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ON HOMOGENEOUS ALGEBRAS

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If A is an algebra over a field K let $\mathrm{Aut}(A)$ denote the group of algebra automorphisms of A. Then A is said to be extremely homogeneous if $\mathrm{Aut}(A)$ act transitively on $A\backslash\{0\}$. Also A is said to be homogeneous if $\mathrm{Aut}(A)$ acts transitively on the one-dimensional subspaces of A. The purpose of this paper is to investigate some of the basic properties of homogeneous algebras. In particular, the alternative homogeneous algebras and the homogeneous algebras of dimension 2 are classified.

All algebras are assumed to be finite dimensional and not necessarily associative.

We now include a brief historical account of this topic. The concept of an extremely homogeneous algebra arose from a particular problem in the structure of certain finite p-groups as studied by Boen, Rothaus and Thompson [1]. Extremely homogeneous algebras have been investigated by Kostrikin [4]. Homogeneous algebras over finite fields other than $GF(2)$ have been investigated by Shult [6], [7], and his results completed the work on the related p-groups. The case of homogeneous algebras over $GF(2)$ was considered by Gross [3]. Swierczkowski classified all real homogeneous Lie algebras [9] and finally Dyokovic classified all real homogeneous algebras [2]. A homogeneous algebra A is said to be nontrivial if $A^2 \neq 0$ and $\dim A > 1$. The author has shown that there are no nontrivial homogeneous algebras over an algebraically closed field [8].

The paper is divided into five sections: arbitrary homogeneous algebras, alternative homogeneous algebras, power-associative homogeneous algebras, homogeneous quasi-division algebras and finally homogeneous algebras of dimension 2.

I. Arbitrary homogeneous algebras.

Let A be an arbitrary algebra over a field K. Then left multiplication by a fixed element $a \in A$ induces a linear map on A which is denoted by L_a. Similarly right multiplication by a induces a linear map on A denoted by R_a. We do not distinguish between the map L_a and its matrix representation relative to some fixed basis. By $\mathrm{End}(A)$ we indicate the vector space of all linear maps on A. By L we indicate the subspace of $\mathrm{End}(A)$ consisting of all L_x as x runs through A and similarly for R. An algebra A is said to be nonzero if $A^2 \neq 0$.

THEOREM 1. *Let A be a nonzero homogeneous algebra over a* *field K. Then*

(i) $\dim L = \dim R = \dim A$

(ii) *If $a, b \in A \backslash \{0\}$ then L_a and L_a are projectively similar and similarly for R_a and R_b,*

(iii) $\mathrm{Aut}(A)$ *acts as a transitive group of collineations on the points of the projective geometry $P(A)$.*

Proof. (1) Let $a \in A\backslash\{0\}$. Then if $aA = 0$ the homogeneity condition implies that $A^2 = 0$ which is a contradiction. This fact implies that the map $\phi: x \rightarrow L_x$ is a linear isomorphism and so $\dim L = \dim A$. Similarly it can be shown that $\dim R = \dim A$.

(2) The proof is a simple generalization of a related result found in the introduction of the paper by Boen, Rothaus, and Thompson [1].

(3) This is obvious since the points of $P(A)$ are exactly the one-dimensional subspaces of A.

THEOREM 2. *Let A be a nontrivial homogeneous algebra over a field K. Then*

$$\mathrm{tr}\, L_x = \mathrm{tr}\, R_x = 0 \qquad \forall x \in A$$

Proof. Let $\dim A = n$. It is well known that $\mathrm{tr}: \mathrm{End}(A) \rightarrow K$ is a linear functional and that $\dim \ker(\mathrm{tr}) = n^2 - 1$. But then since $\dim L = \dim A = n > 1$ it follows that $L \cap \ker(\mathrm{tr}) \neq 0$ and so there must exist at least one nonzero map $L_a \in L$ such that $\mathrm{tr}\, L_a = 0$. But now the second result of the previous theorem implies that $\mathrm{tr}\, L_x = 0$ for all $x \in A$. Similarly $\mathrm{tr}\, R_x = 0$ for all $x \in A$.

THEOREM 3. *Let A be a homogeneous algebra over a field K and let $a \in A\backslash\{0\}$. If $\langle a \rangle$ denotes the subalgebra of A generated by a then $\langle a \rangle$ is also a homogeneous algebra enjoying the property that it is generated by each of its nonzero elements. Also $A = \bigcup A_i$ where each $A_i = \langle a_i \rangle$ for some $a_i \in A\backslash\{0\}$ and $A_i \cap A_j = \{0\}$ for $i \neq j$.*

Proof. Let $b \in \langle a \rangle$. Clearly $\langle b \rangle \subseteqq \langle a \rangle$. But there must exist $\alpha \in \mathrm{Aut}(A)$ such that $\alpha(a) = \lambda b$ for some nonzero $\lambda \in K$ and this implies that $\langle a \rangle \subseteqq \langle b \rangle$ and so $\langle a \rangle = \langle b \rangle$. That is $\langle a \rangle$ is generated by each of its nonzero elements. Now let c and d be any nonzero elements in $\langle a \rangle$. Again there must exist $\beta \in \mathrm{Aut}(A)$ such that $\beta(c) = \lambda d$ for some nonzero $\lambda \in K$. But the fact that both c and d generate $\langle a \rangle$ implies that $\langle a \rangle$ is invariant under β and so the restriction of β to $\langle a \rangle$ is in $\mathrm{Aut}(\langle a \rangle)$. That is, $\langle a \rangle$ is also a homogeneous algebra. The final statement again follows directly from the fact that $\langle a \rangle$ is generated by each of its nonzero elements.

The above theorem implies that in some situations it is sufficient to consider the case where a homogeneous algebra A is generated by each of its nonzero elements.

DEFINITION. Let V be a vector space over a field K and suppose H is a subgroup of $GL(V)$ where $GL(V)$ is the general linear group. Then $C(H)$ is defined as

$$C(H) = \{u \in \mathrm{End}\,(A) \mid uv = vu \text{ for all } V \in H\} .$$

DEFINITION. Let A be an algebra over a field K and suppose $S, T \in C(\mathrm{Aut}\,(A))$. Then $A(S, T)$ indicates a new algebra which coincides with A when considered as a vector space over K but possesses a new multiplication defined by

$$a \circ b = S(a)b + T(b)a \text{ for all } a, b \in A$$

Note that the fact that S and T are linear maps on A ensure that $\circ: A \times A \to A$ is a bilinear map. Also the algebras $A(1, 1)$, $A(1, -1)$ and $A(0, 1)$ are well known and are usually denoted as A^+, A^- and A^{opp} respectively.

THEOREM 4. *Let A be a homogeneous algebra over a field K and suppose $S, T \in C(\mathrm{Aut}\,(A))$. Then $A(S, T)$ is also a homogeneous algebra.*

Proof. Let $\sigma \in \mathrm{Aut}\,(A)$. Then

$$\begin{aligned} \sigma(a \circ b) &= \sigma(S(a)b + T(b)a) \\ &= \sigma(S(a)b) + \sigma(T(b)a) \\ &= (\sigma S(a))\sigma(b) + (\sigma T(b))\sigma(a) \\ &= (S\sigma(a))\sigma(b) + (T\sigma(b))\sigma(a) \\ &= \sigma(a) \circ \sigma(b) \end{aligned}$$

an so the result is true since $\mathrm{Aut}\,(A) \subset \mathrm{Aut}\,(A(S, T))$

DEFINITION. Let A be an algebra over a field K. Then A is left (right) simple if A possesses no nonzero proper left (right) ideals. Also A is simple if A possesses no nonzero, proper, two-sided ideals and $A^2 \neq 0$.

THEOREM 5. *If A is a nonzero homogeneous algebra then A is left simple and right simple.*

Proof. Assume that A has proper nonzero left ideals. When

B runs through minimal left ideals then the sets $B\backslash\{0\}$ form a partition of $A\backslash\{0\}$. Suppose $a \in A\backslash\{0\}$ and let $I(a)$ denote the minimal left ideal which contains a. Now R_a map $A \rightarrow I(a)$ and since $I(a) \neq A$ it follows that R_a has a nonzero kernel. That is, there exists $b \in A\backslash\{0\}$ such that $ba = 0$. Let c be any point in $A\backslash I(a)$. Then $I(c) \cap I(a) = \{0\}$ which implies that $I(c) \cap I(c + a) = \{0\}$. But $b(c + a) = bc$ and so $bc \in I(c) \cap I(c + a)$ which implies that $bc = 0$. Now fix some nonzero $c \in A\backslash I(a)$ and let d be any point in $I(a)\backslash\{0\}$. Then $c + d \in A\backslash I(a)$ and so $b(c + d) = bd = 0$. Hence $bA = 0$ which is impossible since A is a nonzero homogeneous algebra. Hence A has no proper nonzero left ideals and similarly A has no proper nonzero right ideals.

II. Alternative homogeneous algebras.

The following definition is well known.

DEFINITION. An algebra A over a field K is said to be alternative if

$$a^2b = a(ab)$$
$$ab^2 = (ab)b$$

for all $a, b \in A$.

THEOREM 6. *There are no nontrivial alternative homogeneous algebras.*

Proof. Let A be a nontrivial alternative homogeneous algebra. Then the previous theorem implies that A is simple. But it is known that a simple alternative algebra has an identity element 1 (see Corollary 3.11 of Schafer's book [5]). But then A is certainly not homogeneous since $\alpha(1) = 1$ for all $\alpha \in \mathrm{Aut}(A)$.

Note that the above theorem of course implies that there are no nontrivial associative homogeneous algebras.

III. Power-associative homogeneous algebras.

THEOREM 7. *Let A be a power-associative nontrivial homogeneous algebra over a field K. Then either $a^2 = 0$ for all $a \in A$ or $a^2 = a$ for all a in A and in the latter case A is a Jordan algebra and $K = GF(2)$.*

Proof. Let a be some fixed element in $A\backslash\{0\}$. Then Theorem 3 implies that $\langle a\rangle$ is an associative homogeneous algebra and so the previous theorem implies that $\langle a\rangle$ is a trivial homogeneous algebra.

It follows that either $a^2 = 0$ or $a^2 = \lambda a$ for some nonzero $\lambda \in K$. In the former case the homogeneity condition implies that $x^2 = 0$ for all $x \in A$ and so we may assume the latter case. The homogeneity condition implies that $x^2 = \lambda(x)x$ where $\lambda(x)$ is a nonzero scalar in K possibly depending on x. Since $\dim A > 1$ we may choose two independent vectors in A, say e_1 and e_2. Since $a^2 = \lambda a$ implies that $(a/\lambda)^2 = a\backslash\lambda$ we may assume without loss of generality that both e_1 and e_2 are idempotents. It is now necessary to perform several simple calculations. First

$$(e_1 + e_2)^2 = e_1 + e_2 + e_1e_2 + e_2e_1 = \lambda(e_1 + e_1)(e_1 + e_2)$$
$$(e_1 - e_2)^2 = e_1 + e_2 - e_1e_2 - e_2e_1 = \lambda(e_1 - e_2)(e_1 - e_2)$$

Now adding and comparing coefficients gives

$$2 = \lambda(e_1 + e_2) + \lambda(e_1 - e_2)$$
$$2 = \lambda(e_1 + e_2) - \lambda(e_1 - e_2)$$

or

$$2\lambda(e_1 - e_2) = 0$$

which implies that char $K = 2$.

For convenience let $\mu = \lambda(e_1 + e_2)$. Then from above

$$e_1e_2 + e_2e_1 = (\mu + 1)(e_1 + e_2) .$$

Now consider

$$\begin{aligned}(e_1 + e_2)^2 &= e_1 + \mu^2e_2 + \mu(e_1e_2 + e_2e_1)\\ &= (\mu^2 + \mu + 1)e_1 + \mu e_2\end{aligned}$$

from which it follows that $\mu^2 + \mu + 1 = 1$ which implies with char $K = 2$ that $\mu = 1$ and so

$$e_1e_2 + e_2e_1 = 0 .$$

Now let δ be any nonzero scalar in K. Then

$$(e_1 + \delta e_2)^2 = e_1 + \delta^2e_2 + \delta(e_1e_2 + e_2e_1) = e_1 + \delta^2e_2$$

which implies that $\delta^2 = \delta$ and so $\delta = 1$ and indeed $K = GF(2)$. Hence $x^2 = x$ for all $x \in A$. But then

$$(x + y)^2 = x + y + xy + yx = x + y$$

and so $xy = yx$ for all $x, y \in A$ and thus A is a commutative algebra. The second identity for a Jordan algebra is trivally satisfied and so A is a Jordan algebra over $GF(2)$.

It is interesting to note that Dyokovic has shown that all nontrivial real homogeneous algebras are of the first type [2] and Gross has shown that some, but not all, of the known homogeneous algebras over $GF(2)$ are of the second type [3].

IV. Homogeneous quasi-division algebras.

DEFINITION. An algebra A over a field K is said to be a quasi-division algebra if the nonzero elements of A form a quasi-group under multiplication.

One of the reasons for devoting a separate section to homogeneous quasi-division algebras is that Shult [6] and Gross [3] have shown that all nontrivial finite homogeneous algebras are in fact quasi-division algebras.

THEOREM 8. *Let A be a nontrivial homogeneous quasi-division algebra with the property that A is generated by each of its nonzero elements. Then*

(i) $\mathrm{Aut}(A)$ *is sharply transitive on the one-dimensional subspaces of A*

(ii) *If a is any element in $A\backslash\{0\}$ then L_a has precisely one eigenvalue denoted by $\lambda_a \in K$ and the corresponding eigenspace is one-dimensional*

(iii) *Finally $\lambda_a = \lambda_b$ if and only if there exists some $\alpha \in \mathrm{Aut}(A)$ such that $\alpha(a) = b$.*

Proof. (1) It is sufficient to show that no automorphism of A, except the identity Id, can have an eigenvalue in K. Let $\alpha \in \mathrm{Aut}(A)$ and suppose that α has eigenvalue $\lambda \in K$. Then there exists $a \in A\backslash\{0\}$ such that

$$\alpha(a) = \lambda a \ .$$

Since A is not associative by Theorem 6 we define inductively

$$a^n = L_a^{n-1}(a) \qquad n = 2, 3, 4, \cdots$$

But now

$$\alpha(a^n) = \lambda^n a^n \qquad n = 1, 2, 3, \cdots$$

and so there must exists positive integers m, n with $m > n$ such that $\lambda^m = \lambda^n$ since α can only have a finite number of eigenvalues. Letting $k = m - n$ we have

$$\alpha(a^k) = \lambda^k a^k = a^k \neq 0$$

and so $\alpha = \mathrm{Id}$ since from the hypothesis we are assuming that a^k generates A.

(2) Let a and b be any two nonzero elements of A. Since A is a quasi-division algebra the equation

$$xb = b$$

must have a solution, say c and the homogeneity condition implies that there exists $\alpha \in \mathrm{Aut}(A)$ such that

$$\alpha(c) = \lambda a \quad \text{for some} \quad \lambda \in K\backslash\{0\} .$$

But then

$$a\alpha(b) = 1/\lambda\alpha(b)$$

and so L_a has at least one eigenvalue.

Now suppose there exist nonzero elements $b, c \in A$ such that

$$ab = \lambda b$$
$$ac = \mu c$$

where $\lambda, \mu \in K$. If $\{b, c\}$ is an independent set then there must exist $\alpha \in \mathrm{Aut}(A)$ such that

$$\alpha(c) = \delta b \quad \text{for some} \quad \delta \in K .$$

But then

$$\alpha(a)b = \mu b$$

and thus

$$(\lambda\alpha(a) - \mu a)b = 0$$

which implies that $\alpha = Id$ by the previous part of this theorem. Thus L_a has precisely one eigenvector (up to a scalar multiple) which completes the proof of the second statement.

(3) Finally if $\alpha \in \mathrm{Aut}(A)$ then $ax = \lambda_a x$ for some $\lambda \in A\backslash\{0\}$ implies that

$$\alpha(a)\alpha(x) = \lambda_a\alpha(x)$$

and so

$$\lambda_{\alpha(a)} = \lambda_a$$

also if $\lambda_a = \lambda_b$ then there exists $x, y \in A\backslash\{0\}$ such that

$$ax = \lambda_a x$$
$$by = \lambda_b y = \lambda_a y .$$

Now choose $\beta \in \mathrm{Aut}(A)$ such that $\beta(x) = \mu y$ for some $\mu \in K \backslash \{0\}$ and applying β we obtain

$$\beta(a)y = \lambda_a y = by$$

and so it follows that $\beta(a) = b$ as required.

IV. On homogeneous algebras of dimension 2. We now investigate arbitrary homogeneous algebras of dimension 2.

THEOREM 9. *Let A be a nonzero, 2-dimensional, homogeneous algebra over a field K. Then $K = GF(2)$ and A has a basis $\{a, b\}$ so that A is isomorphic to one of the following algebras.*

$$\begin{array}{c|cc} & a & b \\ \hline a & a & a+b \\ b & a+b & b \end{array} \qquad \begin{array}{c|cc} & a & b \\ \hline a & b & a \\ b & a & a+b \end{array} .$$

Proof. Let $a \in A \backslash \{0\}$. Then there are exactly three possibilities which will be considered separately

(i) $a^2 = 0$

(ii) $a^2 = \lambda a$ for some nonzero $\lambda \in K$

(iii) $\{a, a^2\}$ is a basis of A

(1) If $a^2 = 0$ then the homogeneity condition implies that $x^2 = 0$ for all $x \in A$ and the linearized form of this identity implies that A is anticommutative. Extend a to a basis of A, say $\{a, b\}$. Using the fact that $\mathrm{tr}\, L_a 0$ and $L_a \neq 0$ it follows that $ab = \lambda a$ for some nonzero $\lambda \in K$. But now $ab = \lambda a$ and $b^2 = 0$ imply that $\mathrm{tr}\, L_b = -\lambda \neq 0$ which is impossible. Hence this case does not occur.

(2) If $a^2 = \lambda a$ where $\lambda \neq 0$ then the homogeneity condition implies that A is power-associative and so Theorem 7 implies that $K = GF(2)$. Again extend a to a basis of A, say $\{a, b\}$. Using the fact that $\mathrm{tr}\, L_a = \mathrm{tr}\, L_b = 0$ and $L_a \neq 1$ and $L_b \neq 1$ it follows that A must be of the form

$$\begin{array}{c|cc} & a & b \\ \hline a & a & a+b \\ b & a+b & b \end{array} .$$

By direct computations it can be shown that $\mathrm{Aut}(A) = GL(2, 2)$ and so $\mathrm{Aut}(A)$ is in fact triply transitive on $A \backslash \{0\}$.

(3) Suppose that $\{a, a^2\}$ is a basis of A. First pass from A to A^-. By Theorem 4, A^- is also a homogeneous algebra and clearly A^- is of type (1) as defined above and so A^- must be a zero algebra

which implies that A is commutative. If $aa^2 = 0$ then $\operatorname{tr} L_{a^2} = 0$ and $L_{a^2} \neq 0$ implies that L_a is nilpotent but L_{a+a^2} is invertible and so A is a quasi-division algebra generated by each of its nonzero elements and so we may apply Theorem 8. Assume $aa^2 = \mu a$.

Let b be any fixed nonzero element of A. The equation $xb = b$ must have a solution and without loss of generality we may assume that $x = a$. Hence the only eigenvalue of L_a is 1 and it follows that $\mu = 1$ and char $K = 2$. Also $a^2a^2 = va + a^2$ for some nonzero $v \in K$. Now since L_a and L_{a^2} both have eigenvalue 1 it follows from Theorem 8 that there must exist $\alpha \in \operatorname{Aut}(A)$ such that $\alpha(a) = a^2$. But then

$$\begin{aligned}\alpha(a^2) &= \alpha(a)\alpha(a) = a^2a^2 = va + a^2 \\ \alpha(a^2a^2) &= \alpha(va + a^2) = va^2 + va + a^2 \\ &= \alpha(a^2)\alpha(a^2) = (va + a^2)(va + a^2) = v^2a^2 + va + a^2 .\end{aligned}$$

It follows that $v = 1$ and so the multiplication table of A is of the form

$$\begin{array}{c|cc} & a & b \\ \hline a & a^2 & a \\ b & a & a + a \end{array} .$$

If $K = GF(2)$ it is easily shown that A is in fact a homogeneous algebra. If $K = GF(4)$ it can be shown that det $(L_a + \lambda L_{a^2}) = 1 + \lambda + \lambda^2 = 0$ for some $\lambda \in GF(4)$ and so A is not homogeneous since it is not a quasi-division algebra. Now assume that $K \neq GF(2)$ and $K \neq GF(4)$. Then there must exist $\lambda_0 \in K$ such that λ_0 is not a root of the polynomial $x^2 + x + 1$ or of the polynomial $x^4 + x^3 + x^2 + 1$. Since A is homogeneous there must exist $\alpha \in \operatorname{Aut}(A)$ such that

$$\alpha(a) = \lambda(a + \lambda_0 a^2) \quad \text{for some nonzero} \quad \lambda \in K .$$

But then

$$\begin{aligned}\alpha(aa^2) &= \lambda^3(1 + \lambda_0 + \lambda_0^2)a + \lambda^3\lambda_0(1 + \lambda_0 + \lambda_0^2)a^2 \\ &= \alpha(a) = \lambda a + \lambda\lambda_0 a^2\end{aligned}$$

and so

$$\lambda^2 = \frac{1}{1 + \lambda_0 + \lambda_0^2} . \tag{1}$$

Also

$$\begin{aligned}\alpha(a^2a^2) &= \lambda^4(1 + \lambda_0^4)a + \lambda^4 a^2 \\ &= \alpha(a) + \alpha(a^2) \\ &= (\lambda + \lambda^2\lambda_0^2)a + [\lambda\lambda_0 + \lambda^2(1 + \lambda_0^2)]a^2\end{aligned}$$

which implies using (1) that

$$\lambda^2 = \frac{1 + \lambda_0^2 + \lambda_0^4}{1 + \lambda_0^4 + \lambda_0^6} \tag{2}$$

and together (1) and (2) imply that

$$\lambda_0^4 + \lambda_0^3 + \lambda_0^2 + 1 = 0$$

which contradicts our choice of λ_0. Hence A is a homogeneous algebra if and only if $K = GF(2)$.

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REFERENCES

1. J. Boen, O. Rothaus and J. Thompson, *Further results on p-automorphic p-groups*, Pacific J. Math., **12** (1962), 817-821.

2. D. Z. Dyokovic, *Real Homogeneous algebras*, Proc. Amer. Math. Soc., **41** (1973), 457-462.

3. F. Gross, *Finite automorphic algebras over GF(2)*, Proc. Amer. Math. Soc. **4** (1971), 349-353.

4. A. I. Kostrikin, *On homogeneous algebras*, Izvestia Acad. Nauk. U.S.S.R., **29** (1965), 471-484.

5. R. D. Schafer, *An Introduction to Nonassociative Algebras*, Academic Press, New York, 1966.

6. E. E. Shult, *On finite automorphic algebras*, Illinois J. Math., **13** (1969), 625-653.

7. ———, *On the triviality of finite automorphic algebras*, Illinois J. Math., **13** (1969), 654-659.

8. L. G. Sweet, *On the triviality of homogeneous algebras over an algebraically closed field*, Proc. Amer. Math. Soc., **48** (1975), 321-324.

9. S. Swierczkowski, *Homogeneous Lie Algebras*, Bull. Austral. Math. Soc., **4** (1971), 349-353.

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