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PLESSNER'S THEOREM FOR RIESZ CONJUGATES

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Plessner's theorem states that if a trigonometric series converges everywhere in a set E of positive measure, then its conjugate series converges almost everywhere in E. Recently, Ash and Gluck have shown that this theorem is false in two dimensions by exhibiting a Fourier series of an L^1 function which converges almost everywhere, but each of its conjugates is divergent almost everywhere. We show that if instead of the usual conjugates in two dimensions, one uses Riesz conjugates, then Plessner's theorem remains true provided the conjugates are required only to be restrictedly convergent almost everywhere in E. The techniques used to obtain this result are similar to those used in the one-dimensional case and involve the notions of stable convergence, nontangential convergence, the theory of Riesz conjugates as developed by E. M. Stein and G. Weiss, and a Tauberian theorem for Abel summability.

1. Introduction. In [1], J. M. Ash and L. Gluck presented some results for Fourier series in several variables. They proved in dimension 2 that each of the conjugate series of a Fourier series of a function in $L^p (p > 1)$ converges almost everywhere in the set where the Fourier series converges. In the case $p = 1$, however, they exhibited a function whose Fourier series converges almost everywhere such that each of its conjugates is also a Fourier series of an L^1 function, but is square divergent almost everywhere. Furthermore, in dimension 3 or greater, they found a continuous function whose Fourier series converges almost everywhere such that each of its conjugates is also a Fourier series of a continuous function, but is restrictedly divergent almost everywhere.

On a philosophical level, this distressing state of affairs can be explained by the fact that the "singularity" of each conjugate transformation they use, thought of as a "singular integral operator", has changed from a point to a pair of lines as the dimension of the space was increased from 1 to 2. This can be altered by using instead of the ordinary conjugate series, the Riesz conjugates. This is done also to take advantage of the theory of conjugate transformations developed by Stein and Weiss in [3] or [4] and [5]. By doing this, we are able to retain Plessner's theorem in its original form except that the conjugates will be required only to converge restrictedly almost everywhere in the set where the original series converges.

The arguments will be presented in two dimensions. However, similar arguments should obtain for higher dimensional spaces.

2. Definitions and statement of the main theorem. Bold face letters such as $\mathbf{N}$ will represent two-dimensional vectors with coordinates N_1 and N_2. However, we will not use bold face letters for variables x, t in the torus. The norm $\|\mathbf{N}\|$ is $(N_1^2+N_2^2)^{1/2}$. The notation $\mathbf{N}>M$ means $N_1>M$ and $N_2>M$, whereas $\mathbf{N}>\mathbf{k}$ means that $N_1>k_1$ and $N_2>k_2$. For each vector $\mathbf{N}$ of integers, let $S_{\mathbf{N}}$ be a scalar. Then, we speak of $\{S_{\mathbf{N}}\}$ as a sequence. By $S_{\mathbf{N}}\to S$ as $\mathbf{N}\to\infty$, we will mean that for every $\epsilon>0$ there exists M such that $\mathbf{N}>M$ implies $|S_{\mathbf{N}}-S|<\epsilon$ (this is unrestricted rectangular convergence and in this case we speak of convergence without qualifiers). We will say that $S_{\mathbf{N}}\to S$ as $\mathbf{N}\to\infty$ restrictedly if for every $\delta>0$ and $\epsilon>0$ there exists M such that $\mathbf{N}>M$ and $\delta^{-1}<N_1/N_2<\delta$ imply $|S_{\mathbf{N}}-S|<\epsilon$. To say that $S_{\mathbf{N}}$ is restrictedly bounded means that for every $\delta>0$ there exists H such that $\delta^{-1}<N_1/N_2<\delta$ implies $|S_{\mathbf{N}}|<H$. The notation $S_{\mathbf{N}}=o(A_{\mathbf{N}})$ will mean that $S_{\mathbf{N}}/A_{\mathbf{N}}$ is *bounded* and $\to 0$ as $\mathbf{N}\to\infty$. Finally $\Sigma_{\mathbf{k}=0}^{\infty}$ means $\Sigma_{k_1=0}^{\infty}\Sigma_{k_2=0}^{\infty}$.

Let

$$\begin{aligned} t &= \sum_{\mathbf{k}\geqq 0}(a_{\mathbf{k}}\cos k_1x_1\cos k_2x_2+b_{\mathbf{k}}\sin k_1x_1\cos k_2x_2 \\ &\quad + c_{\mathbf{k}}\cos k_1x_1\sin k_2x_2+d_{\mathbf{k}}\sin k_1x_1\sin k_2x_2), \\ T_{\mathbf{k}} &= (\cos k_1x_1\cos k_2x_2,\ \sin k_1x_1\cos k_2x_2, \\ &\quad \cos k_2x_2\sin k_2x_2,\ \sin k_1x_1\sin k_2x_2) \end{aligned}$$

and $V_{\mathbf{k}}=(a_{\mathbf{k}},b_{\mathbf{k}},c_{\mathbf{k}},d_{\mathbf{k}})$, then we can write $t=\Sigma(V_{\mathbf{k}},T_{\mathbf{k}})\equiv\Sigma A_{\mathbf{k}}(x)$, where $(\,,)$ is the standard euclidean inner product in 4 dimensional space, $\mathbf{E}^4$.

Let

$$M_1=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\qquad M_2=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}$$

and $M_3=M_1M_2$. By using M_1 and M_2 as transformations on $\mathbf{E}^4$ we can define the Riesz conjugate series

$$t_1=\sum\frac{k_1}{\|\mathbf{k}\|}(M_1V_{\mathbf{k}},T_{\mathbf{k}})\equiv\sum\frac{k_1}{\|\mathbf{k}\|}B_{\mathbf{k}}(x)$$

and

$$t_2 = \sum \frac{k_2}{\|\mathbf{k}\|} (M_2 V_{\mathbf{k}}, T_{\mathbf{k}}) \equiv \sum \frac{k_2}{\|\mathbf{k}\|} C_{\mathbf{k}}(x).$$

We will also use the double conjugate series

$$t_3 = \sum \frac{k_1 k_2}{\|\mathbf{k}\|^2} (M_3 V_{\mathbf{k}}, T_{\mathbf{k}}) \equiv \sum \frac{k_1 k_2}{\|\mathbf{k}\|^2} D_{\mathbf{k}}(x).$$

In these definitions and elsewhere, 0/0 is interpreted as 0. The essential difference in these definitions and those used by Ash and Gluck is that the factors $k_1/\|k\|$, $k_2/\|k\|$ and $k_1k_2/\|\mathbf{k}\|^2$ do not appear in the definitions of conjugates they use.

THEOREM 1. *Suppose* $\Sigma A_{\mathbf{k}}(x)$ *converges in a set* E *of positive measure. Then*

$$\sum \frac{k_1}{\|\mathbf{k}\|} B_{\mathbf{k}}(x), \quad \sum \frac{k_2}{\|\mathbf{k}\|} C_{\mathbf{k}}(x), \quad \text{and} \quad \sum \frac{k_1 k_2}{\|\mathbf{k}\|^2} D_{\mathbf{k}}(x),$$

each converge restrictedly almost everywhere in E.

3. LEMMAS. *Let* $S_{\mathbf{N}}(x) = \Sigma_{\mathbf{k}\leqq\mathbf{N}} A_{\mathbf{k}}(x)$, *then straight forward calculations show that*

$$S_{\mathbf{N}}(x_1 + t_1, x_2) = \sum_{\mathbf{k}\leqq\mathbf{N}} (A_{\mathbf{k}}(x) \cos k_1 t_1 + B_{\mathbf{k}}(x) \sin k_1 t_1), \tag{2.1}$$

$$S_{\mathbf{N}}(x_1, x_2 + t_2) = \sum_{\mathbf{k}\leqq\mathbf{N}} (A_{\mathbf{k}}(x) \cos k_2 t_2 + C_{\mathbf{k}}(x) \sin k_2 t_2), \tag{2.2}$$

$$\begin{aligned} S_{\mathbf{N}}(x_1 + t_1, x_2 + t_2) = \sum_{\mathbf{k}\leqq\mathbf{N}} (A_{\mathbf{k}}(x) &\cos k_1 t_1 \cos k_2 t \\ &+ B_{\mathbf{k}}(x) \sin k_1 t_1 \cos k_2 t_2 + C_{\mathbf{k}}(x) \cos k_1 t_1 \sin k_2 t_2 \\ &+ D_{\mathbf{k}}(x) \sin k_1 t_1 \sin k_2 t_2). \end{aligned} \tag{2.3}$$

The sequence $\{S_{\mathbf{N}}(x)\}$ is said to converge stably to s at x° as $\mathbf{N} \to \infty$ (unrestrictedly) if for each sequence $t_{\mathbf{N}} = (t_{N_1}, t_{N_2})$ for which $t_{N_i} = O(1/N_i)(i = 1, 2)$, $S_{\mathbf{N}}(x^\circ + t_{\mathbf{N}}) \to s$.

We need the following lemmas for which the proofs follow in much the same way as those in [7, vol. 2, pp. 216–219]. In these lemmas it is to be understood that convergence or stable convergence of double series also means that the partial sums are bounded.

LEMMA 1. *A necessary and sufficient condition for* $\Sigma\, a_{\mathbf{n}} \cos n_1 x_1 \cos n_2 x_2$ *or* $\Sigma\, a_{\mathbf{n}} \cos n_1 x_1$ *or* $\Sigma\, a_{\mathbf{n}} \cos n_2 x_2$ *to converge stably to* s *at* $x = 0$ *is that* $\Sigma\, a_{\mathbf{n}}$ *converges to* s.

LEMMA 2. (i) *A necessary and sufficient condition for* $\Sigma\, b_{\mathbf{n}} \sin n_1 x_1$ *or* $\Sigma\, b_{\mathbf{n}} \sin n_1 x_1 \cos n_2 x_2$ *to converge stably to zero at* $x = 0$ *is that*

$$\frac{1}{n_1} \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} \nu_1 b_\nu = o(1).$$

(ii) *A necessary and sufficient condition for* $\Sigma\, c_{\mathbf{n}} \sin n_2 x_2$ *or* $\Sigma\, c_{\mathbf{n}} \cos n_1 x_1 \sin n_2 x_2$ *to converge stably to zero at* $x = 0$ *is that*

$$\frac{1}{n_2} \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} \nu_2 c_\nu = o(1).$$

LEMMA 3. *A necessary and sufficient condition that* $\Sigma\, d_{\mathbf{n}} \sin n_1 x_1 \sin n_2 x_2$ *converge stably to zero at* $x = 0$ *is that*

$$\frac{1}{n_1 n_2} \sum_{\nu_1=0}^{n_1} \sum_{\nu_2=0}^{n_2} \nu_1 \nu_2 d_\nu = o(1).$$

LEMMA 4. *The series* $\Sigma\, A_{\mathbf{k}}(x)$ *is stably convergent at* x° *to the sum* s *if and only if*

(i) $\Sigma\, A_{\mathbf{k}}(x^\circ)$ *converges to* s,
(ii) $\Sigma_{0 \le \mathbf{k} \le \mathbf{N}}\, k_1 B_{\mathbf{k}}(x^\circ) = o(N_1)$,
(iii) $\Sigma_{0 \le \mathbf{k} \le \mathbf{N}}\, k_2 C_{\mathbf{k}}(x^\circ) = o(N_2)$,
(iv) $\Sigma_{0 \le \mathbf{k} \le \mathbf{N}}\, k_1 k_2 D_{\mathbf{k}}(x^\circ) = o(N_1 N_2)$.

Proof. Suppose $\Sigma\, A_{\mathbf{k}}(x)$ converges stably at x° to s. Then (2.1), (2.2) and (2.3) (with $x = x^\circ$) each converge stably to s at $t = 0$. Part (i) is obvious. Since $\Sigma\, A_{\mathbf{k}}(x^\circ)$ converges, Lemma 1 implies that $\Sigma\, A_{\mathbf{k}}(x^\circ) \cos k_1 t_1$ is stably convergent at $t = 0$ and by (2.1), $\Sigma\, B_k(x^\circ) \sin k_1 t_1$ is stably convergent to 0 at $t = 0$ and Lemma 2 gives (ii). Similarly we obtain (iii). Using these results and similar reasoning applied to (2.3) gives (iv). The converse follows easily.

LEMMA 5. *If* $\Sigma\, A_{\mathbf{k}}(x)$ *converges stably at* x° *to the sum* s, *then the harmonic function* $\Sigma\, A_{\mathbf{k}}(x)\, r^{\|\mathbf{k}\|}$ *tends to* s *as* (x, r) *tends to* $(x^\circ, 1)$ *nontangentially; that is, with* $\|x - x^\circ\| \leqq C(1 - r)$ *as* $x \to x^\circ$ *and* $r \to 1$.

LEMMA 6. *If* $\Sigma\, A_{\mathbf{k}}(x)$ *converges for* $x \in E$, *where* E *is of positive measure, then it converges stably at almost all points of* E.

The final two lemmas come from different sources.

LEMMA 7. [2, Theorem 2.1 and Lemma 2.3] *Suppose* $\Sigma A_{\mathbf{k}}(x)$ *converges* (no hypothesis on the nature of the partial sums) *in a set* E, $|E|>0$. *Then, for almost all points* $x \in E$, *all the partial sums of* $\Sigma A_{\mathbf{k}}(x)$ *are bounded. Furthermore, the coefficients of* $\Sigma A_{\mathbf{k}}(x)$ *are bounded.*

LEMMA 8. *If* $t(x, r) = \Sigma A_{\mathbf{k}}(x) r^{\|\mathbf{k}\|}$ *converges nontangentially in a set* E, $|E|>0$ *then*

$$t_1(x, r) = \sum \frac{k_1}{\|\mathbf{k}\|} B_{\mathbf{k}}(x) r^{\|\mathbf{k}\|}$$

and

$$t_2(x, r) = \sum \frac{k_1}{\|\mathbf{k}\|} C_{\mathbf{k}}(x) r^{\|\mathbf{k}\|}$$

converge nontangentially for almost every point in the set E.

The proof is achieved by appealing to the following theorem which we list as a lemma.

LEMMA 8′. [3, page 213] *Let* $u(x, y)$ *be a function which is defined and harmonic on* $\mathbf{E}_3^+ = \{(x, y) \mid x \in \mathbf{E}^2, y > 0\}$. *Let* u_1 *and* u_2 *be the conjugate harmonic functions associated with* u (see [3] for definitions). *Assume* u *converges nontangentially* ($(x, y) \to (x^\circ, 0)$ *with* $\|x - x^\circ\| < Cy$) *in a set* E, $|E|>0$. *Then* u_1 *and* u_2 *converge nontangentially almost everywhere in the set* E.

In order to see how Lemma 8 follows from this, we first point out that after a simple change of variable, we may think of $\Sigma A_{\mathbf{k}}(x)$ as a distribution on $\mathbf{T}^2 = [0, 1) \times [0, 1)$, since by the hypothesis of our theorem and Lemma 7 the coefficients of $\Sigma A_{\mathbf{k}}(x)$ are bounded. Extend $\Sigma A_{\mathbf{k}}(x)$ periodically so that it is defined on $\mathbf{E}^2$. In this case, we will also denote the resulting distribution by $t(x)$. Since we now have a tempered distribution on $\mathbf{E}^2$, we will be able to "convolve" it with the Poisson kernel for the upper half-plane $\mathbf{E}_3^+$. In general, suppose that φ is a rapidly decreasing function and that Λ is the 2-dimensional lattice plane. Define $\Phi(\cdot) = \Sigma_{\mathbf{m} \in \Lambda} \varphi(\cdot + \mathbf{m})$. In this case, Φ is an infinitely differentiable function which is periodic on $\mathbf{E}^2$ and hence defined on $\mathbf{T}^2$. We then obtain $(t * \varphi)(x) = \Sigma \hat{\Phi}(\mathbf{m}) A_{\mathbf{m}}(x)$ where the $\hat{\Phi}(\mathbf{m})$ are the Fourier coefficients of Φ expressed in the real form. In particular, if $\mathbf{P}_y(\cdot)$ is the Poissson kernel for the half-plane $\mathbf{E}_3^+$ and $P_r(\cdot)$ is the

Poisson kernel for the torus $\mathbf{T}^2$, we have $P_r(\cdot) = \Sigma_{\mathbf{m}\in\Lambda} \mathbf{P}_y(\cdot + \mathbf{m})$ [see 6, page 255] and $(t * \mathbf{P}_y)(x) = \Sigma_{\mathbf{m}\in\Lambda} \hat{P}_r(\mathbf{m}) A_{\mathbf{m}}(x)$. The following identification between r and y is necessary for the above formulas, $r = e^{-2\pi y}$. With these preliminaries one can see that $t * \mathbf{P}_y$ is a periodic function on $\mathbf{E}^2$ which as a function of (x, y) is a harmonic function on $\mathbf{E}_3^+$. With the additional remark that $\hat{P}_r(\mathbf{m}) = r^{\|\mathbf{m}\|}$ we see that nontangential limits for $t * \mathbf{P}_y(x)$ and $\Sigma A_{\mathbf{k}}(x) r^{\|\mathbf{k}\|}$ for $x \in \mathbf{T}^2$ are the same.

Again for φ a rapidly decreasing function and with the Fourier transform defined in the appropriate normalization, the Fourier coefficients $\hat{\Phi}(\mathbf{m}) = \hat{\varphi}(\mathbf{m})$ where these are understood now in complex form. It then follows with $u = t * \tilde{\mathbf{P}}_y$ that the conjugate functions u_1 and u_2 are $t * \tilde{\mathbf{P}}_{y,1}$ and $t * \tilde{\mathbf{P}}_{y,2}$ with

$$\tilde{\mathbf{P}}_{y,1}(\xi) = \frac{i\xi_1}{\|\xi\|}\hat{\mathbf{P}}_y \quad \text{and} \quad \tilde{\mathbf{P}}_{y,2} = \frac{i\xi_2}{\|\xi\|}\hat{\mathbf{P}}_y.$$

Expressing these results in series form and writing the coefficients in real form gives $u_1(x) = \Sigma (k_1/\|\mathbf{k}\|) B_{\mathbf{k}}(x) r^{\|\mathbf{k}\|}$ and

$$u_2(x) = \sum (k_2/\|\mathbf{k}\|) B_{\mathbf{k}}(x) r^{\|\mathbf{k}\|}.$$

The nontangential convergence of these series now follows directly from Lemma 8′. By repeating an application of this lemma we get that $\Sigma (k_1 k_2/\|\mathbf{k}\|^2) D_{\mathbf{k}}(x) r^{\|\mathbf{k}\|}$ also converges nontangentially almost everywhere in E.

4. The Tauberian theorem. Before we can prove Theorem 1, we must have available a Tauberian theorem for Abel summability so that results about nontangential convergence can be translated to results about restricted convergence. This is the purpose of Theorem 2. We need a preliminary lemma.

LEMMA 9. *Suppose $A_{\mathbf{nk}}$ is a scalar for each two vectors of nonnegative integers $\mathbf{n}$ and $\mathbf{k}$. If*

$$\text{(4.1)} \qquad \sum_{k_i=0}^{\infty} |A_{\mathbf{nk}}| \to 0 \text{ as } \mathbf{n} \to \infty \textit{ restrictedly for each } k_{3-i} \, (i = 1, 2),$$

and

$$\text{(4.2)} \qquad \sum_{\mathbf{k}=0}^{\infty} |A_{\mathbf{nk}}| \textit{ is restrictedly bounded,}$$

then $\sigma_{\mathbf{n}} = \Sigma_{\mathbf{k}=0}^{\infty} A_{\mathbf{nk}} \epsilon_{\mathbf{k}} \to 0$ as $\mathbf{n} \to \infty$ restrictedly whenever $\epsilon_{\mathbf{k}} = o(1)$.

Proof. Choose $\delta > 0$ and suppose during the rest of this proof that $\delta^{-1} < N_1 | N_2 < \delta$. Suppose $\epsilon_{\mathbf{k}} = o(1)$. By (4.2) $\Sigma |A_{\mathbf{nk}}|$ is bounded, say by H, and since $\epsilon_{\mathbf{k}}$ is bounded, $\sigma_{\mathbf{n}}$ exists for each $\mathbf{n}$. Choose $\epsilon > 0$. Since $\epsilon_{\mathbf{k}} \to 0$ we may choose an M such that $\mathbf{k} > M$ implies $|\epsilon_{\mathbf{k}}| < \epsilon/2H$. By (4.1) we can choose M' such that $\mathbf{N} > M'$ implies $\sum_{N_1 \leqq M \text{ or } N_2 \leqq M} |A_{\mathbf{N}\mathbf{k}}\, \epsilon_{\mathbf{k}}| < \epsilon/2$. If $\mathbf{N} > M'$, then

$$|\sigma_{\mathbf{N}}| \leqq \sum_{\mathbf{k}=M+1}^{\infty} |A_{\mathbf{N}\mathbf{k}}\, \epsilon_{\mathbf{k}}| + \sum_{N_1 \leqq M \text{ or } N_2 \leqq M} |A_{\mathbf{N}\mathbf{k}}\, \epsilon_{\mathbf{k}}|$$

$$< \frac{\epsilon}{2H} \cdot H + \frac{\epsilon}{2} = \epsilon.$$

THEOREM 2. *Suppose*

$$\lim_{r \to 1^-} \sum_{\mathbf{k}=0}^{\infty} a_{\mathbf{k}}\, r^{\|\mathbf{k}\|} = S \tag{4.3}$$

and

$$\epsilon_{\mathbf{N}} \equiv \frac{1}{\|\mathbf{N}\|^i} \sum_{\mathbf{k}=0}^{\mathbf{N}} \|\mathbf{k}\|^i a_{\mathbf{k}} = o(1), \qquad i = 1 \quad or \quad i = 2, \tag{4.4}$$

then $S_{\mathbf{N}} \equiv \sum_{\mathbf{k}=0}^{\mathbf{N}} a_{\mathbf{k}} \to S$ *as* $\mathbf{N} \to \infty$ *restrictedly.*

Proof. We will first prove the theorem in the case $i = 1$. Let $r = 1 - 1/N_1$ and consider

$$A_{\mathbf{N}} = \sum_{\mathbf{k}=0}^{\mathbf{N}} a_{\mathbf{k}} - \sum_{\mathbf{k}=0}^{\infty} a_{\mathbf{k}}\, r^{\|\mathbf{k}\|} = \sum_{\mathbf{k}=0}^{\infty} \|\mathbf{k}\|\, a_{\mathbf{k}}\, B_{\mathbf{N}\mathbf{k}}$$

where $B_{\mathbf{N}\mathbf{k}} = 0$ if $\mathbf{k} = 0$, $B_{\mathbf{N}\mathbf{k}} = (1 - r^{\|\mathbf{k}\|})/\|\mathbf{k}\|$ if $0 \leqq \mathbf{k} \leqq \mathbf{N}$, $\mathbf{k} \neq 0$, and $B_{\mathbf{N}\mathbf{k}} = -r^{\|\mathbf{k}\|}/\|\mathbf{k}\|$ otherwise. The proof will be completed by showing that $A_{\mathbf{N}} \to 0$ as $\mathbf{N} \to \infty$ restrictedly. Using summation by parts we obtain (with $t_{\mathbf{k}} = \|\mathbf{k}\|\, \epsilon_{\mathbf{k}}$)

$$A_{\mathbf{N}} = \lim_{\mathbf{J} \to \infty} \left[\sum_{\mathbf{k}=0}^{\mathbf{J}-1} t_{\mathbf{k}}\, \Delta^{11} B_{\mathbf{N}\mathbf{k}} + \sum_{k_1=0}^{J_1-1} t_{k_1, J_2}\, \Delta^{10} B_{\mathbf{N}; k_1, J_2} \right.$$
$$\left. + \sum_{k_2=0}^{J_2-1} t_{J_1, k_2}\, \Delta^{01} B_{\mathbf{N}, J_1, k_2} + t_{\mathbf{J}}\, B_{\mathbf{N}\mathbf{J}} \right]$$
$$= \lim_{\mathbf{J} \to \infty} [C_1 + C_2 + C_3 + C_4].$$

However, $\lim_{\mathbf{J} \to \infty} C_4 = \lim_{\mathbf{J} \to \infty} (-\epsilon_{\mathbf{J}}\, r^{\|\mathbf{J}\|}) = 0$ by (4.4). Furthermore,

$$\Delta^{10} B_{\mathbf{N}; k_1, J_2} = \int_{k_1}^{k_1+1} \frac{d}{dx} \left[\frac{r^{\|(x, J_2)\|}}{\|(x, J_2)\|} \right] dx$$

$$= \int_{k_1}^{k_1+1} \frac{x r^{\|(x, J_2)\|}}{\|(x, J_2)\|^3} \left[\|(x, J_2)\| \log \frac{1}{r} + 1 \right] dx$$

$$\leqq \frac{D\, r^{\|(k_1, J_2)\|}}{\|(k_1, J_2)\|}$$

where D is independent of $\mathbf{J}$. Therefore, since $k_1 + J_2 \leqq \sqrt{2}\|(k_1, J_2)\|$,

$$|C_2| \leqq D \sum_{k_1=0}^{J_1-1} |\epsilon_{k_1,J_2}|\, r^{\|(k_1,J_2)\|}$$

$$\leqq D\rho^{J_2} \sum_{k_1=0}^{J_1-1} |\epsilon_{k_1,J_2}|\, \rho^{k_1}$$

where $\rho = r^{1/\sqrt{2}}$. Since ϵ_{k_1,J_2} is assumed bounded, it follows that $\lim_{\mathbf{J}\to\infty} C_2 = 0$. Similarly, $\lim_{\mathbf{J}\to\infty} C_3 = 0$. This leaves

$$A_{\mathbf{N}} = \sum_{\mathbf{k}=0}^{\infty} \epsilon_{\mathbf{k}} \|\mathbf{k}\|\, \Delta^{11} B_{\mathbf{Nk}},$$

and we may complete the proof by showing that $\|\mathbf{k}\|\Delta^{11} B_{\mathbf{Nk}}$ satisfies conditions (4.1) and (4.2) of Lemma 9.

We will first obtain bounds on $\Delta^{11} B_{\mathbf{Nk}}$. If $\mathbf{k} < \mathbf{N}(\mathbf{k} \neq 0)$, then

$$|\Delta^{11} B_{\mathbf{Nk}}| = \left| \int_{k_1}^{k_1+1} \int_{k_2}^{k_2+1} \frac{\partial^2}{\partial x_1 \partial x_2} \int_r^1 y^{\|x\|-1}\, dy\, dx_1\, dx_2 \right|$$

$$= \left| \int_{k_1}^{k_1+1} \int_{k_2}^{k_2+1} \int_r^1 \frac{x_1 x_2}{\|x\|^2} y^{\|x\|-1} \log\frac{1}{y}\left(\log\frac{1}{y} + \frac{1}{\|x\|}\right) dy\, dx_1\, dx_2 \right|$$

$$\leqq (1-r)\log\frac{1}{r}\left(\log\frac{1}{r} + \frac{1}{\|\mathbf{k}\|}\right)$$

$$\leqq 2(1-r)^2\left(2(1-r) + \frac{1}{\|\mathbf{k}\|}\right)$$

whenever $r > \frac{1}{2}$.

If $k_1 > N_1$ or $k_2 > N_2$, then using integration by parts we obtain

$$|\Delta^{11} B_{\mathbf{Nk}}| = \left| \int_{k_1}^{k_1+1} \int_{k_2}^{k_2+1} \int_0^r \frac{x_1 x_2}{\|x\|^2} y^{\|x\|-1} \log\frac{1}{y}\left(\log\frac{1}{y} + \frac{1}{\|x\|}\right) dy\, dx_1\, dx_2 \right|$$

$$\leqq \int_{k_1}^{k_1+1} \int_{k_2}^{k_2+1} \frac{r^{\|x\|}\log^2\frac{1}{r}}{\|x\|}\, dx_1\, dx_2$$

$$+ 3 \int_{k_1}^{k_1+1} \int_{k_2}^{k_2+1} \int_0^r \frac{\left(\log\frac{1}{y}\right) y^{\|x\|-1}}{\|x\|}\, dy\, dx_1\, dx_2$$

$$\leqq \log^2\frac{1}{r}\, \frac{r^{\|\mathbf{k}\|}}{\|\mathbf{k}\|} + 3\log\frac{1}{r}\, \frac{r^{\|\mathbf{k}\|}}{\|\mathbf{k}\|^2} + 3\, \frac{r^{\|\mathbf{k}\|}}{\|\mathbf{k}\|^3}$$

$$\equiv Q(r, \mathbf{k}).$$

A short calculation shows that if $k_1 = N_1$, $k_2 < N_2$ or if $k_1 < N_1$, $k_2 = N_2$, then

$$|\Delta^{11} B_{\mathbf{Nk}}| \leqq Q(r, \mathbf{k}) + \frac{1}{\|\mathbf{k}\|^2}.$$

Finally, if $\mathbf{k} = \mathbf{N}$, it is easily seen that

$$|\Delta^{11} B_{\mathbf{Nk}}| \leqq Q(r, \mathbf{k}) + \frac{1}{\|\mathbf{k}\|}.$$

Combining these estimates, we find that

$$\begin{aligned}
\sum_{\mathbf{k}=0}^{\infty} \|\mathbf{k}\| \, |\Delta^{11} B_{\mathbf{Nk}}| &= \sum_{\mathbf{k}=0}^{\mathbf{N}-1} \|\mathbf{k}\| \, |\Delta^{11} B_{\mathbf{Nk}}| + \sum_{k_1 \geqq N_1 \, or \, k_2 \geqq N_2} \|\mathbf{k}\| \, |\Delta^{11} B_{\mathbf{Nk}}| \\
&\leqq 4(1-r)^3 \sum_{\mathbf{k}=0}^{\mathbf{N}-1} \|\mathbf{k}\| + 2(1-r)^2 \sum_{\mathbf{k}=0}^{\mathbf{N}-1} 1 \\
&\quad + \sum_{k_1 \geqq N_1 \text{ or } k_2 \geqq N_2} \left[\left(\log^2 \frac{1}{r} \right) r^{\|\mathbf{k}\|} + 3 \log \frac{1}{r} \frac{r^{\|\mathbf{k}\|}}{\|\mathbf{k}\|} + 3 \frac{r^{\|\mathbf{k}\|}}{\|\mathbf{k}\|^2} \right] \\
&\quad + \sum_{k_2=0}^{N_2-1} \frac{1}{\|(N_1, k_2)\|} + \sum_{k_1=0}^{N_1-1} \frac{1}{\|(k_1, N_2)\|} + 1 \\
&\leqq \frac{4}{N_1^3} (N_1^2 N_2 + N_1 N_2^2) + \frac{2}{N_1^2} N_1 N_2 \\
&\quad + \left[\log^2 \frac{1}{r} + \frac{3 \log \frac{1}{r}}{\min(N_1, N_2)} + \frac{3}{(\min(N_1, N_2))^2} \right] \left[\frac{1}{1-r} + \frac{1}{\log \frac{1}{2}} + \frac{\pi}{2 \log^2 r} \right] \\
&\quad + \frac{1}{N_1} + \log \left(\frac{N_2}{N_1} + \sqrt{\left(\frac{N_2}{N_1} \right)^2 + 1} \right) + \frac{1}{N_2} + \log \left(\frac{N_1}{N_2} + \sqrt{\left(\frac{N_1}{N_2} \right)^2 + 1} \right) + 1
\end{aligned}$$

and if $\delta^{-1} \leqq N_1/N_2 \leqq \delta$, then this is easily seen to be bounded. Thus condition (4.2) of Lemma 9 is satisfied. In a similar, but easier, manner, one can also show that condition (4.1) of Lemma 9 is satisfied. The proof of Theorem 2 in case $i = 1$ is, therefore, complete.

The proof is similar in case $i = 2$. Only the changes will be noted. The $B_{\mathbf{nk}}$ will be defined as in the previous case except that the denominator will be $\|\mathbf{k}\|^2$ instead of $\|\mathbf{k}\|$. In showing that $\lim_{J \to \infty} C_2 = 0$ it is necessary to estimate

$$\int_{k_1}^{k_1+1} \frac{d}{dx} \frac{r^{\|(x, J_2)\|}}{\|(x, J_2)\|^2} \, dx \tag{4.5}$$

instead of the similar term of the previous case. Carrying out the differentiation and proceeding as before we find that (4.5) is majorized by

$$\frac{C\, r^{\|(k_1, J_2)\|}}{\|(k_1, J_2)\|^2}$$

and the rest of the proof of this part of the theorem proceeds as before. In obtaining bounds on $\Delta^{11} B_{\mathbf{nk}}$, note first that if $\mathbf{k} < \mathbf{N}(\mathbf{k} \neq 0)$, then

$$|\Delta^{11} B_{\mathbf{nk}}| = \left| \int_{k_1}^{k_1+1} \int_{k_2}^{k_2+1} \frac{\partial^2}{\partial x_1\, \partial x_2} \frac{1 - r^{\|x\|}}{\|x\|^2}\, dx_1\, dx_2 \right|.$$

Carrying out the indicated differentiations and using estimates as before, we find that

$$|\Delta^{11} B_{\mathbf{nk}}| \leqq C \frac{(1-r)^2}{\|\mathbf{k}\|^2} + \frac{(1-r)}{\|\mathbf{k}\|^3}.$$

If $k_1 > N_1$ or $k_2 > N_2$, then we have

$$|\Delta^{11} B_{\mathbf{nk}}| = \left| \int_{k_1}^{k_1+1} \int_{k_2}^{k_2+1} \int_0^r \frac{1}{y} \int_0^y \frac{\partial^2}{\partial x_1\, \partial x_2}\, s^{\|x\|-1}\, ds\, dy\, dx_1\, dx_2 \right|$$

and we may proceed through several integrations by parts and some simple estimates to obtain

$$|\Delta^{11} B_{\mathbf{nk}}| \leqq \log^2 \frac{1}{r} \frac{r^{\|\mathbf{k}\|}}{\|\mathbf{k}\|^2} + 5 \log \frac{1}{r} \frac{r^{\|\mathbf{k}\|}}{\|\mathbf{k}\|^3} + 8 \frac{r^{\|\mathbf{k}\|}}{\|\mathbf{k}\|^4}$$

$$\equiv P(r, \mathbf{k}).$$

In case $k_1 = N_1$, $k_2 < N_2$ or $k_1 < N_1$, $k_2 = N_2$ we find

$$|\Delta^{11} B_{\mathbf{nk}}| \leqq P(r, \mathbf{k}) + \frac{1}{\|\mathbf{k}\|^3},$$

and if $\mathbf{k} = \mathbf{N}$, then

$$|\Delta^{11} B_{\mathbf{nk}}| \leqq P(r, \mathbf{k}) + \frac{1}{\|\mathbf{k}\|^2}.$$

Multiplying these estimates by $\|\mathbf{k}\|^2$, summing and proceeding as before will complete the proof for the case $i = 2$.

5. Proof of Theorem 1. Suppose $\Sigma A_{\mathbf{k}}(x)$ converges in a set E with positive measure. By Lemma 7 the partial sums of $\Sigma A_{\mathbf{k}}(x)$ are bounded almost everywhere in E. By Lemma 6, $\Sigma A_{\mathbf{k}}(x)$ converges stably at almost all points of E. By Lemma 5, $\Sigma A_{\mathbf{k}}(y)r^{\|\mathbf{k}\|}$ tends to $\Sigma A_{\mathbf{k}}(x)$ as (y, r) tends to $(x, 1)$ nontangentially almost everywhere in E. By Lemma 8,

$$\sum \frac{k_1}{\|\mathbf{k}\|} B_{\mathbf{k}}(x)r^{\|\mathbf{k}\|}, \quad \sum \frac{k_2}{\|\mathbf{k}\|} C_{\mathbf{k}}(x)r^{\|\mathbf{k}\|} \quad \text{and} \quad \sum \frac{k_1k_2}{\|\mathbf{k}\|^2} D_{\mathbf{k}}(x)r^{\|\mathbf{k}\|}$$

each converge as $r \to 1^-$ almost everywhere in E. Furthermore, the Tauberian conditions

$$\sum_{\mathbf{k}=0}^{\mathbf{N}} k_1B_{\mathbf{k}}(x) = o(\|\mathbf{N}\|), \qquad \sum_{\mathbf{k}=0}^{\mathbf{N}} k_2C_{\mathbf{k}}(x) = o(\|\mathbf{N}\|)$$

and

$$\sum_{\mathbf{k}=0}^{\mathbf{N}} k_1k_2D_{\mathbf{k}}(x) = o(\|\mathbf{N}\|)$$

follow from Lemma 4. Thus Theorem 2 is applicable and yields $\Sigma k_1/\|\mathbf{k}\| B_{\mathbf{k}}(x)$ and $\Sigma k_2/\|\mathbf{k}\| C_{\mathbf{k}}(x)$ converge restrictedly almost everywhere in E by the use of case $i = 1$, and that $\Sigma k_1k_2/\|\mathbf{k}\|^2 D_{\mathbf{k}}(x)$ converges restrictedly almost everywhere in E follows from an application of Theorem 2 in the case $i = 2$.

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