PACIFIC JOURNAL OF MATHEMATICS
Vol. 59, No. 2, 1975

A CHARACTERIZATION OF THE SYMPLECTIC GROUPS $PSp(2m,q)$ AS RANK 3 PERMUTATION GROUPS

ARTHUR YANUSHKA

In this paper the following characterization of the symplectic groups $PSp(2m, q)$ for $m > 2$ as rank 3 permutation groups is obtained:

THEOREM. Let G be a transitive rank 3 group of permutations of a finite set X such that the orbit lengths for G_b, the stabilizer of a point b in X are $1, q(q^{r-2}-1)/(q-1)$ and q^{r-1} for integers $q > 1$ and $r > 4$. Let $b^\perp$ denote the union of b and the G_b orbit of length $q(q^{r-2}-1)/(q-1)$. Let $R(bc)$ denote $\cap\{z^\perp : b, c \in z^\perp\}$. Assume $R(bc) \neq \{b, c\}$, for all distinct pairs of points, b and c. Assume that the pointwise stabilizer of $b^\perp$ is transitive on the points unequal to b of $R(bc)$ for $c \notin b^\perp$. Then r is even, q is a prime power and $G \cong H$, a group of symplectic collineations of projective $r-1$ space over the finite field of q elements and $PSp(r, q) \trianglelefteq H$.

The rank of a transitive permutation group is the number of orbits of the stabilizer of a point. The projective classical groups of symplectic type $PSp(2m, q)$ for $m \geqq 2$ and for a prime power q are transitive groups of rank 3 when considered as groups of permutations of the absolute points of the corresponding projective space. Indeed, the pointwise stabilizer of $PSp(2m, q)$ has 3 orbits of lengths $1, q(q^{2m-2}-1)/(q-1)$ and q^{2m-1}.

Let G be any rank 3 group of permutations of a set X such that the pointwise stabilizer has orbit lengths of $1, q(q^{r-2}-1)/(q-1)$ and q^{r-1} for any integers $r \geqq 4$ and $q \geqq 2$. The characterization problem is to impose some restrictions on G and on X to force the conclusion that X is a projective space and that G is a group of symplectic collineations. Let $b^\perp$ denote the union of b and the G_b-orbit of length $q(q^{r-2}-1)/(q-1)$. There are several rank 3 characterizations of the symplectic groups. Assume that q is a prime power and that $r \geqq 6$. Kantor [5] proved that G can be regarded as an automorphism group of a symplectic geometry, acting on the set of singular points. Next assume that q is any integer, that $r = 4$ and that the pointwise stabilizer of $b^\perp$ contains at least q elements. D. G. Higman [4] proved that G can be regarded as a group of symplectic collineations of projective 3-space over the finite field of q elements and that G contains $PSp(4, q)$. Later Tsuzuku [6] extended Higman's theorem to $r \geqq 4$ under the additional assumption that q is a prime power. This paper essentially generalizes Higman's theorem to all higher dimensions, without the assumption that q is a prime power.

A brief outline of the proof follows. The assumption that the pointwise stabilizer in G of $b^\perp$ is transitive on the points unequal to b of the "hyperbolic line" $R(bc)$ for $c \notin b^\perp$ yields that G_{ab}, the stabilizer of the points a and b of X, is transitive on the points of $a^\perp \cap b^\perp - R(ab)$. This fact implies that G_a is a rank 3 permutation group on $\{R(ab): b \in a^\perp - a\}$, and the set of "totally singular lines" carry $q + 1$ points. We then show that X together with its totally singular lines forms a nondegenerate Shult space [1] of rank $\geqq 3$. Next we use a theorem of Buekenhout and Shult [1] to conclude that X is isomorphic to the set of points of a classical geometry of symplectic type. Therefore G is a group of symplectic collineations. Finally we show that the nontrivial elements of the pointwise stabilizer of $b^\perp$ correspond to symplectic elations with center b and that G contains $PSp(r, q)$.

In §2 we collect the necessary facts about rank 3 groups from the basic papers of D. G. Higman [3], [4]. We refer the reader to a paper of Buekenhout and Shult [1] for the definition of Shult space and a brief introduction to polar spaces. In §3 we prove the characterization theorem. Finally the author wishes to thank Donald Higman for making him aware of the work of Buekenhout and Shult whose theorem makes the proof of the characterization of $PSp(2m, q)$ given here considerably shorter than the original version.

2. Rank 3 permutation groups.

In this section we collect the necessary facts about rank 3 permutation groups which will be used in the proof of the characterization theorem.

Let G be a finite transitive group of permutations of a finite set X. Then the *rank* of G is the number of orbits of the stabilizer of a point. Rank 3 means that for $b \in X$ the stabilizer of b, G_b has exactly 3 orbits on X, denoted b, $D(b)$ and $C(b)$. Choose the notation in such a way that $g(D(a)) = D(g(a))$ and $g(C(a)) = C(g(a))$ for all $a \in X$, $g \in G$. Let $|Y|$ denote the number of elements in a set Y. Set

$$|X| = n,\ |D(b)| = k \text{ and } |C(b)| = l$$

so that $n = 1 + k + l$. Set

$$|D(a) \cap D(b)| = \begin{cases} \lambda & \text{for } b \in D(a) \\ \mu & \text{for } b \in C(a) \end{cases}$$

The *parameters* of G are the triple (n, k, l).

LEMMA 2.1. *Let G be a rank 3 permutation group. Then*

(i) $\mu l = k(k - \lambda - 1)$.

(ii) *G is primitive iff $0 < \mu < k$. If G is primitive, then $(l, k) > 1$ where (l, k) denotes the greatest common divisor of and k.*

(iii) *IfG is imprimitive, then either $(l + 1) | k$ or $(k + 1) | l$ where $a | b$ denotes that a divides.*

(iv) *If $|G|$ is odd, $k = l$.*

(v) *If $|G|$ is even, then $D(\lambda - \mu)^2 + 4(k + \mu)$ is a square.*

(vi) *If $|G|$ is even, then $a \in D(b)$ iff $b \in D(a)$.*

Proof. See [3] and [4].

Assume $|G|$ is even. Define the "lines" of X as follows: for $a \neq b$ in X define

$$R(ab) = \bigcap \{z^\perp : a, b \in z^\perp\}$$

where $z^\perp = z \cup D(z)$. Call $R(ab)$ *totally singular* (resp. *hyperbolic*) if $a \in b^\perp$ (resp. $a \notin b^\perp$).

LEMMA 2.2. *Let G be a rank 3 group of even order. Then*

(i) *$g(R(ab)) = R(g(a)g(b))$ for all $a, b \in X$, $g \in G$.*

(ii) *If $x \in R(ab)$ and $x \neq a$, then $R(ax) = R(ab)$ if $b \in D(a)$ or if $b \in C(a)$ and $\mu > \lambda + 1$.*

(iii) *$x \in R(ab) - \{a\}$ iff $a^\perp \cap x^\perp = a^\perp \cap b^\perp$.*

(iv) *$|R(ab)| - 1$ divides k if $b \in D(a)$.*

Proof. See [4].

Let $T(a)$ denote the pointwise stabilizer of $a^\perp$. Then $T(a)$ is a normal subgroup of G_a.

LEMMA 2.3. *Let G be a primitive rank 3 group of even order such that $\mu > \lambda + 1$. Then*

(i) *$T(a)$ fixes all lines through a.*

(ii) *$T(a)_x = 1$ for $x \in C(a)$.*

(iii) *$|T(a)|$ divides $|R(ab)| - 1$, if $b \in C(a)$.*

NOTATION. If $Y \subseteq X$, let G_Y denote the global stabilizer of Y. If $Y, Z \subseteq X$, then $G_{Y,Z}$ denotes $G_Y \cap G_Z$.

If $Y \subseteq X$, let $X - Y$ denote the set of elements of X which do not belong to Y.

For a natural number r, let v_r denote $(q^r - 1)/(q - 1)$.

3. The proof of the theorem. We now begin the proof of the characterization theorem. Assume that G is a rank 3 permutation group of a set X which satisfies the hypotheses of the theorem.

LEMMA 3.1. (i) *G is primitive of even order.*
(ii) $\mu = \lambda + 2 = v_{r-2}$.
(iii) $a^\perp \cap b^\perp \neq R(ab)$ *for* $b \in D(a)$.

Proof. (i) Assume G is imprimitive. By Lemma 2.1 (iii) either $(k+1)|l$ or $(l+1)|k$. The first case does not occur because $k+1 = v_{r-1}$, $l = q^{r-1}$ and $(v_{r-1}, q^{r-1}) = 1$. The second case does not occur because $l + 1 > k$. So G is primitive. Since $k \neq l$, $|G|$ is even by Lemma 2.1 (iv).

(ii) By Lemma 2.1 (i), $\mu q^{r-1} = qv_{r-2}(qv_{r-2} - \lambda - 1)$. By Lemma 2.1 (ii), $\mu > 0$. Since $(q^{r-2}, v_{r-2}) = 1$, there is a natural number t such that $v_{r-2}t = \mu$. So $\lambda + 1 = q(v_{r-2} - tq^{r-3})$ and $v_{r-2} - tq^{r-3} \geqq 1$. If $t > 1$, then

$$v_{r-2} - 1 = qv_{r-3} \geqq tq^{r-3} \geqq 2q^{r-3}$$

which implies $2q^{r-4} \geqq q^{r-3} + 1$, a contradiction because $q \geqq 2$. So $t = 1$, $\mu = v_{r-2}$ and $\lambda - 1 = qv_{r-3}$.

(iii) Assume $a^\perp \cap b^\perp = R(ab)$ for $b \in D(a)$. Let $|R(ab)| = s + 1$. So $\lambda + 2 = s + 1 = \mu$. Since $s|k = q\mu$ by Lemma 2.2 (iv) and $(q, \mu) = 1$, there is a natural number t such that $st = q$. Then $\mu - 1 = qv_{r-3} = s$ implies $tv_{r-3} = 1$ and $r = 4$, a contradiction. This completes the proof of the lemma.

LEMMA 3.2. (i) $|a^\perp \cap C(b)| = q^{r-2}$ *for* $b \in D(a)$.
(ii) *G_{ab} is transitive on the points of $a^\perp \cap C(b)$ for $b \in D(a)$.*

Proof. (i) Since $a^\perp \cap C(b) = a^\perp - (a^\perp \cap b^\perp)$, by Proposition 3.1 (iii) $|a^\perp \cap C(b)| = k + 1 - (\lambda + 2) = q^{r-2}$.

(ii) Let $b \in D(a)$ and let $d \in a^\perp \cap C(b)$. Now

$$\begin{aligned} qv_{r-2} \cdot |G_{ab}: G_{abd}| &= |G_b: G_{ab}| \cdot |G_{ab}: G_{abd}| = |G_b: G_{bd}|\,|G_{bd}: G_{abd}| \\ &= q^{r-1} \cdot |G_{bd}: G_{abd}| \,. \end{aligned}$$

Let $x = |G_{ab}: G_{abd}|$. Since $(v_{r-2}, q^{r-2}) = 1$, it follows that $q^{r-2}|x$. But $x \leqq q^{r-2}$ because

$$d^{G_{ab}} \subseteqq a^\perp \cap C(b) \,.$$

So $x = q^{r-2}$ and the proof is complete.

PROPOSITION 3.3. *G_{ab} is transitive on the points of $a^\perp \cap b^\perp - R(ab)$ for $b \in D(a)$.*

Proof. Let c and e be distinct points of $a^\perp \cap b^\perp - R(ab)$. Since

$c \notin R(ab)$, by Lemma 2.2 (iii) $c^\perp \not\supseteq a^\perp \cap b^\perp$. There is $u \in a^\perp \cap b^\perp \cap C(c)$. Since $e \notin R(ab)$, there is $v \in a^\perp \cap b^\perp \cap C(e)$. There are 4 possible cases to consider: (1) $u \in C(e)$, (2) $v \in C(c)$, (3) $u = e$ or $v = c$ and (4) $u \in D(e)$ and $v \in D(c)$.

(1) $u \in a^\perp \cap b^\perp \cap C(c) \cap C(e)$. Since $|R(uc)| > 2$, there is $y \in R(uc) - \{u, c\}$. By Proposition 3.1 (ii) and Lemma 2.2 (ii) it follows that $R(yc) = R(uc) \subseteqq a^\perp \cap b^\perp$. Because $R(uc)$ is a hyperbolic line and $T(y)$ is transitive on the points unequal to y of $R(yc)$, there exists t in $T(y)$ such that $t(c) = u$. Since $a, b \in y^\perp$, it follows that $t \in G_{ab}$. Similarly there is $z \in R(ue) - \{u, e\}$ and then $R(ze) = R(ue) \subseteqq a^\perp \cap b^\perp$. Because $R(ue)$ is a hyperbolic line and $T(z)$ is transitive on the points unequal to of z $R(ze)$, there exists s in $T(z) \leqq G_{ab}$ such that $s(u) = e$. Thus $st(c) = e$ and $st \in G_{ab}$.

(2) $v \in a^\perp \cap b^\perp \cap C(c) \cap C(e)$. This case has a proof similar to that of case (1).

(3) If $u = e$ or $v = c$, then $R(ce)$ is a hyperbolic line in $a^\perp \cap b^\perp$. Pick $z \in R(c, e) - \{c, e\}$. There exists t in $T(z) \leqq G_{ab}$ such that $t(c) = e$.

(4) $u \in a^\perp \cap b^\perp \cap C(c) \cap D(e)$ and $v \in a^\perp \cap b^\perp \cap D(c) \cap C(e)$. Since $|R(ce)| > 2$, there is $w \in R(ce) - \{c, e\}$. Note that $w \in C(u)$, for if $w \in u^\perp$, then $c \in R(ce) = R(we) \subseteqq u^\perp$, a contradiction. Now $w \in R(ce) \subseteqq a^\perp \cap b^\perp$. But $w \notin R(ab)$ because $u \in a^\perp \cap b^\perp \cap C(w) \cap C(c)$. By case (1) there exists $g \in G_{ab}$ such that $g(c) = w$. Note that $w \in C(v)$ for if $w \in v^\perp$, then $e \in R(ce) = R(we) \subseteqq v^\perp$, a contradiction. Now $v \in a^\perp \cap b^\perp \cap C(w) \cap C(e)$. By case (1) there exists $h \in G_{ab}$ such that $h(w) = e$. So $hg(c) = e$ and $hg \in G_{ab}$. This completes the proof of the proposition.

PROPOSITION 3.4. *The group G_a is a rank 3 permutation group on the set of totally singular lines through a.*

Proof. Clearly G_a is transitive on the set of totally singular lines through a since $D(a)$ is an orbit of G_a. For $b \in D(a)$ define the sets $D(R(ab))$ and $C(R(ab))$ as follows:

$$D(R(ab)) = \{R(ac) : c \in a^\perp \cap b^\perp - R(ab)\}$$
$$C(R(ab)) = \{R(ac) : c \in a^\perp \cap C(b)\} .$$

We claim that these sets are well-defined, form a partition of the set of totally singular lines through a unequal to $R(ab)$ and are nontrivial orbits of $G_{aR(ab)}$.

These sets are well-defined. Indeed suppose $R(ab) = R(ad)$ for $b, d \in D(a)$. By Lemma 2.2 (iii), $a^\perp \cap b^\perp = a^\perp \cap d^\perp$ and so $a^\perp \cap C(b) = a^\perp \cap C(d)$. Thus $D((ab)) = D(R(ad))$ and $C(R(ab)) = C(R(ad))$.

Let $R(az)$ be a totally singular line. Either $z \in C(b)$ in which case $R(az) \in C(R(ab))$ or $z \in b^\perp$. If $z \in b^\perp$, then either $z \in R(ab)$ in which case $R(az) = R(ab)$ or $z \notin R(ab)$ in which case $R(az) \in D(R(ab))$. Thus

$$R(ab) \cup D(R(ab)) \cup C(R(ab))$$

is a partition of the set of totally singular lines through a.

If $R(ac)$ and $R(ae)$ belong to $D(R(ab))$, then c and e are elements of $a^\perp \cap b^\perp - R(ab)$. By Proposition 3.3 there is $g \in G_{ab}$ such that $g(c) = e$. So $g(R(ab)) = R(ab)$ and $g(R(ac)) = R(ae)$. Thus $D(R(ab))$ is an orbit of $G_{aR(ab)}$.

If $R(ac)$ and $R(ae)$ belong to $C(R(ab))$, then c and e are elements of $a^\perp \cap C(b)$. By Lemma 3.2 (ii) there is $g \in G_{ab}$ such that $g(c) = e$. Thus $C(R(ab))$ is an orbit of $G_{aR(ab)}$ and G_a is a rank 3 group on the set of totally singular lines through a, as desired.

PROPOSITION 3.5. *Totally singular lines carry $q + 1$ points.*

Proof. Let $|R(ab)| = s + 1$. We will show that $s = q$ by determining the rank 3 parameters of G_a on the set of totally singular lines through a. Let $k_2 = |D(R(ab))|$ and $l_2 = |C(R(ab))|$. Then by Lemma 3.1

$$k_2 = (\lambda + 2 - (s + 1))/s = (qv_{r-3}/s) - 1$$

and

$$l_2 = (k + 1 - (\lambda + 2))/s = q^{r-2}/s .$$

So there is a natural number t such that $st = q$. We claim that $t = 1$.

Now G_a is a rank 3 group with $k_2 = tv_{r-3} - 1$ and $l_2 = tq^{r-3}$. We claim that G_a is primitive. If G_a is imprimitive, then by Lemma 2.1 (iii) either $k_2 + 1 = tv_{r-3}$ divides $l_2 = tq^{r-3}$, a contradiction since $(v_{r-3}, q^{r-3}) = 1$ or $l_2 + 1$ divides k_2, a contradiction since $k_2 < l_2 + 1$. So G_a is primitive. By Lemma 2.1 (ii), $(k_2, l_2) > 1$. Let $z = (k_2, l_2)$. We claim that $z = q + t - 1$.

Since $(1 - q)v_{r-3} + q^{r-3} = 1$, it follows that $(1 - q)k_2 + l_2 = q + t - 1$ and that there is a natural number u such that $zu = q + t - 1$. By Lemma 2.1 (i) $\mu_2 l_2 = k_2(k_2 - \lambda_2 - 1)$. So there is a natural number w such that $wk_2/z = \mu_2$. Then $wl_2/z = k_2 - \lambda_2 - 1$ and

$$2 \leqq \lambda_2 + 2 = tv_{r-3} - tq^{r-3}w/z .$$

Now $(z, t) = 1$ for if the prime $p \mid (z, t)$, then $p \mid k_2 = tv_{r-3} - 1$ and $p \mid t$, a contradiction. So $v_{r-3} - q^{r-3}w/z$ is a natural number. From the substitution of $z = (q + t - 1)/u$ into $v_{r-3} - q^{r-3}w/z \geqq 1$, it follows that

$$(uw - 1)q^{r-4} + 1 \leqq tv_{r-4} \leqq (q - 1)v_{r-4}$$

because $t \leqq q - 1$ as $s > 1$. Then $0 \geqq (uw - 2)q^{r-4} + 2$ which forces $u = w = 1$. So $z = q + t - 1$, $\mu_2 = k_2/z$ and $\lambda_2 + 2 = t\mu_2$.

Now $|G_a|$ is even. For if $|G_a|$ is odd, then $k_2 = l_2$ and $tv_{r-3} - 1 = tq_{r-3}$, which is impossible. By Lemma 2.1 (v)

$$\begin{aligned} D &= (t\mu_2 - 2 - \mu_2)^2 + 4(z\mu_2 - \mu_2) = (t - 1)^2\mu_2^2 + 4(q - 1)\mu_2 + 4 \\ &= ((t - 1)\mu_2 + b + 2)^2 \end{aligned}$$

for some nonnegative integer b. If $b = 0$, then $t = q$ and $s = 1$, a contradiction. So $b \geqq 1$ and

$$4(q - 1)\mu_2 + 4 = 2(t - 1)\mu_2(b + 2) + (b + 2)^2$$

implies $b = 2c$ for some natural number c. It follows that

$$((q - 1) - (t - 1)(c + 1))\mu_2 = c(c + 2) .$$

Assume $t > 1$. Since $(q - 1) - (t - 1)(c + 1) > 0$, it follows that $q \geqq (t - 1)c + t \geqq c + 2$ and $\mu_2 \leqq c(c + 2) < q^2$. But

$$\mu_2 = k_2/z \geqq (2v_{r-3} - 1)/2q .$$

If $r \geqq 7$, then $\mu_2 \geqq (2v_4 - 1)/2q > q^2$, a contradiction.

If $r = 5$, then $\mu_2 = t - (t - 1)^2/(q + t - 1)$. Since $t > 1$, there is a natural number f such that $(q + t - 1)f = (t - 1)^2$. Since $q = st$, it follows that $stf = (t - 1)(t - 1 - f)$ and that $t|(t - 1 - f)$, a contradiction.

If $r = 6$, then

$$\mu_2 = (tv_3 - 1)/(q + t - 1) = (t - 1)q^2/(q + t - 1) + q + 1 .$$

Note $t > 2$ for $t = 2$ implies $q^2/(q + 1)$ is a natural number. So $\mu_2 \geqq 2q^2/2q + q + 1 = 2q + 1$. Since μ_2 divides $c(c + 2)$ and $(c, c + 2) = 1$ or 2, it follows that $\mu_2 \leqq 2(c + 2)$. But $c + 2 \leqq q$ and so $\mu_2 \leqq 2q$, a contradiction. Therefore $t = 1$ and $s = q$ for all $r \geqq 5$.

LEMMA 3.6. *If $b \in D(a)$, then $X = \bigcup \{c^\perp : c \in R(ab)\}$.*

Proof. We know $|R(ab)| = q + 1$. Let $R(ab) = \{d_1, d_2, \cdots, d_{q+1}\}$. Let $R = \bigcup\{d_i^\perp : 1 \leqq i \leqq q + 1\}$. Express R as a pairwise disjoint union of $q + 1$ subsets of X.

$$R = d_1^\perp \cup \bigcup_{i=2}^{q+1} \left(d_i^\perp \cap \bigcap_{j=1}^{i-1} C(d_j)\right) .$$

We claim that

$$d_i^\perp \cap \bigcap_{j=1}^{i-1} C(d_j) = d_i^\perp \cap C(d_1) .$$

This is true for $i = 2$. Let $i > 2$. Then

$$d_i^\perp \cap C(d_1) = \left(d_i^\perp \cap C(d_1) \cap \bigcap_{j=2}^{i-1} C(d_j)\right) \cup \left(d_i^\perp \cap C(d_1) \cap \left(\bigcup_{j=2}^{i-1} d_j^\perp\right)\right).$$

Suppose there is $x \in d_i^\perp \cap C(d_1) \cap (\bigcup_{j=2}^{i-1} d_j^\perp)$. Then $x \in d_i^\perp \cap d_j^\perp \cap C(d_1)$ for $i \neq j$. So $d_1 \in R(d_i d_j) \subseteqq x^\perp$, a contradiction. So

$$d_i^\perp \cap C(d_1) \cap \left(\bigcup_{j=2}^{i-1} d_j^\perp\right) = \varnothing$$

and the claim holds for $2 \leqq i \leqq q + 1$. So

$$R = d_1^\perp \cup \bigcup_{i=2}^{q+1} \left(d_i^\perp \cap C(d_1)\right)$$

and this union is pairwise disjoint. Now

$$|R| = k + 1 + q(k + 1 - (\lambda + 2)) = v_r = |X|.$$

Thus $R = X$ and the proof of the lemma is complete.

PROPOSITION 3.7. *X together with the totally singular lines of X forms a nondegenerate Shult space of finite rank $\geqq 3$ in which lines carry $q + 1$ points.*

Proof. It suffices to show that if $x \notin R(ab)$ for $b \in D(a)$, then x is adjacent to either one point or all points of $R(ab)$. By definition two distinct points are adjacent if they determine a totally singular line. By Lemma 3.6, there exists $c \in R(ab)$ such that $x \in c^\perp$. If $x \in d^\perp$ for $d \in R(ab) - \{c\}$, then $R(ab) = R(cd) \subseteqq x^\perp$ and $x \in e^\perp$ for all $e \in R(ab)$. Thus X is a nondegenerate Shult space in which lines carry $q + 1$ points.

It remains to show that X has rank $\geqq 3$. For $b \in D(a)$, there is $c \in a^\perp \cap b^\perp - R(ab)$ by Lemma 3.1 (iii). Define the "plane" $R(abc)$ by

$$R(abc) = \bigcap \{z^\perp : a, b, c \in z^\perp\}.$$

We claim that $R(abc)$ is a subspace of the Shult space X. If so, then X has rank $\geqq 3$ since

$$a \subset R(ab) \subset R(abc)$$

is a chain of subspaces of X. To prove that $R(abc)$ is a subspace, we need the following lemma.

LEMMA 3.8. *$w \in R(abc)$ iff $w^\perp \supseteqq a^\perp \cap b^\perp \cap c^\perp$.*

Proof. Let $w \in R(abc)$. If $u \in a^\perp \cap b^\perp \cap c^\perp$, then $a, b, c \in u^\perp$ and $w \in u^\perp$ since $w \in R(abc)$. So $u \in w^\perp$ and $a^\perp \cap b^\perp \cap c^\perp \subseteqq w^\perp$. Conversely, assume $a^\perp \cap b^\perp \cap c^\perp \subseteqq w^\perp$. Let $a, b, c \in z^\perp$. Then $z \in a^\perp \cap b^\perp \cap c^\perp \subseteqq w^\perp$ and $w \in z^\perp$. By definition of "plane," $w \in R(abc)$ and the lemma is proved.

By definition $R(abc)$ is a subspace if any two points of $R(abc)$ are adjacent and if any line meeting $R(abc)$ in more than one point is contained in $R(abc)$. Let $d, e \in R(abc)$. Since $a, b, c \in a^\perp \cap b^\perp \cap c^\perp$, it follows that $R(abc) \subseteqq a^\perp \cap b^\perp \cap c^\perp$. By Lemma 3.8.

$$d \in R(ab) \subseteqq a^\perp \cap b^\perp \cap c^\perp \subseteqq e^\perp .$$

So any two points of $R(abc)$ are adjacent. Let the line $R(xy)$ meet $R(abc)$ in $\{u, v\}$. Then $R(xy) = R(uv)$ and $x^\perp \cap y^\perp = u^\perp \cap v^\perp$ by Lemma 2.2. If $z \in R(xy)$, then

$$z^\perp \supseteqq x^\perp \cap y^\perp = u^\perp \cap v^\perp \supseteqq a^\perp \cap b^\perp \cap c^\perp$$

since $u, v \in R(abc)$. By Lemma 3.8, $z \in R(abc)$. Thus $R(xy) \subseteqq R(abc)$ and $R(abc)$ is a subspace of the Shult space X, as desired.

PROPOSITION 3.9. (i) *q is a prime power and r is even.*

(ii) *Either X is isomorphic to the polar space S associated with an alternating form f defined on a projective space P of dimension $r-1$ over $GF(q)$ or X is isomorphic to the polar space S associated with a symmetric form f defined on a projective space P of dimension r over $GF(q)$ for q odd.*

Proof. By Proposition 3.7 and Theorem 4 of Buekenhout and Shult [1], X is a polar space of rank $\geqq 3$ in which lines carry $q + 1 \geqq 3$ points. Since $|X| = v_r$ is finite, by Theorem 1 of Buekenhout and Shult [1], X is isomorphic to the set of singular points of a classical symplectic, unitary or orthogonal geometry. Because a line of X carries $q + 1$ points and corresponds to a totally singular line of a classical geometry, it follows that q is a prime power. Note that $|X| = v_r$ equals the number of singular points of a classical geometry. It follows that either the geometry is symplectic or orthogonal and that $r = 2m$ for some $m \geqq 3$ since X has rank $\geqq 3$. Statement (ii) now follows.

PROPOSITION 3.10. (i) *G is isomorphic to a subgroup of $P\Gamma U(f)$, the group of collineations of P which preserve the form f.*

(ii) *For $x \in X$, $\varphi(x^\perp) = \{w \in P\colon f(w, w) = 0, f(w, \varphi(x)) = 0\}$ where $\varphi\colon X \to S$ is a polar space isomorphism.*

(iii) *For $x, y \in X$, $\varphi(R(x, y))$ is the set of singular points of the projective line determined by $\varphi(x)$ and $\varphi(y)$.*

(iv) *X is isomorphic to a symplectic geometry.*

Proof. (i) The group G is a subgroup of the group of automorphisms of the polar space X, which we denote by $\mathrm{Aut}(X)$. If $\varphi: X \to S$ is a polar space isomorphism, then define a map

$$\psi: \mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(S) \quad \text{by}$$
$$\psi(s) = \varphi s \varphi^{-1}$$

for $s \in \mathrm{Aut}(X)$. It follows that ψ is a group isomorphism. Now $P\Gamma U(f) \cong \mathrm{Aut}(S)$ by a natural map defined by

$$P\Gamma U \longrightarrow \mathrm{Aut}(S)$$
$$u \longrightarrow \text{the restriction of } u \text{ to } S\,.$$

See Dieudonné [2] pp. 82–84. So $\psi(G)$ is a subgroup of $P\Gamma U(f)$.

(ii) This statement claims that $\varphi(x \cup D(x))$ is the hyperplane of singular points of S which are perpendicular to $\varphi(x)$. Denote this hyperplane by $\varphi(x)^{\perp}$, where $^{\perp}$ is the polarity determined by the form f.

Since $x^{\perp} = \bigcup\{R(xb): b \in D(x)\}$, it follows that

$$\varphi(x^{\perp}) = \bigcup\{\varphi(R(xb)): b \in D(x)\} \subseteqq \varphi(x)^{\perp}$$

because $\varphi(R(xb))$ is a totally singular line of P. So $\varphi(x^{\perp}) \subseteqq \varphi(x)^{\perp}$.

Conversely for $z \in \varphi(x)^{\perp}$, there exists $b \in X$ such that $z = \varphi(b)$ and $\varphi(b)^{\perp}\varphi(x)$. Suppose $b \notin x^{\perp}$. Then $b \in C(x)$, an orbit of G_x. For $c \in C(x)$ there exists $g \in G_x$ such that $g(b) = c$. Then $\psi(g) \in \mathrm{Aut}(S)$ and $\psi(g)$ preserves the polarity $^{\perp}$. Since $\varphi(b)^{\perp}\varphi(x)$, it follows that $(\psi(g)(\varphi(b))^{\perp}(\psi(g))(\varphi(x))$ and $\varphi(c)^{\perp}\varphi(x)$. So $\varphi(c) \in \varphi(x)^{\perp}$ for all $c \in C(x)$. Since $\varphi(x^{\perp}) \subseteqq \varphi(x)^{\perp}$, it follows that $\varphi(X) = S \subseteqq \varphi(x)^{\perp}$, a contradiction. Thus $b \in x^{\perp}$, $\varphi(b) = z \in \varphi(x^{\perp})$ and $\varphi(x)^{\perp} \subseteqq \varphi(x^{\perp})$.

(iii) Since $R(xy) = \bigcap\{u^{\perp}: x, y \in y^{\perp}\}$, it follows from (ii) that

$$\varphi(R(xy)) = \bigcap\{v^{\perp}: v \in S \quad \text{and} \quad \varphi(x), \varphi(y) \in v^{\perp}\}\,.$$

So $\varphi(R(xy))$ is the set of singular points of the projective line determined by $\varphi(x)$ and $\varphi(y)$.

(iv) Assume X is an orthogonal geometry. If $y \in C(x)$, then $\varphi(R(xy))$ is a hyperbolic line in an orthogonal geometry and so carries just 2 singular points. But $|\varphi(R(xy))| = |R(xy)| > 2$, by hypothesis of the theorem. This contradiction shows that X must be a symplectic geometry. So the proposition is established.

PROPOSITION 3.11. (i) *The nontrivial elements of $T(x)$ correspond to elations of P.*

(ii) *$\psi(G)$ contains $PSp(2m, q)$ as a normal subgroup.*

Proof. (i) Because X is symplectic, all points of P are singular and $S = P$. Because $|R(ax)| > 2$ by Lemma 2.1 (iii), there exists a nontrivial element t of $T(x)$. Then t fixes $x^\perp$ pointwise and t fixes no point outside $x^\perp$ by Lemma 2.3 (ii). It easily follows from Proposition 3.10 (ii) that $\psi(t)$ fixes the hyperplane $\varphi(x)^\perp$ pointwise and $\psi(t)$ fixes no point outside this hyperplane. Thus $\psi(t)$ is an elation of P. Since $|T(x)| \,|\, (|R(xy)| - 1)$ for $y \in C(x)$, since hyperbolic lines of S carry $q + 1$ points and since $T(x)$ is transitive on $R(xy) - \{x\}$, it follows that $|T(x)| = q$.

(ii) $\psi(G)$ contains q elations for each point v of P. Since these elations generate $PSp(2m, q)$, (ii) holds.

References

1. F. Buekenhout and E. Shult, *On the foundations of polar geometry*, Geometriae Dedicata, **3** (1974), 155-170.
2. J. Dieudonné, *La géomêtrie des groupes classiques*, second edition, Springer-Verlag, Berlin, 1963.
3. M. Hestenes and D. Higman, *Rank* 3 *subgroups and strongly regular graphs*, Computers in number theory and algebra, SIAM-AMS Proceedings, American Mathematical Society, Providence, 1971.
4. D. Higman, *Finite permutation groups of rank* 3, Math. Z., **86** (1964), 145-156.
5. W. Kantor, *Rank* 3 *characterizations of classical geometries*, to appear.
6. T. Tsuzuku, *On a prbolem of D. G. Higman*, to appear.

Received April 21, 1975.

University of Michigan