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SPECTRAL INEQUALITIES INVOLVING THE INFIMA AND SUPREMA OF FUNCTIONS

KONG-MING CHONG

This paper presents some spectral inequalities from which some useful tools for the investigations of the permanents of nonnegative matrices are derived as particular cases.

Introduction. In this paper, we establish some spectral inequalities involving the infima and suprema of functions as well as their equimeasurable rearrangements, and obtain as particular cases some rearrangement inequalities which turn out to be useful tools for the investigations of the permanents of non-negative matrices (see [4] and [5]).

1. Preliminaries. Let $M(X, \mu)$ denote the set of all extended real valued measurable functions defined on a finite measure space (X, Λ, μ). Two functions $f \in M(X, \mu)$ and $g \in M(X', \mu')$, where $\mu'(X') = \mu(X)$, are said to be *equimeasurable* (written $f \sim g$) whenever

$$\mu(\{x: f(x) > t\}) = \mu'(\{x: g(x) > t\}) \tag{1.1}$$

for all real t. If $f \sim g$, it follows from (1.1) that

$$\Phi(f) \sim \Phi(g) \tag{1.2}$$

whenever $\Phi: R \to R$ is a Borel measurable function.

If $f \in M(X, \mu)$, let $\delta_f: [0, \mu(X)] \to \bar{R}$ denote the *decreasing rearrangement* of f. It is clear that the function $\iota_f = -\delta_{-f}$, called the *increasing rearrangement* of f, satisfies $\iota_f \sim f$.

It is a direct consequence of (1.2) that

$$\delta_{\psi(f)} = \psi(\delta_f) \qquad (\text{respectively } \iota_{\psi(f)} = \psi(\iota_f)) \tag{1.3}$$

whenever $\psi: R \to R$ is a left continuous (respectively right continuous) and non-decreasing function. Moreover, it is not hard to see that the operation of decreasing or increasing rearrangements preserves a.e. pointwise convergence, convergence in measure and all L^p convergence, $1 \leqq p \leqq \infty$.

If $f, g \in M(X, \mu) \cup M(X', \mu')$ and $f^+, g^+ \in L^1(X, \mu) \cup L^1(X', \mu')$, then we write $f \ll g$ whenever

$$\int (f - t)^+ \leqq \int (g - t)^+$$

for all $t \in R$ and $f \prec g$ whenever $f \ll g$ and $\int f = \int g$.

In the sequel, expressions of the form $f \prec g$ (respectively $f \preccurlyeq g$) are called *strong* (respectively *weak*) *spectral inequalities.*

2. Some spectral inequalities. In what follows, for any given n-tuple $\boldsymbol{x} = (x_1, x_2, \cdots, x_n) \in R^n$, the n-tuples $\boldsymbol{x}^* = (x_1^*, x_2^*, \cdots, x_n^*)$ and $\boldsymbol{x}' = (x_1', x_2', \cdots, x_n')$ respectively denote the decreasing and increasing rearrangements of $\boldsymbol{x}$.

LEMMA 2.1. If $(a_1, a_2, \cdots, a_n) \in R^n$ and $(b_1, b_2, \cdots, b_n) \in R^n$, then

$$\sum_{i=1}^{n} (a_i^* \wedge b_i')^+ \leqq \sum_{i=1}^{n} (a_i \wedge b_i)^+ . \tag{2.1}$$

Proof. Since $(\boldsymbol{a}^*)^+ = (\boldsymbol{a}^+)^*$, $(\boldsymbol{b}')^+ = (\boldsymbol{b}^+)'$, by (1.3), and since $(r \wedge s)^+ = r^+ \wedge s^+$ for all $r, s \in R$, we need only prove the lemma for $a_i \geqq 0$ and $b_i \geqq 0$, $i = 1, 2, \cdots, n$.

Without loss of generality, we may assume that $b_i = b_i'$, $i = 1, 2, \cdots, n$. In this case, if $1 \leqq i < j \leqq n$ and $a_i < a_j$, then it is easily seen that $a_i \wedge b_j + a_j \wedge b_i \leqq a_i \wedge b_i + a_j \wedge b_j$. Thus, for each pair of integers i, j such that $1 \leqq i < j \leqq n$, if $a_i < a_j$, the right-hand sum of the asserted inequality is never increased on interchanging a_i and a_j. We, therefore, conclude that the left-hand sum of (2.1) is the smallest possible value attainable by the right-hand sum as $\boldsymbol{a}$ ranges through all its rearrangements.

In the sequel, we denote the Lebesque measure on R by m.

LEMMA 2.2. *If $f, g \in L^1(X, \Lambda, \mu)$ where $\mu(X) = a < \infty$, then*

$$\int_0^a (\delta_f \wedge \iota_g)^+ \, dm \leqq \int_X (f \wedge g)^+ d\mu . \tag{2.2}$$

Proof. If the measure space (X, Λ, μ) is non-atomic, then there exist sequences $\{f_n\}_{n=1}^\infty$, $\{g_n\}_{n=1}^\infty$ of simple functions with the same sets of constancy such that $f_n \to f$ and $g_n \to g$ both pointwise μ-a.e. and in L^1. Since decreasing and increasing rearrangements preserve a.e. pointwise convergence and L^p convergence, $1 \leqq p \leqq \infty$, we see that δ_{f_n} and ι_{g_n} also converge in L^1 to δ_f and ι_g respectively.

Now by Lemma 2.1, we have

$$\int_0^a (\delta_{f_n} \wedge \iota_{g_n})^+ dm \leqq \int_X (f_n \wedge g_n)^+ d\mu$$

whence (2.2) follows by taking limits and on observing that $f_n \wedge g_n = f_n - (f_n - g_n)^+$.

If (X, Λ, μ) is not non-atomic, we can imbed it into a non-atomic measure space $(\bar{X}, \bar{\Lambda}, \bar{\mu})$ (For details of this device, we refer to [2, pp. 52-54]). Then, $f, g \in L^1(X, \mu)$ can be identified with $\bar{f}, \bar{g} \in L^1(X, \mu)$ by a map $h \to \bar{h}$ which is linear and satisfying $h \sim \bar{h}$ and $\overline{\Phi(h)} = \Phi(\bar{h})$ for every Borel measurable $\Phi: R \to R$. Thus

$$f \wedge g \sim \overline{f \wedge g} = \overline{f - (f - g)^+} = \bar{f} - (\bar{f} - \bar{g})^+ = \bar{f} \wedge \bar{g}$$

and so

$$\int_0^a (\delta_f \wedge \iota_g)^+ dm = \int_0^a (\delta_{\bar{f}} \wedge \iota_{\bar{g}})^+ dm \leqq \int_{\bar{X}} (\bar{f} \wedge \bar{g})^+ d\bar{\mu} = \int_X (f \wedge g)^+ d\mu \, .$$

THEOREM 2.3. *If $f, g \in L^1(X, \mu)$, where $\mu(X) = a < \infty$, then*

$$\delta_f \wedge \iota_g \preccurlyeq f \wedge g \preccurlyeq \delta_f \wedge \delta_g \tag{2.3}$$

where the strong spectral inequality $f \wedge g \prec \delta_f \wedge \delta_g$ (respectively *$\delta_f \wedge \iota_g \prec f \wedge g$) holds if and only if $f \wedge g \sim \delta_f \wedge \delta_g$* (respectively *$\delta_f \wedge \iota_g \sim f \wedge g$).*

Let $\Phi: R \to R$ be any increasing and continuous function, then $\Phi(\delta_f \wedge \iota_g) \preccurlyeq \Phi(f \wedge g) \preccurlyeq \Phi(\delta_f \wedge \delta_g)$. If Φ is strictly increasing and convex such that $\Phi(\delta_f \wedge \delta_g) \in L^1([0, a], m)$ (respectively *$\Phi(f \wedge g) \in L^1(X, \mu)$), then $\Phi(f \wedge g) \prec \Phi(\delta_f \wedge \delta_g)$* (respectively *$\Phi(\delta_f \wedge \iota_g) \prec \Phi(f \wedge g)$) if and only if $f \wedge g \sim \delta_f \wedge \delta_g$* (respectively *$\delta_f \wedge \iota_g \sim f \wedge g$).*

Proof. The spectral inequality $\delta_f \wedge \iota_g \preccurlyeq f \wedge g$ follows immediately from the preceding lemma since, on substituting $f - t$, $g - t$ (where $t \in R$) for f, g respectively in (2.2), we have

$$\int_0^a (\delta_f \wedge \iota_g - t)^+ dm = \int_0^a (\delta_{f-t} \wedge \iota_{g-t})^+ dm \leqq \int_X [(f - t) \wedge (g - t)]^+ d\mu$$
$$= \int_X (f \wedge g - t)^+ d\mu \, .$$

The spectral inequality $f \wedge g \preccurlyeq \delta_f \wedge \delta_g$ is obvious since $\delta_{f \wedge g} \leqq \delta_f \wedge \delta_g$ by the monotonicity of the decreasing rearrangement operator δ. Thus, if $f \wedge g \prec \delta_f \wedge \delta_g$, then $\int_0^a \delta_{f \wedge g} dm = \int_0^a \delta_f \wedge \delta_g dm$ which is the case if and only if $\delta_{f \wedge g} = \delta_f \wedge \delta_g$ m-a.e. or, equivalently, $f \wedge g \sim \delta_f \wedge \delta_g$.

Suppose the strong spectral inequality $\delta_f \wedge \iota_g \prec f \wedge g$ holds. Let $\Psi: R \to R$ be any strictly concave and increasing function such that $\Psi(x) = \log x$ for x large enough and that $\Psi(x)$ approaches x asymptotically as $x \to -\infty$ (the existence of such a function Ψ is geometrically clear; if both f and g are non-negative, we simply choose $\Psi(x) = \log(1 + x)$, $x \geqq 0$). Then $\Psi(h)$ is integrable whenever $h \in L^1(X, \mu) \cup L^1([0, a], m)$ since $\Psi^+(h) = \log h^+ \leqq h^+$ for h^+ large enough and

$h^- - \varepsilon \leqq \Psi^-(h) \leqq h^- + \varepsilon$ for h^- large enough and for some $\varepsilon > 0$. By (1.3), we clearly have $\Psi(\delta_f \wedge \iota_g) = \Psi(\delta_f) \wedge \Psi(\iota_g) = \delta_{\psi(f)} \wedge \iota_{\psi(g)} \preccurlyeq \psi(f) \wedge \psi(g) = \psi(f \wedge g)$. Now it is easily seen that the inverse Φ of ψ is strictly convex and increasing, and so, by (1.2) and [1, Theorem 2.3], we have $\Phi(\Psi(\delta_f \wedge \iota_g)) = \delta_f \wedge \iota_g \prec f \wedge g = \Phi(\Psi(f \wedge g)$ if and only if $\delta_f \wedge \iota_g \sim f \wedge g$.

The rest is easy by virtue of the above result, (1.3) and [1, Theorems 2.3, 2.5 and 2.8].

COROLLARY 2.4. *If $f, g \in L^1(X, \mu)$ where $\mu(X) = a < \infty$, then*

$$-(\delta_f \vee \iota_g) \preccurlyeq -(f \vee g) \preccurlyeq -(\delta_f \vee \delta_g) \tag{2.4}$$

where the strong spectral inequality $-(\delta_f \vee \iota_g) \prec -(f \vee g)$ (respectively $-(f \vee g) \prec -(\delta_f \vee \delta_g)$) *holds if and only if* $\delta_f \vee \iota_g \sim f \vee g$ (respectively $f \vee g \sim \delta_f \vee \delta_g$).

Let $\Phi: R \to R$ be any nonincreasing and continuous function, then

$$\Phi(\delta_f \vee \iota_g) \prec \Phi(f \vee g) \prec \Phi(\delta_f \vee \delta_g) \,. \tag{2.5}$$

If Φ is strictly decreasing and convex such that $\Phi(\delta_f \vee \delta_g) \in L^1([0, a], m)$ (respectively $\Phi(f \vee g) \in L^1(X, \mu)$), *then* $\Phi(f \vee g) \prec (\delta_f \vee \delta_g)$ (respectively $\Phi(\delta_f \vee \iota_g) \prec \Phi(f \vee g)$) *if and only if* $f \vee g \sim \delta_f \vee \delta_g$ (respectively $\delta_f \vee \iota_g \sim f \vee g$).

Proof. The result follows immediately from Theorem 2.3 on substituting $-f$ for f and $-g$ for g.

It is now easy to derive the rearrangement inequalities of Jurkat and Ryser given in [4, Lemma 6.1, p. 353] (cf. [5, p. 498]) and also those of Minc given in [5, Theorems 3-5, pp. 501-502].

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UNIVERSITY OF MALAYA