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ON DOUBLY HOMOGENEOUS ALGEBRAS

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The algebras to be discussed are assumed to be finite dimensional and not necessarily associative. If A is an algebra over a field K let $\mathrm{Aut}(A)$ denote the group of algebra automorphisms of A. We define A to be doubly homogeneous if $\mathrm{Aut}(A)$ is doubly transitive on the one-dimensional subspaces of A. Also a doubly homogeneous algebra A is said to be nontrivial if $A^2 \neq 0$ and dimension $A > 1$. It is shown that the only nontrivial doubly homogeneous algebra is unique up to isomorphism.

An algebra A is said to be homogeneous if $\mathrm{Aut}(A)$ acts transitively on the one-dimensional subspaces of A. The reader is referred to the author's previous paper [1] for a discussion of homogeneous algebras and a bibliography of the related literature.

An arbitrary algebra A is said to be nonzero if $A^2 \neq 0$. If the nonzero elements of A form a quasi-group under multiplication then we say that A is a quasi-division algebra.

LEMMA. *If A is a nonzero doubly homogeneous algebra over a field K then A is a quasi-division algebra.*

Proof. Let $\dim A = n$. If $n = 1$ then A is isomorphic to K and the result is obvious and so we assume that $n > 1$. Let a be any element of A. We claim that if $b \notin Ka$ then $ab \neq 0$. For if $ab = 0$ the doubly homogeneity condition implies that $ac = 0$ for all c such that $c \notin Ka$. But then in particular $b + a \notin Ka$ and so $a(b + a) = 0$ which implies that $a^2 = 0$ and thus $aA = 0$. In this case the homogeneity condition implies that $A^2 = 0$ which is a contradiction and the claim is verified.

Now suppose that $a^2 = 0$. Then the homogeneity condition implies that $x^2 = 0$ for all $x \in A$. Suppose there exists $b \notin Ka$ such that

$$ab \in Ka .$$

Then by doubly homogeneity we would also have

$$(a + b)b \in K(a + b)$$

and $b^2 = 0$ implies that

$$ab \in Ka \cap K(a + b) = \{0\}$$

which is impossible. Fix some $b \notin Ka$. Let c be any nonzero element of A. Then there must exist $\alpha \in \operatorname{Aut}(A)$ such that

$$\alpha(ab) \in Kc$$

and

$$\alpha(a) \in Ka .$$

This implies that L_a (left multiplication by a) is a surjective map which is impossible and so $a^2 \neq 0$. Hence L_a is invertible and the homogeneity condition implies that A is a quasi-division algebra.

THEOREM. *If A is a nonzero doubly homogeneous algebra over a field K then either $A \cong K$ or $K = GF(2)$ and A is isomorphic to the following algebra*

	a	b
a	a	$a+b$
b	$a+b$	b

Proof. If $\dim A = 1$ then clearly $A \cong K$. If $\dim A = 2$ then A must be contained in the authors list of 2-dimensional homogeneous algebras [1] and it is easily checked that the only possibility is that $K = GF(2)$ and A is isomorphic to the following algebra

	a	b
a	a	$a+b$
b	$a+b$	b

Hence to prove the theorem it is sufficient to show that there exist no nonzero doubly homogeneous algebras of dimension $n > 2$.

Let A be a nonzero doubly homogeneous algebra of dimension $n > 2$. If a is any fixed nonzero element in A then the lemma implies that the equation $ax = a$ must have a unique solution, say b and the doubly homogeneity condition now implies that $b \in Ka$. It follows that A is a nonzero, power-associative, homogeneous algebra and so Theorem 7 of the author's previous paper [1] implies that $K = GF(2)$.

Now let a and b be any two distinct nonzero elements of A and let $A_1 = \langle a, b\rangle$ be the subalgebra of A generated by a and b. It can be shown that A_1 is also a doubly homogeneous algebra and it is generated by any two distinct nonzero elements. Hence only the identity automorphism of A_1 can fix two distinct nonzero elements of A_1 and so $\operatorname{Aut}(A_1)$ is sharply doubly transitive on $A_1 \backslash \{0\}$. Hence the order of $\operatorname{Aut}(A_1)$ must be even and so $\operatorname{Aut}(A_1)$ must contain at least one involution, say α. This involution α fixes at most 1 one-dimension subspace of A_1. But since any involution acting on a vector space V over a field of characteristic 2 fixes vectorwise a subspace of dimension $\geqq 1/2 \dim V$ this forces $\dim A_1 = 2$ and so we may assume that

$$ab = a + b .$$

But since A is doubly homogeneous it follows that

$$x^2 = x \qquad \text{for all } x \in A$$
$$xy = x + y \qquad \text{whenever } y \notin Kx .$$

Now since $n > 2$ we can choose three independent vectors $a, b, c \in A$. But then

$$(a + b)c = a + b + c$$

and

$$ac + bc = a + c + b + c = a + b$$

which is impossible and the proof is complete.

Reference

1. L. G. Sweet, *On homogeneous algebras*, (the previous paper).

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