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THE CHARACTERISTIC POLYNOMIAL OF THE MONODROMY

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This paper contains miscellaneous results about the monodromy of a singularity of an algebraic curve $f(z_0, z_1)$, particularly its characteristic polynomial, and its relation to branched cyclic covers of the link of the singularity.

Let $f(z_0, \cdots, z_n)$ for $n \geqq 0$ be a complex polynomial vanishing at the origin, and suppose that f has at most an isolated critical point there. The *link* of this singularity is the manifold $K = f^{-1}(0) \cap S^{2n+1} \subset S^{2n+1}$, where S^{2n+1} is a sufficiently small sphere centered at the origin. K is an $(n-2)$-connected $(2n-1)$-manifold. For $\delta \neq 0$, $f^{-1}(\delta)$ intersected with the corresponding small ball is a smooth $2n$-manifold F whose boundary is diffeomorphic to K. Letting δ travel once about the origin in the positive direction induces a *monodromy* automorphism $h: \tilde{H}_n(F) \rightarrow \tilde{H}_n(F)$. The homology group $\tilde{H}_{n-1}(K)$ of the link is isomorphic to the cokernel of $I - h$ [11, Theorem 8.5]. We will be particularly interested in the characteristic polynomial $\Delta(t) = \det(It - h)$ of the monodromy. For these topics, see [11].

For fixed f, we let $K_k \subset S^{2n+3}$ be the link of the polynomial $f(z_0, \cdots, z_n) + z_{n+1}^k$. K_k is thus an $(n-1)$-connected $(2n+1)$-manifold. It is well-known that K_k is the k-fold cyclic cover of S^{2n+1} branched along K [6].

In the first section we relate the rank and 2-torsion of $H_1(K_2)$ to the number of branches of $f(z_0, z_1)$ at the singular point. As an example, we compute $H_1(K_2)$ of the singularity $(z_0 + z_1^2)(z_0^2 + z_1^5)$ used to foliate odd dimensional spheres [4], thus avoiding resolution of singularities. Section two reproves an old result of Zariski [18] on computing the rank of $H_1(K_k)$ in terms of $\Delta(t)$ (which is closely related to the Alexander polynomial of the link). The rest of the section is a digression on the roots ± 1 of $\Delta(t)$. Section three combines the results of the previous sections to give a simple criterion for the monodromy of $f(z_0, z_1)$ to be of infinite order, namely, that -1 be a root of $\Delta(t)$ of multiplicity greater than or equal to the number of branches of f at the singular point. In particular, $(z_0 + z_1^2)(z_1^2 + z_2^5)$ satisfies this criterion. The first example of monodromy of infinite order was found by A'Campo [1]. Shortly thereafter the author found a mistake in a preliminary version of [4]; this mistake generalized to the present paper. The results here have been expanded by Woods [19]. For an analytic approach to monodromy of infinite order, see [5].

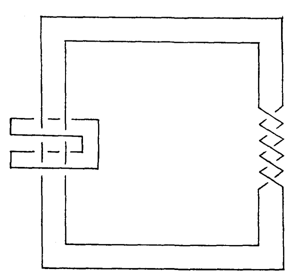
1. The two-fold branched cyclic cover. Let $f(z_0, \cdots, z_n)$ be a complex polynomial as above, with link $K \subset S^{2n+1}$. Let K_2 be the 2-fold cyclic cover of S^{2n+1} branched along K. We claim that *the rank plus the number of 2-torsion coefficients of $\tilde{H}_{n-1}(K)$ is equal to the rank plus the number of 2-torsion coefficients of $H_n(K_2)$*: First consider the polynomial $f'(z_{n+1}) = z_{n+1}^2$ in one variable. The monodromy h' of f' on $\tilde{H}_0(F) \simeq \boldsymbol{Z}$ may be calculated directly, and turns out to be multiplication by -1. (See for example [11, § 9].) Let h'' be the monodromy of the polynomial $f + f'$. By [16], h'' is equivalent to $h \otimes h' = -h$. Since $\tilde{H}_{n-1}(K) \simeq \operatorname{cok}(I - h)$ and $H_n(K_k) \simeq \operatorname{cok}(I - h'')$, we have $\tilde{H}_{n-1}(K) \otimes \boldsymbol{Z}_2 \simeq H_n(K_2) \otimes \boldsymbol{Z}_2$, which proves the italicized assertion.

Next specialize to the case $n = 1$. Let r be the number of branches of $f(z_0, z_1)$ at the origin, so that $K \subset S^3$ has r components. We have proved the following result.

PROPOSITION 1. *When $n = 1$, the rank plus the number of 2-torsion coefficients of $H_1(K_2)$ is $r - 1$.*

The italicized assertion above is true for any simple knot $K \subset S^{2n+1}$ and $K_2 \subset S^{2n+3}$ its suspension, a knot with the same Seifert matrix $\mathscr{L}$ up to sign as K [2]. The homology groups of K and K_2 are determined by $\mathscr{L} \pm \mathscr{L}^t$, and the same proof works. When $n = 1$, the proposition generalizes to $H_1(K_k) \leqq (k - 1)(r - 1)$, for arbitrary links $K \subset S^3$ of r components, and K_k the branched k-fold cyclic cover, for k prime [17, Corollary 5].

For example, consider the singularity $f(z_0, z_1) = (z_0 + z_1^2)(z_0^2 + z_1^5)$. To foliate odd-dimensional spheres as in [4], the key point is that $H_1(K_2) \simeq \boldsymbol{Z}$; here we find this group without resolving the singularity $f(z_0, z_1) + z_2^2$. The polynomial f has two branches at the origin, and its link $K \subset S^3$ is a torus knot of type (2, 5) together with an unknotted circle twice linking its core (see for example [14, § 2.3]):



As remarked above, K_2 is the 2-fold cyclic cover of S^3 branched along this link. The torsion subgroup of $H_1(K)$ is isomorphic to the cokernel of its quadratic form [8]. This quadratic form is computed by coloring alternate regions black and white; by row and column operations its cokernel is seen to have no torsion. In particular, $H_1(K_2)$ has no 2-torsion, so by the proposition, its rank is one. Thus $H_1(K_2) \simeq \boldsymbol{Z}$.

Do all the possibilities of the proposition occur? If $f(z_0, z_1) = z_0^2 + z_1^2$, $H_1(K_2) \simeq \boldsymbol{Z}_2$. Thus if $r = 2$, both possibilities occur. If $r = 3$ there are three possibilities, all of which do occur: If $f(z_0, z_1) = z_0^3 + z_1^3$, then $H_1(K_2) \simeq \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$, if $f(z_0, z_1) = (z_0 + z_1^2)(z_0 - z_1^2)(z_0^2 + z_1^5)$, then $H_1(K_2) \simeq \boldsymbol{Z} \oplus \boldsymbol{Z}_4$, and if $f(z_0, z_1) = z_0^6 + z_1^3$, then $H_1(K_2) \simeq \boldsymbol{Z} \oplus \boldsymbol{Z}$. (Also see [5, § 1].)

2. **The characteristic polynomial of the monodromy.** Again fix a complex polynomial $f(z_0, \cdots, z_n)$ as above, with link $K \subset S^{2n+1}$, monodromy h, and characteristic polynomial $\Delta(t) = \det(It - h)$. Orlik [13, p. 264–5] has noticed that *if the monodromy is of finite order, then the rank of $\tilde{H}_{n-1}(K)$ is the multiplicity of* 1 *as a root of $\Delta(t)$.*

Proof. Let $\Delta_1(t), \cdots, \Delta_n(t)$ be the elementary divisors of $It - h$ over $Q[t]$. These have the usual properties: (1) $\Delta_1(t) \mid \Delta_2(t) \mid \cdots \mid \Delta_r(t)$; (2) $\Delta_r(t)$ is the minimum polynomial of h; (3) $\Delta(t) = \Delta_1(t)\Delta_2(t) \cdots \Delta_r(t)$; and (4) $\operatorname{cok}(It - h) \cong \operatorname{cok} D(t)$ as $\boldsymbol{Q}[t]$-modules, where $D(t)$ has a diagonal matrix with entries $1, \cdots, 1, \Delta_1(t), \cdots, \Delta_r(t)$. Since h is of finite order, there is an N such that $h^N = I$. Thus by (2), $\Delta_r(t) \mid (t^N - 1)$. Hence the factorization in $\boldsymbol{Q}[t]$ of $\Delta_r[t]$ has no repeated factors; by (1), this is true for $\Delta_1(t), \cdots, \Delta_{r-1}(t)$ as well. In particular, the factor $(t - 1)$ can occur at most once. Recall that $\tilde{H}_{n-1}(K) \cong \operatorname{cok}(I - h)$. Setting $t = 1$ in (4), we obtain the $\boldsymbol{Q}$-module isomorphism $\tilde{H}_{n-1}(K) \cong \operatorname{cok} D(1)$. By (3) and the above, the rank of $\operatorname{cok} D(1)$ is equal to the multiplicity of $(t - 1)$ in the factorization of $\Delta(t)$.)

We apply this observation to the k-fold branched cyclic cover K_k.

PROPOSITION 2. *Suppose that the monodromy of $f(z_0, \cdots, z_n)$ is of finite order. Then the rank of $\tilde{H}_n(K_k)$ is equal to the number* (counted with multiplicities) *of roots $\alpha \neq 1$ of $\Delta(t)$ with $\alpha^k = 1$.*

When $n = 1$ and $f(z_0, z_1)$ has one branch at the origin, $\Delta(t)$ is then the Alexander polynomial of the knot $K \subset S^3$ [11, Lemma 10.1], and Zariski [18] showed that $H_1(K_k)$ could be computed in this fashion. Since Lê [9] has shown that such f have monodromy of finite order, our result implies Zariski's. Also see [17].

Proof. First consider the polynomial $f'(z_{n+1}) = z_{n+1}^k$ in one variable. The monodromy h' of f' may be calculated directly, and its characteristic polynomial $\Delta'(t)$ turns out to be $t^{k-1}+\cdots+t+1=\prod_{j=1}^{k-1}(t-\eta^j)$, where η is a primitive k^{th} root of unity. (See for example [11, § 9].)

Let h'' be the monodromy associated to the polynomial $f+f'$, and let $\Delta''(t)$ be its characteristic polynomial. By [16], h'' is equivalent to $h\otimes h'$. Suppose $\Delta(t)$ factors over the complex numbers as $\prod_i(t-\alpha_i)$. Then $\Delta''(t)=\prod_{i,j}(t-\alpha_i\eta^j)$. Hence 1 is a root of $\Delta''(t)$ precisely when there is a j, $1\leqq j\leqq k-1$, and an α_i such that $\alpha_i\eta^j=1$; this will occur if and only if α_i is a k^{th} root of unity different from 1. This proves the proposition.

The rest of this section is a diversion about the roots ± 1 of $\Delta(t)$ when $n=1$. Let r be the number of branches of $f(z_0,z_1)$ at the origin (the number of components of its link $K\subset S^3$).

PROPOSITION 3. (i) *The multiplicity of* 1 *as a root of* $\Delta(t)$ *is* $r-1$.

(ii) *The multiplicity of* -1 *as a root of* $\Delta(t)$ *is even.*

Part (i) says that Orlik's assertion at the beginning of this section is true when $n=1$, even if the monodromy is of infinite order. When $n=2$, the singularity $(z_0+z_1^2)(z_0^2+z_1^5)+z_2^2$ is a counter-example to (i) (with $r-1$ replaced by rank $H_1(K)$) and (ii). (See §§ 1 and 3.) Both parts are true for arbitrary knots $K\subset S^3$ when $r=1$.

Proof. If $r=1$, $\Delta(t)$ is (up to a factor of $\pm t^i$) the Alexander polynomial of the link of $f(z_0,z_1)$ [11, Lemma 10.1], and it is well known that the value of this polynomial at 1 is ± 1. Hence the proposition is true if $r=1$. If $r\geqq 2$, then again according to the above reference, $\pm t^i\Delta(t)=(t-1)\Delta(t,\cdots,t)$, where $\Delta(t_1,\cdots,t_r)$ is the Alexander polynomial of the link. Let $\nabla(t)=(t-1)^{2-r}\Delta(t,\cdots,t)$ be the polynomial defined by Hosokawa [7]. It is shown that $\pm\nabla(1)$ is the $(r-1)$-minor determinant of an $r\times r$ matrix of rank $(r-1)$ whose ij^{th} entry is the linking number l_{ij} of the i^{th} and j^{th} components of the link, $i\neq j$, and whose i^{th} diagonal entry is $-\sum_{1\leqq j\leqq r,i\neq j}l_{ij}$. We claim that $\nabla(1)\neq 0$. In fact, l_{ij}, $i\neq j$, is also the intersection number of the i^{th} and j^{th} branch, which is positive [14]. Thus the upper left-hand $(r-1)$-minor satisfies conditions (a), (b), (c) and (d) of [12, p. 6] and hence is negative definite as a quadratic form. Hence its determinant, which is $\pm\nabla(1)$, is nonzero. Since $\Delta(t)=\pm t^i(t-1)^{r-1}\nabla(t)$, 1 is a root of $\Delta(t)$ of multiplicity $r-1$. This proves part (i).

When $r=1$, $\nabla(t)$ is defined to be the Alexander polynomial. Hence for all $r\geqq 1$, $\Delta(t)=\pm t^i(t-1)^{r-1}\nabla(t)$, so it suffices to show that -1 is a root of $\nabla(t)$ of even multiplicity. In [7] it is proved that $\nabla(t)$ is *symmetric*, that is, $\nabla(t) = t^d\nabla(t^{-1})$ where d is the degree of $\nabla(t)$, and that d is even. Now 0 is not a root of $\nabla(t)$. The symmetry implies that if c is a root, then so is c^{-1}. Thus in the factorization of $\nabla(t)$ over the complex numbers, the factor $(t-c)$ may be paired with the factor $(t-c^{-1})$, except possibly when $c = \pm 1$. Since $c \neq 1$ by part (i), and there are an even number of factors, the factor $(t+1)$ must occur an even number of times. This proves part (ii).

3. Monodromy of infinite order. Let $f(z_0, z_1)$ be a polynomial as above with r branches at the origin (so that its link $K \subset S^3$ has r components), and let $\Delta(t)$ be the characteristic polynomial of its monodromy.

PROPOSITION 4. *The monodromy of $f(z_0, z_1)$ is of infinite order if -1 is a root of $\Delta(t)$ of multiplicity $\geqq r$.*

Proof. If the monodromy were of finite order, then by Proposition 2 this multiplicity would be the rank of $H_1(K_2)$, which by Proposition 1 is $\leqq r-1$.

The Alexander polynomials of all compound torus links with two components have been computed by Burau [3]. (Burau actually computes the Reidemeister-Schumann polynomial, the determinant of a presentation matrix of $H_1(\tilde{X})$, where $\tilde{X}$ is the universal abelian cover of $S^3 - K$. Levine [10, § 8] shows that this is the same as the usual Alexander polynomial. Alternatively, R. H. Fox (private correspondence) applies the free differential calculus to Reidemeister and Schumann's presentation of $\pi_1(S^3 - K)$ and shows that the $(d+1)^{\text{st}}$ elementary ideal of the resulting matrix is the same as the d^{th} elementary ideal of the presentation matrix (7) in [15, p. 258] of $H_1(\tilde{X})$ derived from this. Hence the Alexander polynomial is the same as the determinant of this matrix. He continues, "The calculations in the two Burau papers are almost too painful to contemplate, but I am sure that the results are correct$\cdots$".)

For example, $f(z_0, z_1) = (z_0 + z_1^2)(z_0^2 + z_1^5)$ has monodromy of infinite order: The curve $f = 0$ has two branches at the origin, and according to Burau its Alexander polynomial is

$$\Delta(x, y) = \frac{(1 - x^2y^4)(1 - x^4y^{10})}{(1 - xy^2)(1 - x^2y^5)} \cdot$$

The characteristic polynomial of the monodromy is then $\Delta(t) = \pm(t-1)(1+t^3)(1+t^7)$ by [11, Lemma 10.1], so -1 is a root of $\Delta(t)$ of multiplicity 2.

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[1] Beware of the following misprints: (n, m) on p. 295, line 6 should be (n', m'), and n in line 10 should be n'.