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A REPRESENTATION THEOREM FOR ISOMETRIES OF $C(X, E)$

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Let X, Y be compact Hausdorff spaces and let E, F be Banach spaces such that their duals are strictly convex. We show that a linear map $T: C(X, E) \to C(Y, F)$ is an isometric isomorphism if and only if there exists a homeomorphism $\phi: Y \to X$ and a continuous map λ from Y to the set of isometric isomorphisms from E onto F (with the strong topology) such that $Tf(y) = \lambda(y) \cdot f(\phi(y))$ for all $y \in Y$, $f \in C(X, E)$.

1. Suppose E is a Banach space and X is a compact Hausdorff space, we use $C(X, E)$ to denote the Banach space of continuous functions from X into E. In **[3]**, Jerison gave a generalization of the Banach-Stone theorem, he showed that if X, Y are compact Hausdorff spaces, E is a strictly convex space and $T: C(X, E) \to C(Y, E)$ is an isometric isomorphism, then there exists a homeomorphism $\phi: Y \to X$, a continuous map λ from Y into the set of rotations of E (i.e. the set of isometric isomorphisms from E onto E) under the strong topology such that for each $f \in C(X, E)$, $y \in Y$, we have

$$Tf(y) = \lambda(y) \cdot f(\phi(y)).$$

Makai **[5]** and Sundaresan **[6]** made some improvements of the result. In this paper, we will consider the isometric isomorphisms between $C(X, E)$ and $C(Y, F)$ where E^*, F^* are strictly convex spaces. Let E, F be Banach spaces, we use $S(E)$ to denote the unit ball of $E, \partial S(E)$ the set of extreme points of $S(E), L(E, F)$ the set of bounded linear operators from E into F and $I(E, F)$ the set of isometric isomorphisms from E into F. We will show

THEOREM. *Suppose X, Y are compact Hausdorff spaces and E, F are Banach spaces with E^*, F^* strictly convex. Let*

$$T: C(X, E) \to C(Y, F)$$

be an isometric isomorphism; then there exist a homeomorphism $\phi: Y \to X$ and a continuous map $\lambda: Y \to I(E, F)$ (with the strong topology) such that

(*) $$Tf(y) = \lambda(y) \cdot f(\phi(y)) \quad \textit{for all } y \in Y, f \in C(X, E).$$

Conversely, if we are given ϕ and λ as above, then there exists an isometric isomorphism T from $C(X,E)$ onto $C(Y,F)$ satisfies (*).

We remark that the theorem will not be true for arbitrary Banach spaces (c.f. §3).

2. We will begin by showing the converse part of the theorem. The map T defined by (*) is obviously linear and continuous. For $g \in C(Y,F)$, define $\tau: X \to I(F,E)$ by $\tau(x) = (\lambda(\phi^{-1}(x)))^{-1}$ and let $f \in C(X,E)$ be defined by $f(x) = \tau(x) \cdot g(\phi^{-1}(x))$ for all $x \in X$. Then $Tf = g$ and T is onto. To show that T is an isometry, take any $f \in C(X,E)$, then

$$\begin{aligned} \|Tf\| &= \sup\{\|Tf(y)\| : y \in Y\} \\ &= \sup\{\|\lambda(y) \cdot f(\phi(y))\| : y \in Y\} \\ &= \sup\{\|f(\phi(y))\| : y \in Y\} \\ &= \sup\{\|f(x)\| : x \in X\} \\ &= \|f\|. \end{aligned}$$

The proof of the first part is divided into the subsequent lemmas.

LEMMA 1. *Let X be a compact Hausdorff space and let E be a Banach space; then the set of extreme points of $S(C(X,E)^*)$ is of the form $\delta_{x,u}$ where $x \in X$, $u \in \partial S(E^*)$, and*

$$\delta_{x,u}(f) = u(f(x)), f \in C(X,E)$$

Proof. C.f. **[4]**, Theorem 3.2.

Under the assumption of the Theorem, the adjoint map $T^*: C(Y,F)^* \to C(X,E)^*$ is also an isometric isomorphism. It sends the extreme points of $S(C(Y,F)^*)$ onto the set of extreme points of $S(C(X,E)^*)$, i.e., for $y \in Y$ $v \in \partial S(F^*)$, $T^*(\delta_{y,v})$ is of the form $\delta_{x,u}$, where $x \in X$ and $u \in \partial S(E^*)$.

LEMMA 2. (i) *For any $y \in Y, v \in F^*$, $T^*(\delta_{y,v})$ is of the form $\delta_{x,u}$ where $x \in X, u \in E^*$.*

(ii) *Let $y \in Y, v, \bar{v} \in F^*$ and let $T^*(\delta_{y,v}) = \delta_{x,u}, T^*(\delta_{y,\bar{v}}) = \delta_{\bar{x},\bar{u}}$; then $x = \bar{x}$.*

(iii) *For each fixed* $y \in Y$, *the map* $v \to u$, $F^* \to E^*$ *where* $T^*(\delta_{y,v}) = \delta_{x,u}$ *is an isometric isomorphism. Moreover, this map is weak* continuous.*

Proof. Since F^* is strictly convex, every point of norm 1 in F^* is an extreme point of $S(F^*)$. By the preceding remark, (i) holds for all points of norm 1. Note also that $\alpha\delta_{y,v} = \delta_{y,\alpha v}$ for all $\alpha \in R$, so (i) is true for all $v \in F^*$. To prove (ii), suppose $x \neq \bar{x}$ and consider $T^*(\delta_{y,v+\bar{v}})$; by (i), it is of the form $\delta_{x',u'}$ for some $u' \in E^*, x' \in X$ and

$$\delta_{x',u'} = \delta_{x,u} + \delta_{\bar{x},\bar{u}}.$$

Note that $x' \neq x, \bar{x}$. Indeed, if $x' = x$ (or $\bar{x}$), then we can choose $f \in C(X,E)$, $z \in E$ such that $f(\bar{x}) = z$, $\bar{u}(z) \neq 0$, but $f(x) = 0$, then

$$\delta_{x',u'}(f) \neq \delta_{x,u}(f) + \delta_{\bar{x},\bar{u}}(f).$$

Since $x' \neq x, \bar{x}$, by a similar kind of argument, it is easily shown that there exists a $g \in C(X,E)$ such that

$$\delta_{x',u'}(g) \neq \delta_{x,u}(g) + \delta_{\bar{x},\bar{u}}(g).$$

a contradiction. In (iii), it follows from (i), (ii) that the map is well defined and linear. To show that it is onto, we note that if $T^*(\delta_{y_1,v_1}) = \delta_{x,u_1}$, $T^*(\delta_{y_2,v_2}) = \delta_{x,u_2}$, then $y_1 = y_2$ (for we need only consider $(T^*)^{-1}$ as in (ii)). For $u_1 \in E^*$, consider δ_{x,u_1} where $x \in X$ is such that $T^*(\delta_{y,v}) = \delta_{x,u}$, $v \in F^*$ (by (ii), the point x is well defined). Since T^* is onto, there exists $\delta_{y_1,v_1} \in C(Y,F)^*$ such that $T^*(\delta_{y_1,v_1}) = \delta_{x,u_1}$. By the above remark, $y_1 = y$ and hence $T^*(\delta_{y,v_1}) = \delta_{x,u_1}$ and v_1 is the preimage of u_1. To show that the map is an isometry, we need only observe that for any $v \in F^*$ such that $\|v\| = 1$, the point $\delta_{y,v}$ is an extreme point of $S(C(Y,F)^*)$, hence $\delta_{x,u} = T^*(\delta_{y,v})$ is an extreme point of $S(C(X,E)^*)$ and $\|u\| = 1$. The last assertion of (iii) follows from the weak* continuity of T^*.

From Lemma 2 (ii), we can define a map $\phi: Y \to X$ such that $\phi(y) = x$. For each $y \in Y$, we let $\lambda(y)^*: F^* \to E^*$ be the map in Lemma 2 (iii). Since $\lambda(y)^*$ is weak* continuous, it induces a map $\lambda(y): E \to F$ which is also an isometric isomorphism. Hence we can define the map $\lambda: Y \to I(E,F)$ with $y \to \lambda(y)$. For any $v \in F^*, y \in Y$ and $f \in C(X,E)$, we have

$$\begin{aligned} &v(Tf(y)) \\ &\quad = \delta_{y,v}(Tf) = T^*(\delta_{y,v})f \\ &\quad = (\delta_{\phi(y),\lambda(y)^*v})(f) = (\lambda(y)^*v)(f(\phi(y))) \\ &\quad = v(\lambda(y)\cdot f(\phi(y))). \end{aligned}$$

Thus

$$Tf(y) = \lambda(y) \cdot f(\phi(y)).$$

It remains to show

LEMMA 3. *The map ϕ is a homeomorphism.*

Proof. That ϕ is onto follows from the fact T^* sends the set of elements of the form $\delta_{y,v}$, $y \in Y, v \in F^*$ onto the set of elements of the form $\delta_{x,u}$, $x \in X, u \in E^*$. That ϕ is one-to-one follows from the remark in the proof of the onto part in Lemma 2 (iii). It remains to show that ϕ is continuous. (ϕ^{-1} will then be continuous since X, Y are compact Hausdorff spaces). Let $\{y_\alpha\}$ be a net in Y converging to y. Fix $v \in F^*$ and let $T^*(\delta_{y_\alpha, v}) = \delta_{x_\alpha, u_\alpha}$; then $\{\delta_{x_\alpha, u_\alpha}\}$ converges weak* to $T^*(\delta_{y,v}) = \delta_{x,u}$. We want to show that $\{x_\alpha\}$ converges to x. Let $\{x_\beta\}, \{u_\beta\}$ be subnets of $\{x_\alpha\}, \{u_\alpha\}$ which converge weak* to $\bar{x}, \bar{u}$ respectively. For f in $C(X,E)$,

$$\begin{aligned}
&|\delta_{x,u}(f) - \delta_{\bar{x},\bar{u}}(f)| \\
&\quad \leqq |\delta_{x,u}(f) - \delta_{x_\beta,u_\beta}(f)| + |\delta_{x_\beta,u_\beta}(f) - \delta_{\bar{x},u_\beta}(f)| \\
&\qquad + |\delta_{\bar{x},u_\beta}(f) - \delta_{\bar{x},\bar{u}}(f)| \\
&\quad \leqq |\delta_{x,u}(f) - \delta_{x_\beta,u_\beta}(f)| + |u_\beta(f(x_\beta)) - u_\beta(f(\bar{x}))| \\
&\qquad + |u_\beta(f(\bar{x})) - \bar{u}(f(\bar{x}))| \\
&\quad \leqq |\delta_{x,u}(f) - \delta_{x_\beta,u_\beta}(f)| + \|f(x_\beta) - f(\bar{x})\| \, \|v\| \\
&\qquad + |u_\beta(f(\bar{x})) - \bar{u}(f(\bar{x}))|.
\end{aligned}$$

The right side converges to zero as $\{x_\beta\}$ and $\{u_\beta\}$ converge to $\bar{x}$ and $\bar{u}$ respectively. This shows that $x = \bar{x}$. The net $\{x_\alpha\}$ is in the compact set X and has only one limit point x, thus $\{x_\alpha\}$ converges to x.

LEMMA 4. *The map $\lambda: Y \to I(E,F)$ is continuous with respect to the strong topology on $I(E,F)$.*

Proof. Let $\{y_\alpha\}$ be a net in Y converging to y_0. For each z in E, we can find an f such that $f(x) = z$ for all x in X, thus

$$\|\lambda(y_\alpha)z - \lambda(y_0)z\| = \|Tf(y_\alpha) - Tf(y_0)\|.$$

Since Tf is in $C(Y,F)$, the right side converges to 0 as $\{y_\alpha\}$ converges to y_0. This shows that λ is continuous.

3. We give an example which shows that the theorem is not true if we do not assume that E^*, F^* are strictly convex. Let X be a compact Hausdorff space and let R^2 be the two dimensional linear space with the maximum norm ($\|(r, s)\| = \max\{|r|, |s|\}, r, s \in R$). It is clear that $C(X, R^2)$ is a Banach lattice with an order unit f_e where $f_e(x) = (1, 1)$ for all x in X. Also the norm satisfies $\|f \vee g\| = \|f\| \vee \|g\|$ for all f, g in the positive cone of $C(X, R^2)$. By Kakutani's representation theorem of abstract M spaces **[2]**, $C(X, R^2)$ is isometrically isomorphic to $C(Y, R)$ for some compact Hausdorff space Y. Thus, the theorem does not hold.

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