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a^*-CLOSURES OF COMPLETELY DISTRIBUTIVE LATTICE-ORDERED GROUPS

A. M. W. GLASS, W. CHARLES HOLLAND AND STEPHEN H. MCCLEARY

The a^*-closure of a completely distributive lattice-ordered group is located within a wreath product of primitive components associated with certain transitive representation of the group. For many primitive lattice ordered groups, the a^*-closures are described explicitly.

1. **Introduction.** An a-closure of a totally ordered group is a maximal extension having the same convex subgroups. Every totally ordered group has an a-closure, and in the abelian case, it is unique. In [8], P. F. Conrad extended the notion to lattice-ordered groups by requiring the extension to have the same convex l-subgroups. Khuon [14] proved that every l-group has an a-closure. However, a-closures, even of archimedean l-groups, are not necessarily unique. Bleier and Conrad [2] generalized the totally ordered case in yet another way, called an a^*-extension, by requiring only that the extension have the same *closed* convex l-subgroups. They showed the existence of an a^*-closure in the abelian case, and uniqueness in the archimedean case. More recently, R. N. Ball [1] has shown that every lattice ordered group has an a^*-closure.

The purpose of this paper is to determine as much as possible about the a^*-extensions of completely distributive l-groups, using the techniques of representations as permutation groups. In § 2, we introduce the necessary background information and notation, while in § 3, we show the existence of a^*-closures by a cardinality argument. The existence of a^*-closures of completely distributive l-groups can also be shown by the methods of Khuon [14] and Byrd [4]. In § 4, we study the problem of stabilizer extensions of permutation groups; these are required to have the same stabilizer subgroups. In § 5, we show that for completely distributive l-groups, a^*-extensions are equivalent to stabilizer extensions. We use this equivalence to locate the a^*-closures for transitive permutation groups within a wreath product associated with the o-primitive components. Let $A(S)$ be the l-group of all order-preserving permutations of a totally ordered set S. For certain classes of transitive groups, including nice $A(S)$, we prove in § 6 that the a^*-closure in unique. We also give some limiting examples to show that in certain cases, the a^*-closure is not unique. The major advantage of this paper is that we are able to locate, in a very concrete fashion, all a^*-closures of completely distributive l-groups. However, we have been unable to make any headway in the non-completely distributive case. Research along this line would be very valuable as all that is available is Ball's existence proof.

In view of the great technicality of the statements and proofs in § 6, we have concluded that section with certain consequences of our research which have both an intuitive and a concrete flavor for all interested in a^*-closures and not necessarily in the generalized wreath product. We hope that this will prove valuable.

2. Notation and background information. The expression (G, S) will be used to indicate that G is an l-subgroup of $A(S)$, the lattice ordered group (l-group) of all order-preserving permutations of the totally ordered set S under the point-wise ordering. Therefore, (G, S) will only be written if G is indeed faithful on S. Any such G has a natural extension to $(G, \bar{S})$, where $\bar{S}$ is the Dedekind completion of S (without end points.) For any $X \subseteqq \bar{S}$, $G_X = \{g \in G : xg = x$ for all $x \in X\}$ is called a *stabilizer subgroup* of G. An *o-block* of (G, S) is a nonempty convex subset C of S such that for each $g \in G$, $Cg = C$ or $Cg \cap C = \varnothing$. If C is an o-block of (G, S), then $\{g \in G : Cg = C\} = G_{\sup C}$, where $\sup C$ is the supremum of C in $\bar{S}$ (provided C is not cofinal in S).

Throughout the remainder of this section we assume that (G, S) is transitive. We review some information from [12] and [17]. If C is an o-block of (G, S), the partition comprising the translates of C by elements of G gives rise to a *convex congruence of* (G, S), i.e., an equivalence relation on S which is respected by G and whose equivalence classes are convex subsets of S. Every convex congruence arises in this way. If $\mathscr{B}$ and $\mathscr{C}$ are convex congruences, we set $\mathscr{B} \leqq \mathscr{C}$ if and only if $\mathscr{B}$ refines $\mathscr{C}$. This gives a total order on the set of convex congruences. Moreover, the set of o-blocks containing any given $s \in S$ is totally ordered by inclusion; and if two o-blocks B and C containing s give rise to the congruences $\mathscr{B}$ and $\mathscr{C}$, respectively, then $\mathscr{B} \leqq \mathscr{C}$ if and only if $B \subseteqq C$.

If $\mathscr{B}$ and $\mathscr{C}$ are convex congruences of (G, S) such that $\mathscr{B} < \mathscr{C}$ and no convex congruence of (G, S) lies between $\mathscr{B}$ and $\mathscr{C}$, we say that $(\mathscr{B}, \mathscr{C})$ is a *convering pair of convex congruences of* (G, S). The set of these convering pairs, with the inherited total order, will be denoted by $\Gamma(G, S)$, and the γ^{th} convering pair by $(\mathscr{S}_\gamma, \mathscr{S}^\gamma)$.

A transitive group is said to be *o-primitive* provided its only convex congruences are the two improper convex congruences. Each covering pair $(\mathscr{S}_\gamma, \mathscr{S}^\gamma)$ yields an *o-primitive component* (G_γ, S_γ) in the following way: Choose any $s \in S$ and let $S_\gamma = s\mathscr{S}^\gamma/\mathscr{S}_\gamma$, the $\mathscr{S}^\gamma$ equivalence class of s modulo the $\mathscr{S}_\gamma$ classes contained in $s\mathscr{S}^\gamma$. Let G_γ denote the action of G_x on S_γ where $x = \sup s\mathscr{S}^\gamma$. Note that this is not, in general, a faithful representation of G_x on S_γ. The component (G_γ, S_γ) is o-primitive and independent (to within isomorphism) of the choice of s.

The set $\Gamma(G, S)$ and the o-primitive components of (G, S) will play a central role in locating a^*-extensions of completely distributive l-groups since they are the building blocks of every transitive l-permutation group.

If (G, S) has a minimal o-primitive component (i.e., associated with a minimal covering pair) (G_μ, S_μ), then (G, S) is said to be *locally o-primitive* and the $\mathscr{S}^\mu$ classes are called the *primitive segments*.

If (G, S) is o-primitive, then, by [17] and [20], there are just these four possibilities:

(i) (G, S) is *regular and archimedean*; $G_s = \{e\}$ for each $s \in S$, G is isomorphic to S as an ordered set, and is o-isomorphic to a subgroup of the real numbers [22] (e is the group identity).

(ii) (G, S) is *periodic*; there exists $e < f \in A(\bar{S})$ such that for all $g \in G$, $fg = gf$, and for each $s \in \bar{S}$, G_s fixes only the points of the coterminal subset $\{sf^m : m = 0, \pm 1, \pm 2, \cdots\}$, and G_s is o-2-transitive on the interval (s, sf). The permutation f is the *period* of G and $G \subseteq Z_{A(\bar{S})}(f) \cap A(\bar{S})$, where $Z_{A(\bar{S})}(f)$ denotes the centralizer of $\{f\}$ in $A(\bar{S})$. Either there exists a positive integer n such that for $s \in S$, $sf^m \in S$ if and only if n divides m—in which case (G, S) is said to have *Config*(n)—or $sf^m \in S$ if and only if $m = 0$, and (G, S) is said to have *Config* (∞).

(iii) (G, S) *is o-2-transitive* and contains a nonidentity element of bounded support.

(iv) (G, S) is *pathological*; (G, S) is o-2-transitive and contains no nonidentity element of bounded support.

In cases (i), (ii) and (iii), G is completely distributive and all stabilizers G_X, $X \subseteq \bar{S}$, are closed. In case (iv), G is not completely distributive and if $y \in \bar{S}$, G_y is not closed. Finally, in all cases, each G_y $(y \in \bar{S})$ is a maximal prime subgroup of G.

The *wreath product* of two l-permutation groups $(G, S) Wr(H, T)$ is the l-group of all order-preserving permutations of $S \overleftarrow{\times} T$ of the form $(\{g_t : t \in T\}, h)$ where $g_t \in G$, $h \in H$, and $(s, t)(\{g_t\}, h) = (sg_t, th)$. This can be generalized to the wreath product of infinitely many factors indexed by a totally ordered set Γ, written as $Wr\{(H_\gamma, T_\gamma) : \gamma \in \Gamma\}$ (see [13])).

For other general background metrial, see [6] and [12].

3. a^*-extensions. In this section we show, by a cardinality argument, that every completely distributive l-group has an a^*-closure.

We will adopt the same notation as used in [2]; the lattice of all convex l-subgroups of the l-group G will be denoted by $\mathscr{C}(G)$ and the lattice of all closed convex l-subgroups of G will be written $\mathscr{K}(G)$. Suppose G is an l-subgroup of an l-group H. Then H is an *a^*-extension* of G if and only if intersection with G provides an isomorphism from $\mathscr{K}(H)$ onto $\mathscr{K}(G)$. Bleier and Conrad have shown in [3] that H is an a^*-extension of G provided that intersection with G yields a one-to-one map of $\mathscr{K}(H)$ into $\mathscr{C}(G)$; moreover, the direct limit of a tower of a^*-extensions of an l-group G is still an a^*-extension of G. Consequently, to prove that an l-group G has an a^*-closure (an a^*-extension having itself no proper a^*-extensions), it is enough to show that there exists a bound on the cardinalities of a^*-extensions of G. This we do fairly easily in the completely distributive case. More recently, R. N. Ball has proved the existence of a cardinality bound on a^*-extensions for any l-group G, but his method is much deeper in the general case [1]. Further, observe that if G is an l-subgroup of H and H is an l-subgroup of the l-group K, then K is an a^*-extension of G if and only if K is an a^*-extension of H and H is an a^*-extension of G [3]. Therefore, any a^*-closure of an l-group G is a maximal a^*-extension of G and conversely.

PROPOSITION 3.1. *Let H be an a^*-extension of G. Then H is completely distributive if and only if G is completely distributive.*

Proof. By [5, Corollary 3.8], an l-group G is completely distributive if and only if its distributive radical $D(G) = \{e\}$, where $D(G)$ is the intersection of all closed prime subgroups of G. Since the property of being "closed prime" is distinguishable in $\mathscr{K}(G)$ [2, Proposition 1.4], the result follows.

Conrad [7] showed that every totally ordered group has an a^*-closure (which must again be totally ordered). It is known [10] that a^*-closures need not be unique in general, even for totally ordered groups. The following theorem generalizes Conrad's result, since every totally ordered group is completely distributive.

THEOREM 3.2. *Every completely distributive l-group has an a^*-closure.*

First, we establish a lemma.

LEMMA 3.3. *For any l-group G, $|G/D(G)| \leqq (2^{\aleph_0})^\kappa$, where κ is* *the cardinality of the set of closed regular subgroups of G.*

Proof. Let G_γ be a closed regular subgroup of G and let G^γ be its cover. Then G^γ acts on the totally ordered set of right cosets of G_γ in G^γ as an o-primitive group. Hence $|G^\gamma/G_\gamma| \leqq \max\{2^{\aleph_0}, \aleph_0\kappa\}$ (for, with reference to the characterization of o-primitive groups, if the action is regular, the set is isomorphic to a subgroup of the reals, and if not, there are at most $\aleph_0$ points having a given stabilizer, and each point stabilizer is a conjugate of G_γ and so a closed regular subgroup of G).

Next we construct a one-to-one map $\phi: G/D(G) \to \Pi$ where Π is the set cartesion product $\Pi\{G^\gamma/G_\gamma: G_\gamma$ closed regular subgroup of $G\}$. For each G_γ and each coset $G^\gamma x$ of G^γ in G, pick a fixed one-to-one map $\phi_{\gamma,x}$ from the set of cosets of G_γ lying in $G^\gamma x$ onto G^γ/G_γ (if $G^\gamma x = G^\gamma y$, then $\phi_{\gamma,x} = \phi_{\gamma,y}$). Letting $D = D(G)$, define $\phi: G/D \to \Pi$ by: $(Dg)\phi = (\cdots, (G_\gamma g)\phi_{\gamma,g}, \cdots)$. It can be easily verified that ϕ is well-defined. If $Dg_1 \neq Dg_2$, then $g_1g_2^{-1} \notin D$. So there is a closed prime G_δ such that $g_1g_2^{-1} \notin G_\delta$. Thus there is a regular subgroup of G for $g_1g_2^{-1}$ containing G_δ. By [5, Lemma 3.3], this regular subgroup is closed; call it G_γ. Now $g_1g_2^{-1} \in G^\gamma\backslash G_\gamma$. Hence $G^\gamma g_1 = G^\gamma g_2$ but $g_1 \notin G_\gamma g_2$. Consequently, $(G_\gamma g_1)\phi_{\gamma,g_1} = (G_\gamma g_1)\phi_{\gamma,g_2} \neq (G_\gamma g_2)\phi_{\gamma,g_2}$ and therefore $(Dg_1)\phi \neq (Dg_2)\phi$. It follows that ϕ is one-to-one.

Now $|G/D| \leqq |\Pi| \leqq (\max(2^{\aleph_0}, \aleph_0\kappa\}))^\kappa \leqq (2^{\aleph_0})^\kappa$, proving the lemma.

Proof of Theorem 3.2. It is enough to obtain a bound on the cardinalities of a^*-extensions of G. Let H be any a^*-extension of G. Since the closed regular subgroups of G are distinguishable in $\mathscr{K}(G)$, those of H correspond to those of G; hence H and G have the same κ. By Proposition 3.1, H is completely distributive, so that $D(H) = \{e\}$. By Lemma 3.3, $|H| = |H/D(H)| \leqq (2^{\aleph_0})^\kappa$.

Observe that a similar procedure, using the set of all (not necessarily closed) regular subgroups of G and letting κ be its cardinality, yields a bound of $(2^{\aleph_0})^\kappa$ on the cardinalities of a-extensions of an l-group G. That bound, which establishes that every l-group has an a-closure, was obtained at the same time by S. H. McCleary (using the above argument) and by D. Khuon using a different argument (giving rise to a bound one cardinal number greater than the above bound). In fact, using Hahn groups, one can show that $|G|$ attains the bound $(2^{\aleph_0})^\kappa$, so it is the best possible. We mention that Khuon's techniques can also be adapted to prove Lemma 3.3 (with the bound raised by one cardinal number).

It is abundantly clear that this section (as [1]) does not give us any concrete idea of what the a^*-closures of a given completely distributive l-group may look like; it is purely an existence proof.

The rest of this paper will be devoted to trying to locate the a-closures of completely distributive l-groups and to obtaining positive and negative results concerning uniqueness.

4. **Stabilizer extensions.** (G, S) is said to be a *permutation subgroup* of (H, T), written $(G, S) \subseteqq (H, T)$ if $S \subseteqq T$ and there is an "auxiliary" subgroup G' of H such that $SG' = S$ and the faithful restriction of G' to S gives G. We say that (H, T) is a *stabilizer extension* of (G, S), written $(G, S) \dagger (H, T)$, if $(G, S) \subseteqq (H, T)$, and

(1) whenever $X, Y \subseteqq \bar{T}$ and $G'_X = G'_Y$, then $H_X = H_Y$ and

(2) $SH = T$.

(1) is equivalent to: whenever $X, Y \subseteqq \bar{T}$ and $G'_X \subseteqq G'_Y$, then $H_X \subseteqq H_Y$. (1) is also equivalent to: whenever $X \subseteqq \bar{T}$, $y \in \bar{T}$ and $G'_X \subseteqq G'_y$, then $H_X \subseteqq H_y$.

Actually, we have the following proposition, showing that the definition is independent of the choice of auxiliary subgroup and justifying identifying G' with G.

PROPOSITION 4.1. *Suppose that $(G, S) \dagger (H, T)$ via some auxiliary subgroup G'. Then if h_1 and h_2 agree on S $(h_1, h_2 \in H)$, $h_1 = h_2$. In particular, each element of G can be uniquely extended to an element of H acting on T.*

Proof. If h_1 and h_2 agree on S, then $h_1h_2^{-1}$ is the identity on S. But $G'_S = \{e\} = G'_T$, so $H_S = H_T = \{e\}$. Hence $h_1h_2^{-1}$ is the identity on T; consequently, $h_1 = h_2$.

From now on, we will suppress all mention of the auxiliary subgroup. We say that (G, S) is *$\dagger$-closed* if $(G, S) \dagger (H, T)$ only when $(G, S) = (H, T)$. Without condition (2) above, no (G, S) would be $\dagger$-closed; for example, extra points could be added to S and left fixed by G. Now suppose $(G, S) \subseteqq (H, T)$ satisfies (1). Then H acts faithfully on SH (for if $h \in H$ is the identity on $SH \supseteqq S$, then $h = e$ by Proposition 4.1). Hence $(G, S) \dagger (H, SH)$. Thus (2) serves to eliminate any extraneous orbits of (H, T) which fail to meet S. In particular, if $(G, S) \dagger (H, T)$ and (G, S) is transitive, then so is (H, T). Moreover, we have the following lemma:

LEMMA 4.2. *If $(G, S) \dagger (H, T)$, C is an o-block of (H, T) on which H acts transitively, and $C \cap S \neq \varnothing$, then $C \cap S$ is coterminal in C.*

Proof. Suppose that $C \cap S$ is not coinitial in C. Let $m \in \bar{T}$ be the greatest lower bound of C if C is not coinitial in T, and otherwise let m denote the empty set. Let $n \in \bar{T}$ be the greatest lower bound of $G \cap S$. Clearly $G_m = G_n$. Hence $H_m = H_n$. Since $m \neq n$, the transitivity of H on C implies the existence of $h \in H_m \backslash H_n$, a contradiction. Therefore $C \cap S$ is coinitial in C. Similarly, $C \cap S$ is cofinal in C.

We now wish to find the connection between $\Gamma(G, S)$ and $\Gamma(H, T)$ when $(G, S) \dagger (H, T)$ and also the relation between the o-primitive components (recall § 2).

LEMMA 4.3. *If $(G, S) \dagger (H, T)$ and C is an o-block of (H, T), then $C \cap S$ is an o-block of (G, S); if $\mathscr{C}$ is a convex congruence of (H, T), then $\mathscr{C} \cap (S \times S)$ is a convex congruence of (G, S).*

THEOREM 4.4. *Let $(G, S) \dagger (H, T)$ with (G, S) (and thus also (H, T)) transitive. Then restriction to S provides an o-isomorphism from the tower of convex congruences of (H, T) onto the tower of convex congruences of (G, S); and thus from $\Gamma(G, S)$ onto $\Gamma(H, T)$.*

Proof. Lemma 4.3 establishes that restriction gives a function ψ from the tower of convex congruences of (G, S) into that of (H, T).

If $\mathscr{C} \subsetneqq \mathscr{K}$ are convex congruences of (H, T), choose any $s \in S$. Then $s\mathscr{C} \subsetneqq s\mathscr{K}$. By Lemma 4.2, the points of S are coterminal in $s\mathscr{K}$, and thus $(s\mathscr{C}) \cap S \subsetneqq (s\mathscr{K})) \cap S$, proving that $\mathscr{C}$ and $\mathscr{K}$ are still distinct when restricted to S. Hence ψ is an order-embedding.

Now suppose $\mathscr{C}$ is a nontrivial convex congruence of (G, S). Let C be a $\mathscr{C}$-class and $\hat{C}$ the convexification of C in T (written $\hat{C} = \operatorname{Conv}(C)$). Clearly, $\hat{C}$ is an o-block of (G, T). Indeed, $\hat{C}$ is an o-block of (H, T). For let $m = \sup \hat{C}$ and $n = \inf \hat{C}$, $m, n \in \bar{T}$. Then $G_m = G_n$, so $H_m = H_n$. Thus if $e < h \in H$, $\hat{C}h \neq \hat{C}$ and $\hat{C}h \cap \hat{C} \neq \varnothing$, it follows that $n < nh < m < mh$ and so $n < mh^{-1} < m$. Therefore, $G_{mh^{-1}} \subseteqq G_n$. Hence $hH_mh^{-1} = H_{mh^{-1}} \subseteqq H_n = hH_{nh}h^{-1}$, so $H_m \subseteqq H_{nh}$ and $G_m \subseteqq G_{nh}$. This is a contradiction. Let $\hat{\mathscr{C}}$ be the partition of T whose classes are the H-translates of $\hat{C}$. Since H is transitive, this is a partition of T. Then $\hat{\mathscr{C}}$ is a convex congruence of (H, T) whose intersection with $S \times S$ is $\mathscr{C}$, showing that ψ is onto. Therefore restriction to S maps $\Gamma(H, T)$ onto $\Gamma(G, S)$.

Since $\Gamma(G, S)$ and $\Gamma(H, T)$ are merely indexing sets for the set of convering pairs of convex congruences, we will identify $\Gamma(G, S)$ and $\Gamma(H, T)$ if the hypotheses of Theorem 4.4 prevail. More generally, if $(G\ S) \subseteqq (H, T)$ with (G, S) and (H, T) transitive, if $\Gamma(H, T)$ and $\Gamma(G, S)$ are identified via restriction to S, and if $s\mathscr{T}_\gamma = \operatorname{Conv}(s\mathscr{S}_\gamma)$ and $s\mathscr{T}^\gamma = \operatorname{Conv}(s\mathscr{S}^\gamma)$ for each $s \in S$ and $\gamma \in \Gamma(G, S) = \Gamma(H, T)$, we will write $(G, S) \leqq (H, T)$. Note that the relation $\leqq$ is transitive. Also, if (G, S) is transitive and $(G, S) \dagger (H, T)$, then $(G, S) \leqq (H\ T)$ (by Theorem 4.4).

COROLLARY 4.5. *Suppose* $(G, S) \dagger (H, T)$ *and that* (G, S) *is transitive. Then* (H, T) *is o-primitive if and only if* (G, S) *is o-primitive.*

Now assume that (G, S) is transitive and $(G, S) \dagger (H, T)$. Let $(\mathscr{T}_\gamma, \mathscr{T}^\gamma) \in \Gamma(H, T)$ correspond to $(\mathscr{S}_\gamma, \mathscr{S}^\gamma) \in \Gamma(G, S)$ and let $s \in S$. Then $s\mathscr{T}^\gamma = \mathrm{Conv}\,(s\mathscr{S}_\gamma)$ and for each $r \in s\mathscr{S}^\gamma = s\mathscr{T}^\gamma \cap S$, $r\mathscr{T}_\gamma = \mathrm{Conv}\,(r\mathscr{S}_\gamma)$. We may, therefore, consider $S_\gamma = s\mathscr{S}^\gamma/\mathscr{S}_\gamma$ as a totally ordered subset of the totally ordered set $T_\gamma = s\mathscr{T}^\gamma/\mathscr{T}_\gamma$ (see § 2).

THEOREM 4.6. *Let* $(G, S) \dagger (H, T)$ *with* (G, S) *transitive. Then for each pair of corresponding o-primitive components,* $(G_\gamma, S_\gamma) \dagger (H_\gamma, T_\gamma)$.

Proof. Since $(G, S) \dagger (H, T)$, each $g_\gamma \in G_\gamma$ has an extension to T_γ which is induced by some $g \in G$ (i.e., the image of g agrees on T_γ with the extension). We claim that the only such extension of the identity of G_γ is the identity map on T_γ. where $T_\gamma = s\mathscr{T}^\gamma/\mathscr{T}_\gamma$ for some fixed $s \in S$. For let $u \in T_\gamma \backslash S_\gamma$. Now $u \subseteqq \bar{T}$, so write $\sup u$ for the supremum of u in $\bar{T}$. Let $v = \inf\{w \in S_\gamma : w > u\} \in \bar{T}_\gamma$, and $\tilde{v} = \inf\{r\mathscr{T}_\gamma : r \in S,\ r\mathscr{T}_\gamma \subseteqq s\mathscr{T}_\gamma,$ and $r\mathscr{T}_\gamma > v\} \in \bar{T}$, these being well defined by Lemma 4.2. Then $G_{\sup u} \subseteqq G_{\tilde{v}}$ so $H_{\sup u} \subseteqq H_{\tilde{v}}$. Therefore $(H_\gamma)_u \subseteqq (H_\gamma)_v$. Thus $(H_\gamma)_u = (H_\gamma)_v$, since (H_γ, T_γ) is o-primitive. Hence $H_{\sup u} = H_{\tilde{v}}$ and, consequently, $G_{\sup u} = G_{\tilde{v}}$. If g_γ is the identity on S_γ, then $g \in G_{\tilde{v}}$ where $g \in G$ induces g_γ. It follows that $g \in G_{\sup u}$ and so $g_\gamma \in (G_\gamma)_u$. Hence the extension of g_γ to T_γ is indeed the identity. It follows that each $g_\gamma \in G_\gamma$ has a unique extension to T_γ which is induced by some $g \in G$, so that $(G_\gamma, S_\gamma) \subseteqq (H_\gamma, T_\gamma)$.

Now if $(G_\gamma)_X \subseteqq (G_\gamma)_y$ for some $X \subseteqq \bar{T}_\gamma$ and $y \in \bar{T}_\gamma$, then $G_{\tilde{X}} \subseteqq G_{\tilde{y}}$ (where $\tilde{X} = \{\tilde{x} : x \in X\}$), so $H_{\tilde{X}} \subseteqq H_{\tilde{y}}$, and $(H_\gamma)_X \subseteqq (H_\gamma)_y$. Therefore $(G_\gamma, S_\gamma) \dagger (H_\gamma, T_\gamma)$.

We must now, therefore, consider stabilizer extensions of o-primitive l-permutation groups.

THEOREM 4.7. *Let* (G, S) *be o-primitive. If G is nonpathological, then* (G, S) *has unique* †*-closure (to within isomorphism over* (G, S)*), viz.:*

(i) $(\boldsymbol{R}, \boldsymbol{R})$ *is* (G, S) *is regular and archimedean.*

(ii) *(Z, SZ) if (G, S) is periodic with period f and $Z = Z_{A(\bar{S})}(f)$.*

(iii) *$(A(\bar{S}), SA(\bar{S}))$ if (G, S) is* (nonpathologically) *o-2-transitive. If (G, S) is pathological, then every $\dagger$-extension of (G, S) is a pathological permutation l-subgroup of $(A(\bar{S}), SA(\bar{S}))$ and (G, S) has a $\dagger$-closure.*

Proof. Let (H, T) be a $\dagger$-extension of (G, S). By Corollary 4.5, (H, T) is o-primitive.

If (G, S) is regular and archimedean, and thus is the regular representation of some subgroup of $\boldsymbol{R}$, then $G_s = \{e\}$ for all $s \in S$. Hence $H_s = \{e\}$ for all $s \in S$; so H cannot be o-2-transitive or periodic. It follows that H must, therefore, be regular and archimedean. Thus H is o-isomorphic over G to a subgroup of $\boldsymbol{R}$. But it is obvious that $(G, S) \dagger (\boldsymbol{R}, \boldsymbol{R})$, and (i) follows.

If (G, S) is not regular, we first show that S must be dense in T, so that, to within isomorphism over (G, S), $T \subseteqq \bar{S}$. Let D be an interval of T maximal with respect to containig no points of S, and let $m = \inf D$. Then for every $t \in D$, $G_t \subseteqq G_m$, so $H_t \subseteqq H_m$, and thus $H_t = H_m$ since (H, T) is o-primitive. (H, T) is not regular because (G, S) is not. Thus D cannot contain more than one point since in the periodic and o-2-transitive cases, each H_t moves points arbitrarily close to t. Similarly, no point in $T \backslash S$ can cover or be covered by a point in S. Hence S is dense in T.

If (G, S) is periodic with period $f \in A(\bar{S})$, then for each $s \in S$, $G_s = G_r$ if and only if $r \in \{sf^n : n \text{ an integer}\}$. Hence, for each $s \in S$, $H_s = H_r$ if and only if $r \in \{sf^n\}$. Thus (H, T) must be periodic with period k, say. But then $H_s = H_r$ if and only if $r \in \{sk^n\}$, which shows that for each $s \in S$, $\{sf^n\} = \{sk^n\}$. Therefore $k = f$, and so $(H, T) \subseteqq (Z, \bar{S})$.

We now show that $(G, S) \dagger (Z, SZ)$. Let $X \subseteqq \bar{S}$, $y \in \bar{S}$ and $G_X \subseteqq G_y$. Then y must lie in the topological closure of the set $X' = \bigcup \{Xf^n : n = 0, \pm 1, \pm 2, \cdots\}$. Otherwise, there exist $s_1, s_2 \in S$ with $s_1 < y < s_2 < s_1 f$ and no point of X' lying between s_1 and s_2. By [20, Lemma 5], there exists $g \in G$ such that $yg \neq y$ and

$$(\text{support } g) \cap (s_1, s_1 f) \subseteqq (s_1, s_2) .$$

It follows that $(\text{support } g) \subseteqq \bigcup \{(s_1, s_2)f^n : n = 0, \pm 1, \pm 2, \cdots\}$, and as $(\bigcup \{s_1, s_2)f^n : n = 0, \pm 1, \pm 2, \cdots\}) \cap X = \varnothing$, $g \in G_X \backslash G_y$, a contradiction. Since Z also has period f, $Z_X \subseteqq Z_y$. Hence (Z, SZ) is the unique $\dagger$-closure of (G, S).

We remark in passing that a $\dagger$-closed periodic group must have Config (1)—since $(G, S) \dagger (Z, SZ)$ then $(G, S) = (Z, SZ)$; and since the period f belongs to Z, $t \in SZ$ implies $tf \in SZ$.

If (G, S) is o-2-transitive and contains an element of bounded support, and $G_X \subseteqq G_y$ for some $X \subseteqq \bar{S}$, $y \in \bar{S}$, then y is in the topological closure of X, as one sees by considerations similar to those in the previous case. Hence even $A(\bar{S})_X \subseteqq A(\bar{S})_y$, so $(G, S) \dagger (A(\bar{S}), SA(\bar{S}))$, and $(A(\bar{S}), SA(\bar{S}))$ is the unique $\dagger$-closure of (G, S).

Finally, if (G, S) is pathological, then (H, T) can be neither regular nor periodic, and so must be o-2-transitive. Choose any $s_1, s_2 \in S$ and let $X = \{x \in T: s_1 < x \text{ or } x < s_2)$. Then $G_X \subseteqq G_{X \cap S} = \{e\}$, so $H_X = \{e\}$ and H must be pathological. A Zorn's lemma argument shows that there must exists a maximal $\dagger$-extension of (G, S) within $(A(\bar{S}), \bar{S})$, which must be a $\dagger$-closure of (G, S).

Beyond this we have been able to decide little about $\dagger$-extensions of a pathological group. We do not know whether $\dagger$-closures of pathological groups must be unique.

EXAMPLES 4.8(a). Let $\boldsymbol{R}$ be the real line and

$$P = \{g \in A(\boldsymbol{R}): (\exists \text{ positive integer } n)(\forall x \in \boldsymbol{R})((x + n)g = xg + n)\} .$$

4.8(b). (McCleary [19]).

$$M = \{g \in A(\boldsymbol{R}): (\forall x \in \boldsymbol{R})(\exists \text{ positive integer } n)(\forall \text{ integer } m)$$
$$((x + mn)g = xg + mn)\} .$$

4.8(c).

$$B = \{g \in A(\boldsymbol{R}): (\forall \varepsilon > 0)(\forall x \in \boldsymbol{R})(\exists \text{ positive integer } n)$$
$$(\forall \text{ integer } m)(|(x + mn)g - xg - mn| < \varepsilon \;\&$$
$$|(x + mn)g^{-1} - xg^{-1} - mn| < \varepsilon)\} .$$

P, M and B are pathological o-2-transitive and $(P, \boldsymbol{R}) \dagger (M, \boldsymbol{R}) \dagger (B, \boldsymbol{R})$. $P_X \subseteqq P_y$ or $M_X \subseteqq M_y$ or $B_X \subseteqq B_y$ holds precisely when the following is true: there exists a subgroup S of the additive group of integers such that for every subgroup S' of S, the distance of X from $y + S'$ is 0. Actually, $(B, \boldsymbol{R})$ is the unique $\dagger$-closure of both $(P, \boldsymbol{R})$ and $(M, \boldsymbol{R})$.

The reader may wish to consider finding the $\dagger$-closures of the more complicated pathological o-2-transitive examples given in [9]. Since the pathological groups are the o-primitive groups which are not completely distributive, they are at the heart of understanding what is true in the general case.

Let (G, S) be a transitive l-permutation group. We embed (G, S) in the wreath product $(W_1, R_1) = Wr\{(G_\gamma, S_\gamma): \gamma \in \Gamma\}$ of its o-primitive components using, say, an immediate embedding of the sort in [13]. For each $\gamma \in \Gamma$, let $(H_\gamma, T_\gamma) \supseteqq (G_\gamma, S_\gamma)$ with (H_γ, T_γ) o-primitive. Let $(W, R) = Wr\{(H_\gamma, T_\gamma): \gamma \in \Gamma\}$. Then $R_1 \subseteqq R$, and we embed W_1 in W by taking each $(w_1)_{\gamma,x}$ to be the identity if $x\mathscr{R}^\gamma \cap R_1 = \varnothing$. In this way we may assume $(G, S) \leqq (W, R)$.

THEOREM 4.9. *Let (G, S) be transitive. For each o-primitive component (G_γ, S_γ), let (H_γ, T_γ) be the l-permutation group given by Theorem* 4.7 (where $(H_\gamma, T_\gamma) = (A(\bar{S}_\gamma), S_\gamma A(\bar{S}_\gamma))$ *if (G_γ, S_γ) is pathological). Let $(W, R) = Wr\{(H_\gamma, T_\gamma): \gamma \in \Gamma\}$. Then every †-extension of (G, S) is isomorphic over (G, S) to some (K, U) such that $(G, S) \leqq (K, U) \leqq (W, R)$.*

Proof. We consider (G, S) to be contained in (W, R) as indicated above. Let (L, V) be a †-extension of (G, S). Then (L, V) is transitive and $\Gamma(L, V) = \Gamma(G, S)$ by Theorem 4.4. Then, by the techniques of [16, Lemma 16], (L, V) can be immediately embedded in $(W_2, R_2) = Wr\{(L_\gamma, V_\gamma): \gamma \in \Gamma\}$ so that the embedding of V in R_2 is the identity on S (this follows from the fact that if ϕ is a map having domain $s\mathscr{S}^\gamma/\mathscr{S}_\gamma (s \in S)$, which is induced by some $g \in G$, then some $k \in L$ induces an extension of ϕ to $(s\mathscr{V}^\gamma/\mathscr{V}_\gamma)$. Now, in view of Theorem 4,7, we may consider $(W_2, R_2) \leqq (W, R)$, and the theorem follows.

Observe that although we take the embedding of G to be the identity on S (so that $g\psi = g$ on S), it cannot, in general, be arranged that $g\psi \in K$ and $g \in G \subseteqq W$ agree on U in the statement of Theorem 4.9.

COROLLARY 4.10. *Any wreath product of †-closed o-primitive groups is †-closed.*

Proof. With †-closed o-primitive components, we may take (H_γ, T_γ) to be (G_γ, S_γ) in the pathological case. The corollary now follows.

Even if (G, S) is †-closed, it need not be the case that each of its o-primitive components is †-closed, as can be seen in the following example.

EXAMPLE 4.11. Let $\boldsymbol{R}$ be the totally ordered set of real numbers, and $G = \{(\{g_r\}, \bar{g}) \in (A(\boldsymbol{R}), \boldsymbol{R})\, Wr(A(\boldsymbol{R}), \boldsymbol{R})$: support $(\bar{g})$ is bounded, and for all large r, $g_r = \bar{g}\}$. Then G is an l-subgroup of $A(\boldsymbol{R})\, WrA(\boldsymbol{R})$ with o-primitive components G_1 and G_2, where $G_1 = A(\boldsymbol{R})$ and $G_2 =$ those members of $A(\boldsymbol{R})$ with bounded support. Let $S = \boldsymbol{R} \overleftarrow{\times} \boldsymbol{R}$ be the set which G permutes. Let $X \subseteqq \bar{S}$ and $y \in \bar{S}$. It is easily seen that $G_X \subseteqq G_y$ if and only if

(1) $y \in$ topological closure of X, or

(2) $y = z_m$ or z_M, where $z_m = \inf (\boldsymbol{R} \overleftarrow{\times} \{z\})$ and $z_M = \sup \boldsymbol{R} \overleftarrow{\times} \{z\})$ and the topological closure of $\boldsymbol{R} \overleftarrow{\times} \{z\}$ meets the topological clousure of X, or

(3) $y = z_m$ or z_M and $\forall \varepsilon > 0$, $\forall r \in \boldsymbol{R}$, $\exists (x, \bar{x}) \in X \cap S$ such that $r < \bar{x}$ and $|z - x| < \varepsilon$.

Observe that if $G \dagger H$ and $H \subseteqq A(\boldsymbol{R}) Wr A(\boldsymbol{R})$ and if $h = (\{h_r\}, \bar{h}) \in H$, then the support of $\bar{h}$ is bounded above. For we may assume $e < h$ and choose $\bar{g} \in A(\boldsymbol{R})$, of bounded support, such that $O\bar{h} < O\bar{g}$; for each $r \in \boldsymbol{R}$, let $g_r = \bar{g}$. Then $g = (\{g_r\}, \bar{g}) \in G$ and $gh^{-1} \vee e \in H$. If $m = \inf (\boldsymbol{R} \overleftarrow{\times} \{0\}) \in \bar{S}$, then $m(gh^{-1} \vee e) = m(gh^{-1}) > m$. If $\bar{h}$ where to have support unbounded above, there would be a cofinal subset $F \subseteqq \boldsymbol{R}$ of the support of $\bar{h}$, but missing the support of $\bar{g}$. Hence for $X = \boldsymbol{R} \overleftarrow{\times} F$, $gh^{-1} \vee e \in H_X$. But it is clear that $G_X \subseteqq G_m$, contradicting the fact that $G \dagger H$. Hence the "upper" component of H is not $A(\boldsymbol{R})$, and therefore, by Theorem 4.7, is not $\dagger$-closed.

Now let $H \subseteqq A(R) Wr A(\boldsymbol{R})$ consist of those $(\{h_r\}, \bar{h})$ such that support $(\bar{h})$ is bounded above and $\forall x \in \boldsymbol{R}$, $\forall \varepsilon > 0$, $\exists \delta > 0$, $\exists r \in \boldsymbol{R}$ such that if $|y - x| < \delta$ and $s > r$, then $|yh_s - x\bar{h}| < \varepsilon$ and $|yh_s^{-1} - x\bar{h}^{-1}| < \varepsilon$. Then H is an l-subgroup of $A(\boldsymbol{R}) Wr A(\boldsymbol{R})$, as can be determined by a little computation, and $(G, S) \dagger (H, S)$ because $H_X \subseteqq H_y$ if and only if (1), (2), or (3), as before. Using the conditions (1), (2), and (3), it can be shown that (H, S) is the unique (to within isomorphism over (G, S)) $\dagger$-closure of (G, S).

EXAMPLE 4.12. Even if each o-primitive component of G is $\dagger$-closed, G need not be. Let $G \subseteqq A(\boldsymbol{R}) Wr A(\boldsymbol{R})$ be the "small" wreath product consisting of those $(\{g_r\}, \bar{g})$ such that $g_r = e$ for all but finitely many $r \in \boldsymbol{R}$. Then each o-primitive component of G is $A(\boldsymbol{R})$, which is $\dagger$-closed by Theorem 4.7, but $G \dagger A(\boldsymbol{R}) Wr A(\boldsymbol{R})$.

5. The relation between a^*-extensions and $\dagger$-extensions.

We now make explicit the relationship between a^*-extensions and $\dagger$-extensions. We shall exploit the close connection between closed convex l-subgroups and stabilizer subgroups given in [18]. If (G, S) is an l-permutation group, then every closed convex l-subgroup is a stabilizer G_X, for some $X \subseteqq \bar{S}$. If every point stabilizer G_s $(s \in S)$ is closed, then all stabilizers G_X $(X \subseteqq \bar{S})$ are closed. Hence there is no ambiguity regarding the kind of stabilizer in the statement "(G, S) has closed stabilizers." Indeed, if (G, S) has closed stabilizers the closed convex l-subgroups of G are precisely the stabilizers G_X $(X \subseteqq \bar{S})$. If (G, S) is transitive, the point stabilizers are all conjugate; so if one is closed, they all are. G has a representation (G, S) with closed point stabilizers if and only if G is completely distributive. Moreover, if G is completely distributive and has a transitive representation, it has a transitive representation with closed stabilizers.

The main result that we wish to prove in this section is the following:

THEOREM 5.1. *If* $(G, S) \dagger (H, T)$, *then* H *is an* a^*-*extension of* G. *Conversely, if* $(G, S) \subseteqq (H, T)$, (G, S) *has closed stabilizers, and* H *is an* a^*-*extension of* G, *then* $(G, S) \dagger (H, SH)$.

Note that in the statement of Theorem 5.1, H should actually be an a^*-extension of an auxiliary subgroup G'; but since G' is uniquely determined, we have identified it with G.

Once we have established this theorem, we will be able to make use of the results in the previous section.

To prove Theorem 5.1, we need a lemma.

PROPOSITION 5.2. *Let* $(G, S) \dagger (H, T)$. *If* (G, S) *has closed stabilizers, then so does* (H, T).

Proof. We first consider $s \in S$. Let $\bar{\tilde{G}}_s$ be the closure in H of the convexification in H of G_s. Since $(G, S) \dagger (H, T)$,

$$Y = \{y \in \bar{T}: H_s \subseteqq H_y\} = \{y \in \bar{T}: G_s \subseteqq G_y\} = \{y \in \bar{T}: \tilde{G}_s \subseteqq \tilde{G}_y\} .$$

If K is a convex l-subgroup of an l-permutation group (H, T), then $\bar{K} \supseteqq \{h \in H: yh = y$ whenever $y \in \bar{T}$ and y is fixed by $K\}$ [17, Theorem 9]. Hence $\bar{\tilde{G}}_s \supseteqq \{h \in H: yh = y$ for all $y \in Y\} \supseteqq H_s$ since H_s fixes each $y \in Y$. Thus $\tilde{G}_s \subseteqq H_s \subseteqq \bar{\tilde{G}}_s$, so $\bar{\tilde{G}}_s = \bar{H}_s$. Since G_s is closed, $G_s = \bar{\tilde{G}}_s \cap G = \bar{H}_s \cap G$ by [2, Lemma 2.1].

By [17, Theorem 8], $\bar{H}_s = H_{\sup B}$ for some o-block B of H such that $sH_{\sup B}$ is coterminal in B. If B is not a single point, then $H_{\sup B}$ moves s; since $(G, S) \dagger (H, T)$, $G_{\sup B}$ moves s; i.e., $\bar{H}_s \cap G = G_s$ moves s. This is clearly impossible. Hence $B = \{s\}$ and so $\bar{H}_s = H_s$. Consequently, H_s is closed.

Now let $t \in T$. Since $(G, S) \dagger (H, T)$, $sh = t$ for some $s \in S$, $h \in H$. Thus H_t is conjugate to H_s and so is closed in H. This completes the proof that (H, T) has closed stabilizers.

We now prove Theorem 5.1:

Suppose that $(G, S) \dagger (H, T)$. Let C and K be closed convex l-subgroups of H such that $C \cap G = K \cap G$. There exist $X, Y \subseteqq \bar{T}$ such that $C = H_X$ and $K = H_Y$. Hence $G_X = H_X \cap G = C \cap G =$

$K \cap G = H_Y \cap G = G_Y$. Therefore $C = H_X = H_Y = K$, and H is an a^*-extension of G.

Conversely, suppose that H is an a^*-extension of G, $(G, S) \subseteqq (H, T)$ and (G, S) has closed stabilizers. Let G' be any auxiliary subgroup. Assume $G'_X = G'_Y$ for some $X, Y \subseteqq \bar{T}$. By Proposition 5.2., H_X and H_Y are closed convex l-subgroups of H. Thus $H_X \cap G' = G'_X = G'_Y = H_Y \cap G'$, so that $H_X = H_Y$. Hence $(G, S) \subseteqq (H, T)$ satisfies condition (1) in the definition of †-extension, and, consequently, $(G, S) \dagger (H, SH)$. Moreover, G' is unique by Proposition 4.1.

COROLLARY 5.3. *Let* $(G, S) \dagger (H, T)$. *If* (G, S) *has closed stabilizers, then so does* (G, T).

Proof. By Theorem 5.1, H is an a^*-extension of G, and by Proposition 5.2, (H, T) has closed stabilizers. Hence $G_t = H_t \cap G$ is closed in G.

COROLLARY 5.4. *Let* $(G, S) \dagger (H, T)$ *and* $(H, T) \dagger (K, U)$. *If* (G, S) *has closed stabilizers, then* $(G, S) \dagger (K, U)$.

Proof. K is an a^*-extension of H and H is an a^*-extension of G (by Theorem 5.1), so K is an a^*-extension of G. Moreover, $(G, S) \subseteqq (H, T)$ and $(H, T) \subseteqq (K, U)$. Let G' and H' be the auxiliary subgroups. We obtain an auxiliary subgroups G'' of G in K by taking each $g \in G$ and (uniquely) extending it to T to obtain an element of $G' \subseteqq H$ and (uniquely) extending the resulting permutation of T to U obtaining an element of H'. The image of $G'' \subseteqq K$ is an auxiliary subgroup and so $(G, S) \subseteqq (K, U)$. By Theorem 5.1, $(G, S) \dagger (K, U)$.

A direct proof using only Proposition 5.2 and Corollary 5.3 is also possible. We do not know (even for transitive groups) whether the closed stabilizer hypothesis is essential.

For transitive groups, we can improve on Theorem 5.1.

THEOREM 5.5. *Let* (G, S) *be a transitive* l*-permutation group with closed stabilizers, and let* H *be an* a^**-extension of* G. *Then* H *has a faithful transitive representation* ϕ *such that* $(H\phi, T)$ *has closed stabilizers,* $(G, S) \dagger (H\phi, T)$, *and for each* $g \in G$, $g\phi$ *is the unique extension of* g *in* $H\phi$. *Roughly speaking,* $(G, S) \dagger (H, T)$ *with each* $g \in G$ *identified with its unique extension in* H.

Proof. Let $s_0 \in S$. Then G_{s_0} is a closed prime subgroup of G. Let $H(s_0)$ be the unique closed convex l-subgroup of H such that $H(s_0) \cap G = G_{s_0}$. By [2, Proposition 1.4], $H(s_0)$ is prime. Moreover,

$$\Big(\bigcap_{h\in H} h^{-1}H(s_0)h\Big) \cap G \subseteqq \Big(\bigcap_{g\in G} g^{-1}H(s_0)g\Big) \cap G = \bigcap_{g\in G} g^{-1}(H(s_0)\cap G)g$$
$$= \bigcap_{g\in G} g^{-1}G_{s_0}g = \bigcap_{g\in G} G_{s_0 g} = G_S = \{e\}\ .$$

But $\bigcap_{h\in H} h^{-1}H(s_0)h$ is a closed convex l-subgroup of H and H is an a^*-extension of G. Hence $\bigcap_{h\in H} h^{-1}H(s_0)h = \{e\}$. By [11], we may faithfully represent H on the totally ordered set T of right cosets of $H(s_0)$. The totally ordered set S may be embedded in T in the following way: Let $s \in S$. Then there exists $g \in G$ such that $s_0 g = s$. $s \mapsto H(s_0)g$ is a well-defined one-to-one map since $H(s_0) \cap G = G_{s_0}$. It is straightforward to check that the map preserves both the order of S and the action of G, so henceforth we identify S with its image; thus $S \subseteqq T$. Then $(H\phi)_{s_0} = H(s_0)$ is closed, so that $(H\phi, T)$ has closed stabilizers. Now $(G, S) \subseteqq (H\phi, T)$ so the theorem follows from Theorem 5.1.

The technique used in the above proof is essentially that of [12] and [23].

COROLLARY 5.6. *Let* $(G, S) \subseteqq (H, T)$, *where* (G, S) *and* (H, T) *are transitive with closed stabilizers. Then* (H, T) *is a* $\dagger$*-extension* ($\dagger$-closure) *of* (G, S) *if and only if* H *is an* a^**-extension* (a^*-closure) *of* G.

THEOREM 5.7. *Let* (G, S) *be o-primitive and not pathological. Then* G *has a unique* (to which l-isomorphism over G) a^**-closure, which is*

(i) *The real numbers if* (G, S) *is regular and archimedean,*

(ii) $Z_{A(\bar{S})}(f)$ *if* (G, S) *has period* f,

or

(iii) $A(\bar{S})$ *if* (G, S) *is o-2-transitive.*

Proof. This is immediate from Theorems 4.7 and 5.5 and Corollary 5.6.

Observe that even if (G, S) has closed stabilizers, it may have an o-primitive component which is pathological. This can be seen from the following example:

EXAMPLE 5.8. Let $(G_1, S_1) = (Z, Z)$ be the l-group of integers, permuting itself regularly. Let $(G_2, S_2) = (P, \boldsymbol{R})$ be the pathological group of Example 4.8, and $(G, S) = (G_1, S_1)Wr(G_2, S_2)$. Then (G, S) has closed stabilizers, but has its "upper" o-primitive component l-isomorphic to $(P, \boldsymbol{R})$.

We do not know if it is possible for (G, S) to be a locally pathological transitive l-permutation group with closed stabilizers. Any reasonable added condition prevents it from happening.

THEOREM 5.9. *Let (G, S) be a transitive l-permutation group with closed stabilizers. For each o-primitive component (G_γ, S_γ), let (H_γ, T_γ) be the l-permutation group given by Theorem* 5.7 (with $(H_\gamma, T_\gamma) = (A(\bar{S}_\gamma), S_\gamma A(\bar{S}_\gamma))$ if (G_γ, S_γ) is pathological). *Let $(W, R) = Wr\{(H_\gamma, T_\gamma)\colon \gamma \in \Gamma\}$ and H be an a^*-extension of G. Then H has a faithful representation ϕ on a subchain T of R such that*

(a) $(G, S) \dagger (H\phi, T) \leqq (W, R)$,

(b) *For each $g \in G$, $g\phi$ and g agree on S,*

and

(c) *$(H\phi, T)$ is transitive and has closed stabilizers.*

The transitive groups (G, S) and (H, S) of Example 4.11 have closed stabilizers. Hence H is the unique (to within l-isomorphism over G) a^*-closure of G. Moreover, even though H is a^*-closed its upper component is not. On the other hand, the l-group G of Example 4.12 is not a^*-closed even though its components are.

6. a^*-closures. Our aim in this last section will be to find the unique a^*-closure of certain classes of l-permutation groups. We shall show that our results are sharp by constructing (G, S) with more than one a^*-closure. We will make repeated use of the wreath product and the results contained in earlier portions of this paper, especially Theorem 4.9. Thus we consider $(G, S) \leqq (W, R)$, and letting (K, U) be a $\dagger$-extension of (G, S), we have $(G, S) \leqq (K, U) \leqq (W, R)$ to within isomorphism over (G, S) (see Theorem 4.9 for notation).

We need to determine how new points are added when S is enlarged to U. Let $\Gamma = \Gamma(G, S) = \Gamma(K, U) = \Gamma(W, R)$ and $\bar{s} \in \bar{S}$. If $\bar{s}$ determines a proper Dedekind cut in some $s\mathscr{S}^\gamma/\mathscr{S}_\gamma$, we say that $\bar{s}$ is a *hole* in $s\mathscr{S}^\gamma/\mathscr{S}_\gamma$. Also, let S_A (respectively, S_B) be the set of all $\bar{s} \in \bar{S}\backslash S$ such that the tower $\mathscr{T}$ of $\mathscr{S}_\gamma$-classes whose completions contain $\bar{s}$ has empty intersection and such that $\mathscr{T}$ contains (respectively, does not contain) $\mathscr{S}_\gamma$-classes for arbitrary small $\gamma \in \Gamma$.

Observe that if Γ has a minimal element, then $S_A = \varnothing$. Also note that if (G, S) is the wreath product of o-primitive groups, then $R_A = S_B = \varnothing$, [13, page 713].

The main theorems we wish to establish here are Theorems 6.1 and 6.2.

THEOREM 6.1. *Let $(G, S) = Wr\{(G_\gamma, S_\gamma)\colon \gamma \in \Gamma\}$ be a wreath product of non-pathological o-primitive groups. Suppose, further, that for*

γ nonminimal

(i) *If (G_γ, S_γ) is regular, then the divisible closure of G_γ is $\boldsymbol{R}$, and*

(ii) *If (G_γ, S_γ) is periodic with period f_γ, then it does not have Config (∞).*

Then (G, S) has a unique (to within isomorphism over (G, S)) *†-closure (H, T) and H is the unique* (to within l-isomorphism over G) *a^*-closure of G. Moreover, $(H, T) = Wr\{(L_\gamma, V_\gamma): r \in \Gamma\}$, where if γ is minimal, (L_γ, V_γ) is the †-closure of (G_γ, S_γ)* (cf. Theorem 4.7), *and if γ is not minimal, $V_\gamma = S_\gamma$ and L_γ is G_γ if (G_γ, S_γ) is regular, $Z_{A(S_\gamma)}(f_\gamma)$ if (G_γ, S_γ) has period f_γ, and $A(S_\gamma)$ if (G_γ, S_γ) is o-2-transitive.*

No o-primitive component of $A(S)$ can be pathological or periodic, which simplifies the hypotheses of the next theorem. We could, without much difficulty, permit nonminimal regular components $(G_\gamma, S_\gamma) = (A(S_\gamma), S_\gamma)$ for which the divisible closure of G_γ is $\boldsymbol{R}$, but it seems doubtful that such components can actually occur (S_γ would have to be one of the "uniquely transitive" totally ordered sets (not the integers) described by Ohkuma in [22]).

THEOREM 6.2. *Let $A(S)$ be transitive and have no nonminimal regular o-primitive component. Then $(G, S) = (A(S), S)$ has a unique* (to within isomorphism over $(A(S), S)$) *†-closure (H, T) and H is the unique* (to within l-isomorphism over $(A(S))$ *a^*-closure of $A(S)$. If $(A(S), S)$ has a smallest component $(A(S_\mu), S_\mu)$, then T is S with each $\mathscr{S}^\mu$-class enlarged to $\boldsymbol{R}$ if $(A(S_\mu), S_\mu)$ is regular, and to $S_\mu A(\bar{S}_\mu)$ otherwise; if $(A(S), S)$ has no smallest component, $T = S \cap S_A$. H consists of those $h \in A(T)$ such that h preserves the convex congruences of (G, S). In addition, if $A(S)$ has a minimal component $(A(S_\mu), S_\mu)$ which is regular, then each $h_{\mu,x}$ is a permutation induced by G followed by a translation of $\boldsymbol{R}$.*

COROLLARY 6.3. *Let $A(S)$ be transitive and have no regular components. Suppose, also, that $A(S)$ is locally o-primitive and that the minimal component is a^*-closed. Then $A(S)$ is a^*-closed.*

Since these results are extremely technical, we will close the paper with some direct consequences of them.

Our goal now is to prove Theorems 6.1 and 6.2. Before embarking on this, some comments on the theorems are in order.

The descriptions given of a^*-closures in the two theorems are very similar. However, it is convenient to use the language of wreath products in the first and to partially suppress it in the second. If (G, S) is locally o-primitive, the enlargement of S to T simply adds points at certain cuts in the primitive segments (unless these segments are o-isomorphic to the integers, in which case they are enlarged to the reals, still with regular action). If (G, S) is not locally o-primitive, Theorem 6.2 adds single points at cuts in S_A; and in Theorem 6.1, $T = S$. Thus in each case S is dense in T (unless (G, S) has primitive segments o-isomorphic to the integers), so that each $g \in G$ has a unique extension lying in H. Hence $(G, S) \subseteqq (H, T)$ with no ambiguity about the auxiliary subgroup G'. Also, there is no ambiguity in the notation (G_γ, S_γ) in Theorem 6.1 since, in a wreath product $Wr\{(G_\gamma, S_\gamma): \gamma \in \Gamma\}$ of o-primitive groups, the o-primitive components are precisely $\{(G_\gamma, S_\gamma): \gamma \in \Gamma\}$.

In order to establish that the a^*-closures are as claimed, we shall borrow from [21] a permutation group property common to the two situations. This property will (since the groups in question are not locally pathological) force many stabilizers $G_{\bar{s}}$ to be minimal closed primes, and ultimately will force every †-extension of (G, S) to lie within the desired (H, T). Then we shall show that, under the hypotheses of the two theorems, H is an a^*-extension of G.

(G, S) is said to have the *strong support property* if for all $\gamma \in \Gamma(G, S)$, there exist $g \in G$ and $s \in S$ such that support$(g) \subseteqq s\mathscr{S}^\gamma$ and $(s\mathscr{S}_\gamma)g \neq s\mathscr{S}_\gamma$. Equivalently, whenever $t \in \bar{S}$ lies in the interior of an o-block B of (G, S), there exists $g \in G$ such that support$(g) \subseteqq B$ and $tg \neq t$ (see [21]).

Any transitive $(A(S), S)$ has the strong support property, as does any wreath product of o-primitive factors. So do the groups in Examples 4.11 and 4.12. For this reason, we shall first prove the following theorem:

THEOREM 6.4. *Let (G, S) be transitive and have the strong support property. Suppose that no o-primitive component of (G, S) is pathological and for γ nonminimal,*

(i) *if (G_γ, S_γ) is regular, then the divisible closure of G_γ is $\boldsymbol{R}$, and*

(ii) *if (G_γ, S_γ) is periodic with period f_γ, then it does not have* Config (∞).

Let (H, T) be the group described in Theorem 6.1. *Let T' be T with the points used to fill cuts in S_B deleted, and let H' be the restriction to T' of $\{h \in H: T'h = T'\}$. Then every †-extension of (G, S) is isomorphic over (G, S) to a permutation subgroup of (H', T') and every a^*-extension of G is l-isomorphic over G to an l-subgroup of H'.*

In order to prove Theorem 6.4, we will need some technical lemmas. Recall, in general, $(G, S) \leqq (W, R)$, and letting (K, U) be a

$\dagger$-extension of (G, S), we have $(G, S) \leqq (K, U) \leqq (W, R)$ (see Theorem 4.9).

LEMMA 6.5. *Let (G, S) be transitive. Suppose $(G, S) \dagger (K, U)$ and $(K, U) \leqq (W, R)$. Let $u \in U \backslash S$, let $M(u)$ be the largest segment of U which contains u and fails to meet S and let $S = X \cup Y$, $X < u < Y$. Then one of the following is true:*

(1) *For some $s \in S$ and $\gamma \in \Gamma$, X and Y each meet $s\mathscr{S}^\gamma$, X is a union of $\mathscr{S}_\gamma$-classes and so is Y, there is no largest $\mathscr{S}_\gamma$-class in X and no smallest $\mathscr{S}_\gamma$-class in Y, and $M(u) = u\mathscr{U}_\gamma$.*

(2) *For some $s \in S$ and $\gamma \in \Gamma$, X and Y each meet $s\mathscr{S}^\gamma$, X is a union of $\mathscr{S}_\gamma$-classes and so is Y, $s\mathscr{S}^\gamma/\mathscr{S}_\gamma$ is o-isomorphic to the integers, and $M(u)$ is a union of $\mathscr{U}_\gamma$-classes.*

(3) *The cut in S determined by (X, Y) belongs to S_A and $M(u) = \{u\}$.*

(4) *The cut in S determined by (X, Y) belongs to S_B and $M(u)$ is a nonsingleton o-block of (K, U).*

Proof. Recall that $(G, S) \leqq (W_1, R_1) \leqq (W, R)$ where $(W_1, R_1) = \{Wr\{(G_\gamma S_\gamma): \gamma \in \Gamma(G, S)\}$. We first consider the case in which no element of R_1 lies between X and Y. Then some $u(\gamma) \in T_\gamma \backslash S_\gamma$. There must be a largest such γ, which we call δ, for if $r_1, r_2 \in R$, then $\{\alpha \in \Gamma: r_1(\alpha) \neq r_2(\alpha)\}$ is inversely well-ordered. If S_δ is not o-isomorphic to the integers, then $T_\delta \subseteqq \bar{S}_\delta$. Since $u\mathscr{U}_\delta \cap S = \varnothing$, and any segment of U which contains u and extends outside $u\mathscr{U}_\delta$ would have to meet R_2 and thus also S, $M(u) = u\mathscr{U}_\delta$. A similar argument establishes (2) when S_δ is o-isomorphic to the integers.

Next suppose some $r_1 \in R_1$ lies between X and Y. By [13, page 713], the cut in S determined by (X, Y) belongs to S_A or S_B. Moreover, $M(u)$ is the intersection of a tower of o-blocks of (K, U) and so is itself an o-block. But the o-blocks in that tower are precisely the o-blocks of (K, U) which contain $M(u)$. Now (3) and (4) follow.

LEMMA 6.6. *Suppose that, in Lemma 6.5, (G, S) has closed stabilizers and that $G_{\bar{s}}$ is a minimal closed prime of G. Then $M(u) = \{u\}$ unless (G, S) is locally the integers and $\bar{s}$ is a cut in a primitive segment.*

Proof. By Corollary 5.3, (G, U) has closed stabilizers. Now if $v \in \overline{M(u)}$ and $z = \sup M(u) \in \bar{T}$, then $G_v \subseteqq G_z = G_{\bar{s}}$. Since G_v is a closed prime subgroup of G, $G_v = G_{\bar{s}} = G_z$ by the minimality of $G_{\bar{s}}$. Hence $K_v = K_z$. If $M(u)$ contains an o-block B which contains u, then since any point of B belongs to $\overline{M(u)}$, we see that $K_{\sup B} = K_z = K_b$ for all $b \in B$. Since K is transitive, $B = \{u\}$ (if $b \in B$ and

$b \neq u$, there exists $k \in K$ such that $uk = b$. Now $(\sup B)k = \sup B$, so $k \in K_{\sup B} \backslash K_u$, a contradiction). The lemma now follows from a consideration of the various cases arising in Lemma 6.5.

We now make our first use of the strong support property to find out when the hypotheses of Lemma 6.6 can occur.

If (H, T) is an l-permutation group and $z, y \in \bar{T}$, we shall say that x and y are *tied* if $H_x = H_y$. Observe that if x and y are tied, then so are xh and yh $(h \in H)$.

The following proposition is contained in [21].

PROPOSITION 6.7. *Let (G, S) have the strong support property and not be locally pathological. Then the closed primes of G (other than G) are precisely the stabilizers G_y $(y \in \bar{S})$. Moreover G_y is a minimal closed prime if*

(i) $y \in S_A \cup S_B \subset S$,

(ii) y is a hole in some $s\mathscr{S}^\gamma/\mathscr{S}_\gamma$, where (G_γ, S_γ) is o-2-transitive, or

(iii) y is a hole in some $s\mathscr{S}^\gamma/\mathscr{S}_\gamma$, where (G_γ, S_γ) is periodic and y is not tied to any point of $s\mathscr{S}^\gamma/\mathscr{S}_\gamma$.

COROLLARY 6.8. *Any wreath product of a^*-closed o-primitive groups which is not locally pathological is a^*-closed.*

Proof. By Theorem 5.1, the factors are †-closed and so, by Corollary 4.10, the wreath product is †-closed. Since the wreath product has closed stabilizers (Proposition 6.7), it is a^*-closed by Corollary 5.6.

We now prove Theorem 6.4.

Proof. First we consider †-extensions. Let (K, U) be a †-extension of (G, S). We may suppose that $(G, S) \dagger (K, U) \leqq (W, R)$ as in Lemma 6.5. By Proposition 6.7, (G, S) has closed stabilizers. Also, by Proposition 6.7, if $\bar{s} \in S_B$, then $G_{\bar{s}}$ is a minimal closed prime subgroup of G. Then (4) of Lemma 6.5 is impossible by Lemma 6.6 (no element of S_B can be a cut in a primitive segment!). If γ is not minimal in $\Gamma = \Gamma(G, S)$, then (2) of Lemma 6.5 cannot occur since $s\mathscr{S}^\gamma/\mathscr{S}_\gamma$ is not o-isomorphic to the integers by hypothesis (i) of the theorem. In addition, if γ is not minimal in Γ, then (1) of Lemma 6.5 cannot occur unless (G_γ, S_γ) is regular or periodic with Config (n), for some positive integer n (by Proposition 6.7 and Lemma 6.6).

Assume that $u \in U \backslash S$ fits case (1) of Lemma 6.5 with γ nonminimal. Since G_u is closed in G (Corollary 5.3), Proposition 6.7 guarantees that $G_u = G_y$ for some $y \in \bar{S}$. Thus $G_y = G_u \subseteqq G_{\bar{s}} = G_x$, where $\bar{s}$ is the cut in S determined by u and $x = \sup u\mathscr{U}_\gamma$. Since $\bar{s}$ is a hole in $s\mathscr{S}^\gamma/\mathscr{S}_\gamma$, the strong support property yields a contradiction unless y lies in the completion of $s\mathscr{S}^\gamma$. Moreover, as γ is not minimal, $u\mathscr{U}_\gamma$ is not a singleton. Hence $K_u \neq K_x$. Thus $K_y = K_u \subsetneqq K_x$. In o-primitive groups, every stabilizer of a point or hole is a maximal prime. Hence y must lie in the interior of some $\mathscr{U}_\gamma$-class $y\mathscr{U}_\gamma$.

In short, we have shown that $K_u = K_y$, where the cut y lies in the interior of a $\mathscr{U}_\gamma$-class. Since (G, S) enjoys the strong support property, this $\mathscr{U}_\gamma$-class in uniquely determined.

Let $t = \sup y\mathscr{U}_\gamma$ and $w = \sup u\mathscr{U}^\gamma$. Then K_x and K_t are maximal prime subgroups of K_w which contain the prime subgroup $K_u = K_y$. Hence $K_x = K_t = \hat{K}$, say. The map $uk \mapsto yk$ and the identity on $\hat{K}$ yields a well-defined isomorphism from the pair $(\hat{K}, u\hat{K})$ ($\hat{K}$ acting on $u\hat{K}$—not necessarily faithfully) to $(\hat{K}, y\hat{K})$ ($\hat{K}$ acting on $y\hat{K}$, also not necessarily faithfully). This isomorphism preserves both the action of $\hat{K}$ and the orders. If v belongs to the completion of $u\mathscr{U}_\gamma$, then v is tied to $z = \inf\{yk\colon k \in \hat{K}\ \&\ uk \geqq v\} \in$ completion of $y\mathscr{U}_\gamma$.

Now choose $h \in K$ so that $(y\mathscr{U}_\gamma)h = u\mathscr{U}_\gamma$. Then each cut in $(u\mathscr{U}_\gamma)h$ is tied to some cut in $(y\mathscr{U}_\gamma)h = u\mathscr{U}_\gamma$, which, in turn, is tied to some cut in $y\mathscr{U}_\gamma$. By induction, each cut in $(u\mathscr{U}_\gamma)h^m$ is tied to some cut in $y\mathscr{U}_\gamma$, m any positive integer. If $(u\mathscr{U}_\gamma)h^m \cap S \neq \varnothing$ for some positive integer m, we have a contradiction to the strong support property. But either (G_γ, S_γ) is regular and the divisible closure of G_γ is $\boldsymbol{R}$, or (G, S_γ) is periodic with Config(n), as previously noted. In the former case, (K_γ, U_γ) is also regular and so contained in the regular representation of the reals.

By assumption, some power $p > 1$ of the permutation of $y\mathscr{U}^\gamma/\mathscr{U}_\gamma$ induced by h is back in G_γ; it must map $y\mathscr{U}_\gamma$ onto another class also containing points of S. Then $(u\mathscr{U}_\gamma)h^{p-1} \cap S = \varnothing$, a contradiction. In the latter case, the hole $u\mathscr{U}_\gamma$ in S_γ would be tied in G_γ to $y\mathscr{S}_\gamma$; so $u\mathscr{U}_\gamma = (y\mathscr{S}_\gamma)f_\gamma^p$ for some integer p. Then $m = |p|n$ would yield a contradiction. Consequently, case (1) of Lemma 6.5 cannot occur if γ is nonminimal.

We have now shown that under the hypotheses of Theorem 6.4, any $u \in U\backslash S$ must have been added at a cut lying in S_A or in a primitive segment of (G, S). Since also $(K, U) \leqq (W, R)$, we have $(K, U) \leqq (H', T')$. As S is (almost) dense in T, there is no ambiguity about auxiliary subgroups; so the original embedding of K in W must have been over G. This completes the proof of Theorem 6.4 for †-extensions; for a^*-extensions, apply Theorem 5.5.

Example 4.11 shows that the hypotheses of Theorem 6.4 do not force $(G, S) \dagger (H', T')$; the unique $\dagger$-closure determined there is not (H', T').

We claim that the group (H', T') of Theorem 6.4 is transitive. This is clear if (G, S) is locally o-primitive, so assume that $\Gamma = \Gamma(G, S)$ has no least element. Let $s \in S$ and let t be the single point used to fill in some cut of S belonging to S_A. Since s and t lie in the wreath product set T, $\{\gamma \in \Gamma: s(\gamma) \neq t(\gamma)\}$ is inversely well-ordered. Define $h \in H$ by choosing $h_{\gamma,s}$ so that $s(\gamma)h_{\gamma,s} = t(\gamma)$ whenever $s(\gamma) \neq t(\gamma)$ and taking all other $h_{\delta,x}$'s to be the identity. It can be shown that $h \in H_1$, and, by construction, $sh = t$. Therefore, (H', T') is transitive.

We now show, under the hypotheses of Theorems 6.1 and 6.2, that $(G, S) \dagger (H', T')$.

Assume $G_X \subseteqq G_y$, $X \subseteqq \bar{T}$, $y \in \bar{T}$. Recall that S is (almost) dense in T, so we may assume that $X \subseteqq \bar{S}$ and $y \in \bar{S}$. Since (G, S) has the strong support property, X must meet the completion of every o-block which contains y. Let Y be the intersection of the tower of such o-blocks. If y is an endpoint of Y or if $Y = \varnothing$, then the tower must have contained no smallest o-block. Hence the topological closure of X must meet Y or include one of the endpoints of Y. This forces $H'_X \subseteqq H'_y$, as required. If y lies in the interior of Y, then Y covers some o-block (otherwise Y would be the union of a tower of smaller o-blocks and y would lie in one of these o-blocks). Thus $Y = s\mathscr{S}^\gamma$ for some $s \in S$. By the minimality of Y, y does not lie in any $\mathscr{S}_\gamma$-class. Consequently, y is an endpoint of such a class or a hole in $s\mathscr{S}^\gamma/\mathscr{S}_\gamma = S_\gamma$. In effect, $y \in \bar{S}_\gamma$. Let $\hat{X}$ comprise the $\mathscr{S}_\gamma$-classes in S_γ whose completions (including endpoints) meet X and the holes in S_γ which lie in X. Then $(G_\gamma)_{\hat{X}} \subseteqq (G_\gamma)_y$. For let $z = \sup s\mathscr{S}^\gamma$. If the action of some $g \in G_z$ on $s\mathscr{S}^\gamma/\mathscr{S}_\gamma$ fixes each element of $\hat{X}$ but moves y, then $\bar{g} \in G_X \backslash G_y$, where $\bar{g}$ agrees with g on those $\mathscr{S}_\gamma$-classes in $s\mathscr{S}^\gamma$ which are moved by g, and $\bar{g}$ is the identity on the rest of S. Clearly $\bar{g} \in G$ because $G = A(S)$ or G is a wreath product of o-primitive groups. But $(G_\gamma, S_\gamma) \dagger (H'_\gamma, T'_\gamma)$, so $(H'_\gamma)_{\hat{X}} \subseteqq (H'_\gamma)_y$. Thus $H'_X \subseteqq H'_y$ and, consequently, $(G, S) \dagger (H', T')$.

Now if (K, U) is any $\dagger$-closure of (G, S), so that $(G, S) \dagger (K, U) \leqq (H', T')$, then there is no ambiguity about auxiliary subgroups (S is (almost) dense in T'). Hence $(K, U) \dagger (H', T')$, so that $(K, U) = (H', T')$. Therefore (H, T) of Theorems 6.1 and 6.2 is the unique (to within isomorphism over (G, S)) $\dagger$-closure of (G, S). Thus H is the unique (to within l-isomorphism over G) a^*-closure of G. This completes the proofs of Theorems 6.1 and 6.2.

(G, S) is said to be *depressible* if whenever $sg \neq s$, there exists $g_1 \in G$ such that g_1 agrees with g on the interval of support of g

containing s, and is the identity on the rest of S. The preceding argument yields a generalization of Theorem 6.2 to depressible groups. Nonminimal regular components (G_γ, S_γ) are permitted provided the divisible closure of G_γ is $\boldsymbol{R}$, and then $h_{\gamma,x}$ is required to be induced by G.

We now show that we cannot relax the conditions on nonminimal components in Theorem 6.1. We shall give two examples in which there is more than one a^*-closure of a given (G, S). In the first, we have an "upper" component that is regular but whose divisible clusure is not $\boldsymbol{R}$; in the second, we use an "upper" component that has Config(∞).

EXAMPLE 6.9. Let $(G, S) = A(\boldsymbol{R})Wr(\boldsymbol{Z}, \boldsymbol{Z}) = A(\boldsymbol{R} \overleftarrow{\times} \boldsymbol{Z})$, where $\boldsymbol{R}$ is the reals and $\boldsymbol{Z}$ is the integers. Let C be a complementary group in $\boldsymbol{R}$ of the rationals, $\boldsymbol{Q}$ $(\boldsymbol{R} = \boldsymbol{Q} \oplus C)$. Let $(H, U) = A(\boldsymbol{R})Wr(\boldsymbol{Z} \oplus C, \boldsymbol{Z} \oplus C)$ and let $K = \{h \in H\colon (\forall n \in \boldsymbol{Z})(\forall x, y \in C)(h_{n+x} = h_{n+y})\}$. In (K, U), each $u \in U \backslash S$ is tied to a unique $s \in S$. Hence $(G, S) \dagger (K, U)$.

The arguments used to prove Theorem 6.4 show that (K, U) is a maximal $\dagger$-extension of (G, S) so, by Corollary 5.4, (K, U) is a $\dagger$-closure of (G, S). Therefore K is an a^*-closure of G.

Let K_1 and K_2 be formed in this way from two different complements C_1 and C_2 of the rationals. Then K_1 and K_2 are not l-isomorphic over G. For if they were, pick any $s \in S$ and represent K_i on the chain of right cosets of $\bar{\bar{G}}_s (i = 1, 2)$. Then (K_1, U_1) and (K_2, U_2) are isomorphic over (G, S) forcing $\boldsymbol{Z} \oplus C_1$ and $\boldsymbol{Z} \oplus C_2$ to be isomorphic over $\boldsymbol{Z}$. This is impossible. Indeed, it is possible to choose C_1 and C_2 so that K_1 and K_2 are not l-isomorphic at all.

Actually, we can show that (G, S) has $2^{2^{\aleph_0}}$ nonisomorphic a^*-closures.

EXAMPLE 6.10. Let (G_2, S_2) be a periodic o-primitive group having period f and Config(∞). Let $(G, S) = (\boldsymbol{R}, \boldsymbol{R})Wr(G_2, S_2)$ and F be the cyclic subgroup of $A(\bar{S}_2)$ generated by f. Let

$$(H, U) = (\boldsymbol{R}, \boldsymbol{R})Wr(Z_{A(S_2F)}(f), S_2F).$$

Let $K_1 = \{h \in H\colon (\forall s_2 \in S_2)(\forall m \in \boldsymbol{Z})(h_{s_2f^m} = h_{s_2})\}$. Then K_1 is an a^*-closure of G. Let $\tau\colon x \mapsto 2x$, so $\tau \in A(\boldsymbol{R})$. Let

$$K = \{h \in H\colon (\forall s_2 \in S_2)(\forall m \in \boldsymbol{Z})(h_{s_2f^m} = \tau^{-1}h_{s_2f^{m-1}}\tau)\} .$$

Thus if $h_{s_2f^{m-1}}$ is translation by r, $h_{s_2f^m}$ is translation by $2r$. K_2 is also an a^*-closure of G and it can be shown that K_1 and K_2 are not l-isomorphic at all, much less l-isomorphic over G.

We conclude with some concrete consequences of our theorems.

COROLLARY 6.11. *Let $A(S)$ be transitive and have no nonminimal regular component. Then $A(S)$ has a unique a^*-closure. If, in addition, $A(S)$ is locally o-primitive and the minimal component is a^*-closed, then $A(S)$ is a^*-closed.*

COROLLARY 6.12. *Suppose that $A(S)$ is o-primitive and not regular. Then the unique a^*-closure of $A(S)$ is $A(\bar{S})$. In particular, $A(\boldsymbol{R})$ is a^*-closed and is the unique a^*-closure of $A(\boldsymbol{Q})$.*

COROLLARY 6.13. *Let $(P, \boldsymbol{R})$ be a periodic group with period f. Then the unique a^*-closure of P is $Z_{A(\boldsymbol{R})}(f)$, the unique a^*-closure of $A(\boldsymbol{Q}) Wr P$ is $A(\boldsymbol{R}) Wr Z_{A(\boldsymbol{R})}(f)$, and the unique a^*-closure of $P Wr A(\boldsymbol{Q})$ is $Z_{A(\boldsymbol{R})}(f) Wr A(\boldsymbol{Q})$.*

COROLLARY 6.14. *The unique a^*-closure of $A(\boldsymbol{Q}) Wr A(\boldsymbol{Q})$ is $A(R) Wr A(\boldsymbol{Q}) = A(\boldsymbol{R} \overleftarrow{\times} \boldsymbol{Q})$.*

COROLLARY 6.15. *The unique a^*-closure of $B(\boldsymbol{Q})$ is $A(R)$ where $B(\boldsymbol{Q})$ is the set of all elements of $A(\boldsymbol{Q})$ of bounded support.*

PROPOSITION 6.16. *Let P and B be as given in Example 4.8. Then $A(\boldsymbol{R}) Wr B$ is the unique a^*-closure of $A(\boldsymbol{Q}) Wr P$.*

This proposition is not a special case of any of the theorems. It can be proved by applying Proposition 6.7, Theorem 5.5, Lemma 6.5, Theorem 4.6, and the fact that $(B, \boldsymbol{R})$ is the unique †-closure of $(P, \boldsymbol{R})$.

In conclusion, we note that the methods of this paper cannot be very useful in the investigation of a^*-extensions of l-groups which are not completely distributive. Therefore, two central problems remain, namely:

1. Does every pathological o-2-transitive l-group have a unique a^*-closure, and is every a^*-extension of a pathological o-2-transitive l-group also pathological?

2. Find necessary and sufficient conditions for uniqueness of a^*-closures even in the completely distributive case. The main difficulty seems to be lack of knowledge about subgroups of a wreath product.

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