$ there exists a measurable subset B_t of B such that $p_1(B_t) = 1$ and $g \in L^2(r_t(x, \cdot)$ for all x in B_t.*

REMARK. In a less precise way the above theorem says that if $g \in L^2(p_1)$ then $g \in L^2(r_t(x, \cdot))$ for a.e. $x[p_1]$.

Proof. By Fubini's theorem and the invariance of p_1

$$\begin{aligned}\int_B\left[\int_B |g(y)|^2 r_t(x, dy)\right]p_1(dx) &= \int_B |g(y)|^2 \int_B r_t(x, dy)p_1(dx) \\ &= \int_B |g(y)|^2 p_1(dy) < \infty .\end{aligned}$$

Hence $g \in L^2(r_t(x, \cdot))$ a.e. $x[p_1]$.

Let $g \in L^2(p_1)$. Then by the above theorem, we have $g \in L^2(r_t(x, \cdot))$ for all $x \in B_t$. But

$$\int_B |g(y)|^2 r_t(x, dy) = \int_B |g(y)|^2 p_{1-e^{-2t}}(e^{-t}x, dy) .$$

Hence $g \in L^2(p_{1-e^{-2t}}(e^{-t}x, \cdot))$ for all $x \in B_t$. Observe that $p_{1-e^{-2t}}g(e^{-t}x) = r_t g(x)$. Hence the formulas for first and second derivatives in Piech's theorem follow follow from Theorem 1. Moreover, by Theorem 1, for $x \in B_t$

$$\begin{aligned}\| D^2 r_t g(x)\|_2 &= e^{-2t}\| D^2 p_{1-e^{-2t}} g(e^{-t}x)\|_2 \\ &\leqq e^{-2t}\sqrt{2}(1 - e^{-2t})^{-1}\left\{\int_B |g(y)|^2 p_{1-e^{-2t}}(e^{-t}x, dy)\right\}^{1/2} \\ &= \sqrt{2}(e^{2t} - 1)^{-1}\left\{\int_B |g(y)|^2 r_t(x, dy)\right\}^{1/2} .\end{aligned}$$

Hence

$$\begin{aligned}\int_B \| D^2 r_t g(x)\|_2^2 p_1(dx) &= \int_{B_t} \| D^2 r_t g(x)\|_2^2 p_1(dx) \\ &\leqq 2(e^{2t} - 1)^{-2}\int_{B_t}\int_B |g(y)|^2 r_t(x, dy)p_1(dx) \\ &= 2(e^{2t} - 1)^{-2}\int_B\int_B |g(y)|^2 r_t(x, dy)p_1(dx) \\ &= 2(e^{2t} - 1)^{-2}\int_B |g(y)|^2 p_1(dy) .\end{aligned}$$

On the other hand, it is easy to see that for f in Theorem 1 we have

$$|Dp_t f(x)| \leqq t^{-1/2}\left\{\int_B |f(y)|^2 p_t(x, dy)\right\}^{1/2} .$$

Hence by the same argument above

$$|Dr_t g(x)| \leqq (e^{2t} - 1)^{-1/2}\left\{\int_B |g(y)|^2 r_t(x, dy)\right\}^{1/2}, \ x \in B_t ,$$

and

$$\int_B |Dr_t g(x)|^2 p_1(dx) \leqq (e^{2t} - 1)^{-1} \int_B |g(y)|^2 p_1(dy) .$$

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REFERENCES

1. R. Bonic and J. Frampton, *Differentiable functions on certain Banach spaces*, Bull. Amer. Math. Soc., **71** (1965), 393-395.
2. V. Goodman, *Quasi-differentiable functions on Banach spaces*, Proc. Amer. Math. Soc., **30** (1971), 367-370.
3. L. Gross, *Abstract Wiener spaces*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability, vol. II, part 1 (1965), 31-42.
4. ———, *Potential theory on Hilbert space*, J. Functional Analysis, **1** (1967), 123-181.
5. H.-H. Kuo, *Integration by parts for abstract Wiener measures*, Duke Math. J., **41** (1974), 373-379.
6. M. Ann Piech, *The Ornstein-Uhlenbeck semigroup in an infinite dimensional L^2 setting*, J. Functional Analysis, **18** (1975), 271-285.
7. J. H. M. Whitfield, *Differentiable functions with bounded nonempty support on Banach spaces*, Bull. Amer. Math. Soc., **72** (1966), 145-146.

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