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ON LOOP SPACES WITHOUT P TORSION

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Let (X, m) be a 1-connected H-space whose loop space ΩX has no p torsion. We study the algebra structure of $H_*(\Omega X; Z_p)$ and its relation, via the Eilenberg-Moore spectral sequence, to that of $H^*(X; Z_p)$. The module $Q(H^*(X; Z_p))$ of indecomposables is a module over $A^*(p)$, the Steenrod algebra. Our main result is to show that, when X is finite, lack of torsion in the loop space is reflected in the $A^*(p)$ structure of $Q(H^*(X; Z_p))$.

1. Introduction. It is a standard conjecture that if (X, m) is a 1-connected finite H-space then its loop space ΩX is torsion free. We will show;

THEOREM 1.1. *Let (X, m) be a 1-connected finite H-space. Then $H^*(\Omega X; Z)$ has no p torsion if and only if*

$$Q(H^{\text{even}}(X; Z_p)) = \sum_{m \geqq 1} \beta_p \mathcal{P}^m Q(H^{2m+1}(X; Z_p)).$$

We note that for $p = 2$ our condition on $Q(H^{\text{even}}(X; Z_2))$ amounts to saying that $H^*(X; Z_2)$ has no indecomposables of even degree.

Our method of proving 1.1 differs from the usual approach to finite H-spaces. Rather than constructing "implications" which contradict finite dimensionality we will study the structural consequences for ΩX of lack of p torsion and then reinterpret the results, in terms of X.

The main tool in this proceedure will be the Eilenberg-Moore spectral sequence converging to $H^*(\Omega X; Z_p)$. We will show:

THEOREM 1.2. *Let (X, m) be a 1-connected finite H-space such that $H^*(\Omega X; Z)$ has no p torsion. Then, in the Eilenberg-Moore spectral sequence for the prime p, $E_p^{**}(X) = E_\infty^{**}(X)$.*

We will prove 1.2 by using results of Zabrodsky to severely limit the possible coalgebra structure on the term of convergence and from this conclude that it must agree with the pth term. The following theorem (or, more precisely, its dual in cohomology) is the major result needed in this respect.

THEOREM 1.3. *Let (X, m) be a 1-connected H-space where β_p acts trivially on $H^*(\Omega X; Z_p)$. Given $x \in H_*(\Omega X; Z_p)$, if x is of finite height then $x^p = 0$.*

The equivalences in 1.1 have a number of consequences for finite H-spaces. In particular:

THEOREM 1.4. *Let (X, m) be a 1-connected finite H-spaces such that $H^*(\Omega X; Z)$ has no p torsion. Then $H^*(X; Z)$ has no higher p torsion.*

THEOREM 1.5. *Let (X, m) be a 1-connected finite H-space of rank r such that $H^*(\Omega X; Z)$ has no p torsion. Then, for all $x \in H^*(X; Z_p)$ we have $x^{p^{r+1}} = 0$.*

A proof of 1.4 was asserted by Clark in **[4]**. However, as John Hubbuck has pointed out, Theorem 1.6 of his proof is incorrect. A counterexample is obtained by looking at $H^*(\Omega SU(3); Z)$ The difficulty arises from the fact that $H^*(\Omega SU(3); Z_2)$ has elements of finite height greater than 2. For, a restricted version of Theorem 1.6 is true. Let Q_p be the integers localized at the prime p. Given X, a 1-connected H-space where $H_*(\Omega X; Q_p)$ is torsion free, a splitting of $H_*(\Omega X; Q_p)$, as an algebra, into a tensor product of quasimonogenic Hopf algebras can be obtained from Theorem 1.3 of this paper. Also, by working over Q_p and using this restricted theorem Clark's arguments will go through. However we deduce Theorem 1.4 as a simple consequence of Theorem 1.1.

In §2 we will discuss the Eilenberg-Moore spectral sequence. In §3 we will look at Hopf algebras over the Steenrod algebra. In §4 we use Zabrodsky's work to analyze the algebra structure of $H_*(X; Z_p)$ for certain H-spaces (X, m). In particular we prove Theorem 1.3. In §5 we prove some near collapse results for the Eilenberg-Moore spectral sequence. In particular we prove Theorem 1.2. In §6 we prove Theorems 1.1, 1.4, and 1.5.

All spaces will be assumed to have the homotopy type of CW complexes of finite type. All Hopf algebras are graded, of finite type, and over Zp. A Hopf algebra will not be assumed to be either associative or commutative unless indicated. The dual of A, written A^*, will be defined by

$$(A^*)^m = \mathrm{Hom}(A^{-m}; Zp)$$

In particular the homology of spaces will be considered as being negatively graded. If a Hopf algebra is not connected then its dual is. We use $Q(A)$ and $P(A)$ to indicate indecomposables and primitives respectively.

Given a commutative associative Hopf algebra A it is isomorphic, as an algebra, to a tensor product $\bigotimes_{i\in I} A_i$ where each A_i is a Hopf algebra generated as a algebra by a single element a_i. Such a tensor product is called a Borel decomposition of A. The elements $\{a_i\}_{i\in I}$ are called generators of the decomposition. An element $x \in A$ is said to be of height n if $x^{n-1} \neq 0$ and $x^n = 0$ where $n < \infty$ or of height ∞ if no such n exists. The height of x is a power of p or ∞ unless x is of odd degree and p is odd in which case the height of x is 2. The symbols $E(x)$, $P(x)$, and $\Gamma(x)$ indicate exterior, polynomial, and divided polynomial Hopf algebras respectively, in each case generated by x. The symbol $T(x)$ indicates the Hopf algebra obtained from $P(x)$ by truncating x at height p. A divided polynomial Hopf algebra truncated at height p^n will mean a Hopf algebra whose dual is obtained from a polynomial Hopf algebra $P(x)$ by truncating x at height p^n.

2. The Eilenberg-Moore spectral sequence. Let (X, m) be a 1-connected H-space. Then there exists, for each prime, a second quadrant spectral sequence $\{E_r^{**}(X), d_r\}_{r\geqq 1}$ of commutative, associative, bigraded Hopf algebras where:

(2.1) $E_2^{**}(X) = \mathrm{Tor}_{H^*(X;Zp)}{}^{**}(Zp\,;Zp)$ as Hopf algebras

(2.2) $E_\infty^{**}(X) = E^0(H^*(\Omega Z; Zp))$ as Hopf algebras where $E^0(H^*(\Omega X; Zp))$ is the graded object associated to a filtration on $H^*(\Omega X; Zp))$ The filtration $\{F^{-n}(H^*(\Omega X; Zp))\}_{n\geqq 1}$ is an increasing one. In particular, $F^{-1}(H^*(\Omega X; Zp)$ is the image of the loop map

$$\Omega\colon Q(H^*(X; Zp)) \to P(H^*(\Omega X; Zp))$$

(2.3) The differentials d_r are maps of bidegree $(r, -r+1)$.

For the construction of this spectral sequence consult **[9]**. The spectral sequence has many properties.

(2.4) $\mathrm{Tor}_{H^*(X;Zp)}{}^{**}(Zp\,;Zp)$ is a tensor product $\bigotimes_{i\in I} A_i$ where each A_i is either an exterior Hopf algebra $E(a_i)$ where a_i has external degree -1, or a divided polymonial Hopf algebra $\Gamma(a_i)$ where a_i has external degree -1 or -2.

This decomposition $\otimes_{i\in I} A_i$ of $\mathrm{Tor}_{H^*(X;Zp)}{}^{**}(Zp;Zp)$ can be determined from any Borel decomposition of $H^*(X;Zp)$. For details again consult [9]. We merely note the following:

(2.5) There exists an isomorphism $Q(H^*(X;Zp)) \approx \mathrm{Tor}_{H^*(X;Zp)^{-1}}, {}^*(Zp;Zp)$, While if a_i is of external degree -2 then it has bidegree $(-2, 2pts)$ where $t > 0$ and $Q(H^s(X;Zp)) \neq 0$

Using 2.4 and the Hopf algebra structure of the spectral sequence we can argue as in [3] that:

(2.6) For $r \geqq 2$, d_r acts trivially unless $r = p^k - 1$ or $2p^k - 1$ for some $k > 0$. Furthermore, for such values of r, $E_r{}^{**}(X)$ can be regarded as a tensor product $\otimes_{j\in J} B_j$ where each B_j is either a trivial differential Hopf algebra or a Hopf algebra of the form $\Gamma(b_j) \otimes E(c_j)$ where $d_r(\gamma_k(b_j)) = c_j$ and b_j and c_j have external degrees as determined in 2.4 and 2.5.

We will use these properties in section §5. At the moment we will draw only one immediate conclusion. First of all we notice:

(2.7) Given a differential Hopf algebra $A = \Gamma(a) \otimes E(b)$ where $d(\gamma_k(a)) = b$, then the homology $H(A)$ of A is a divided polymonial Hopf algebra truncated at height p^k.

We use this to prove

PROPOSITION 2.8. *$H^*(\Omega X;Zp)$ and $E_\infty{}^{**}(X) = E^0(H^*(\Omega X;Zp)$ are isomorphic as coalgebras.*

Proof. From 2.4, 2.6, and 2.7 we conclude that any primitive element of $E_\infty{}^{**}(X)$ has external degree -1 or -2. Hence any representative of a primitive element in $E_\infty{}^{**}(X)$ is already primitive in $H^*(\Omega X;Zp)$. Now, by taking a Borel decomposition of $H_*(\Omega X;Zp)$ and dualizing, we see that $H^*(\Omega X;Zp)$ is isomorphic, as a coalgebra, to a tensor product $\otimes_{i\in I} A_i$ of Hopf algebras where each A_i is an exterior Hopf algebra or a divided polynomial Hopf algebra possibly truncated at height p^n cogenerated by an element a_i. Let B_i be the sub Hopf algebra of $E_\infty{}^{**}(X)$ generated by the elements represented by $\{\gamma_k(a_i) \neq 0 \mid k \geqq 0\}$. Then A_i and B_i are isomorphic as Hopf algebras since otherwise, for some $k > 0$, then on primitive element $\gamma_k(a_i)$ would give rise to a primitive element in B_i. Also $\otimes_{i\in I} B_i$ is a sub Hopf algebra of $E_\infty^{**}(X)$, that is, the nonzero monomials in the elements represented by $\{\gamma_k(a_i)\neq 0 \mid k \geqq 0, i \in I\}$ can be assumed to be linearly independent. For, because of the comultiplication map on $E_\infty^{**}(X)$, this will be true if the elements represented by $\{a_i \mid i \in I\}$ are linearly independent. And we can rewrite the elements of $\bigotimes_{i\in I} A_i$ if necessary in order to obtain this property. From the isomorphism of Hopf algebras $\bigotimes_{i\in I} A_i \cong \bigotimes_{i\in I} B_i$ we conclude that $E_\infty^{**}(X) = \bigotimes_{i\in I} B_i$ for reasons of dimension and hence that $H^*(\Omega X; Zp)$ and $E_\infty^{**}(X)$ are isomorphic as coalgebras.

Michael Barratt has also proven this result.

3. Hopf algebra over the Steenrod algebra. Let H be an associative commutative Hopf algebra on which the Steenrod algebra $A^*(p)$ acts so as to satisfy the Cartan formula. Further, suppose there exists N such that $H^i = 0$ if $i > N$. Any 1-connected finite H-space (X, m) gives rise to two examples of the above, namely $H^*(X; Zp)$ with its standard $A^*(p)$ action, and $H_*(\Omega X; Zp)$ with the adjoint $A^*(p)$ action (recall homology is negatively graded). In what follows we will also assume that $A^*(p)$ acts to the left. Any right module over $A^*(p)$ can be converted to a left module via the canonical antiautomorphism defined on $A^*(p)$ (see [7] for its properties). Hence, with the necessary modifications, our results in this section will hold for right modules as well.

$A^*(p)$ is a Hopf algebra and letting $A_*(p)$ be its dual we have:

(3.1) For $p = 2$ $A_*(p) = P(\xi_1, \xi_2, \cdots)$ as an algebra where ξ_1^n is dual to Sq^n
For p odd $A_*(p) = P(\xi_1, \xi_2, \xi_3, \cdots) \otimes E(\tau_0, \tau_1, \tau_2, \cdots)$ as an algebra where ξ_1^n is dual to $\mathcal{P}^n$

The action of $A^*(p)$ on H gives rise to a comodule structure

$$\lambda : H \to H \otimes A_*(p) \tag{3.2}$$

which is defined by the formula

(3.3) Given $x \in H$ such that $\lambda(x) = \Sigma_i x_i \otimes w_i$ then, for any $\theta \in A^*(p)$, $\theta(x) = \Sigma_i (-1)^{|w_i||x_i|} < \theta, w_i > x_i$.

Furthermore, λ is a ring homomorphism. The argument is based on that of [7].

Now pick a Borel decomposition $\bigotimes_{i\in I} A_i$ of H with generators $\{a_i\}_{i\in I}$. Then H has a Zp basis $\{a(T) = a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}\}$ consisting of all nonzero monomials in the generators. Likewise, by 3.1, $A_*(p)$ has a Zp basis $\{\xi(R)\otimes\tau(S)=\xi_1^{r_1}\cdots\xi_m^{r_m}\otimes\tau_0^{s_0}\cdots\tau_q^{s_q}\}$ where $R=(r_1,r_2\cdots)$ and $S=(s_1,s_2\cdots)$ range through all sequences with only finitely many nonzero terms. Hence $H\otimes A_*(p)$ has a basis $B=\{a(T)\otimes\xi(R)\otimes\tau(S)\}$ where R,S,T run through the appropriate sequences. Given $x\in H$, $\lambda(x)$ can be expanded in terms of a finite subset of B. The element $a(T)\otimes\xi(R)\otimes\tau(S)$ is said to appear non trivially in $\lambda(x)$ if the coefficient $\lambda_{T,R,S}$ of $a(T)\otimes\xi(R)\otimes\tau(S)$ is nonzero when we expand $\lambda(x)$ in terms of B.

PROPOSITION 3.4. *Given $x\in H$ suppose $a(T)\otimes\xi(R)$ appears nontrivially in $\lambda(x)$.*

Then $x^{p^k}=0$ for any $k\geqq 1$ implies that $a(T)^{p^k}=0$.

Proof. We have the identity

$$\Sigma\,\lambda_{T,R,S}\,a(T)^{p^k}\otimes\xi(R)^{p^k}\otimes\tau(S)^{p^k}=\lambda(x)^{p^k}=\lambda(x^{p^k})=0$$

Since the elements of B are linearly independent $\lambda_{T,R}\neq 0$ implies $a(T)^{p^k}\otimes\xi(R)^{p^k}=0$. Therefore $a(T)^{p^k}=0$ since $\xi(R)$ belongs to a polynomial algebra.

4. The algebra structure of $H_*(X;Zp)$.

In this section we will prove Theorem 1.3. To prove it we will use techniques of Zabrodsky. He has obtained the following result:

LEMMA 4.1. *Let (X,m) be an H-space where $H_*(X;Zp)$ is commutative and associative and βp acts trivially on $H^{\text{even}}(X;Zp)$. Also, for $p=2$, β_2 acts trivially on $H^{\text{odd}}(X;Z_2)$ as well. Given $x\in P(H_{-2m}(X;Zp))\cap\operatorname{Ker}\mathscr{P}^1$ where $m\not\equiv 1\pmod p$ then $x^p\neq 0$.*

For a proof of this see Proposition 4.3 of **[11]**. For the case $p=2$ we are using a stronger hypothesis regarding the action of the Bockstein than Zabrodsky does. Zabrodsky's weaker hypothesis is not sufficient to ensure the validity of his proof in that case. The problem is that the Bockstein Sq^1 appears in the Cartan formula of Sq^2.

Before proving 1.3 we use Lemma 4.1 to obtain the following:

THEOREM 4.2. *For p odd let (X,m) be an H-space where $H_*(X;Zp)$ is associative and commutative and β_p acts trivially on $H^{\text{even}}(X;Zp)$. Given $x\in H_*(X;Zp)$ if x is of finite height then $x^p=0$.*

Proof. We first observe that all elements of odd degree must be of height 2. We now prove 4.2 by contradiction. Pick the element x of highest degree such that x is of finite height greater than p(say p^{n+1}). Then $x^{p^k} \in H_{-2p^{nk}}(X; Zp)$ for some $k > 0$. Moreover x^{p^n} is primitive. To see this let $\bar{\Delta}_* = \Delta - q_{1*} - q_{2*}$ where Δ, q_1, q_2: $X \to X \times X$ are the diagonal map and the two standard inclusion maps respectively. Then, for $t \geqq 0$

$$(*) \qquad \bar{\Delta}_*(x^{p^t}) = [\bar{\Delta}_*(x)]^{p^t}$$

Since x is of finite height, $\bar{\Delta}_*(x)$ is of finite height. Therefore, by our choice of $x, \bar{\Delta}_*(x)^p = 0$ and x^{p^n} is primitive. Lastly, by the Cartan formula, $x^{p^n} \in \operatorname{Ker} \mathcal{P}^1$. By 4.1 we have $x^{p^{n+1}} \neq 0$, a contradiction.

We will now prove Theorem 1.3. For the case p odd it is an immediate consequence of 4.2. For $p = 2$ we first observe that all elements of odd degree in $H_*(\Omega X; Z_2)$ are of height 2 since they come from elements in $H_*(\Omega X; Z_4)$. Other than on this point our proof is a repetition of that given for 4.2.

We close this section by listing a result of Browder which will be needed in the next two sections. See **[1]** for a proof.

LEMMA 4.3. *Let (X, m) be a 1-connected finite H-space. Then $H^*(\Omega X; Z)$ has no p torsion if and only if $H_{2i+1}(\Omega X; Zp) = H^{2i+1}(\Omega X; Zp) = 0$ for all i.*

5. Near collapse results for the Eilenberg-Moore spectral sequence. In this section we will prove 1.2. We first use the results of the last two sections to conclude that:

THEOREM 5.1. *Let (X, m) be a 1-connected H-space where β_p acts trivially on $H^*(\Omega X; Z_p)$. Then, in the Eilenberg-Moore spectral sequence for the prime p, $E_p{}^{**}(X) = E_\infty{}^{**}(X)$ if p is odd and $E_4{}^{**}(X) = E_\infty{}^{**}(X)$ if $p = 2$.*

To prove 5.1 we first observe from §2 combined with 1.3 that $E_{2p}{}^{**}(X) = E_\infty{}^{**}(X)$ and $E_{2p}{}^{**}(X)$ is a tensor product $\bigotimes_{i \in I} A_i$ where each A_i is a Hopf algebra of the form $E(a_i)$ or $T(a_i)$ or $\Gamma(a_i)$, a_i having external degree -1 or -2 and internal as determined in 2.5. Moreover $E_p{}^{**}(X) = E_{2p}{}^{**}(x)$ if and only if $E_{2p}{}^{**}(X)$ has no factors $T(a)$ where a is of external degree -2. Thus we must exclude the possibility of such factors.

Let A be the Hopf ideal of $H^*(\Omega X; Zp)$ generated by the image of the loop map Ω from 2.2. Consider the quotient Hopf algebra $B =$

$H^*(\Omega X; Zp)/A$. Since the image of Ω is invariant under the action of $A^*(p)$ it follows that B inherits a Steenrod module structure from $H^*(\Omega X; Zp)$. B also inherits a filtration from $H^*(\Omega X; Zp)$. If $E^0(B)$ is the associated graded Hopf algebra we can see that it is obtained from $E_\infty^{**}(X)$ by equating to zero the Hopf ideal generated by the elements of external degree -1. Furthermore B and $E^0(B)$ are isomorphic as coalgebras. Let B^* be the dual of B. Then B^* is a sub Hopf algebra of $H_*(\Omega X; Zp)$ over $A^*(p)$. Also $E_p^{**}(x) = E_{2p}^{**}(X)$ if and only if B^* is a polynomial algebra. For, if the factors $T(a)$ of the last paragraph exist they will give rise to elements of height p in B^*.

To see when B^* must be a polynomial algebra we first prove:

LEMMA 5.2. *If B^* is not a polynomial algebra then there exists a nonzero $x \in P(H_{-2ps+2}(\Omega X; Zp))$ where $x^p = 0$ and $x\theta = 0$ for all $\theta \in A^*(p)$, of positive degree.*

Proof. We need only find such an element in B^*. If B^* is not a polynomial algebra we can find $x \neq 0$ such that $x^p = 0$. Pick such a x of the highest possible degree. By 2.5 $x \in H_{-2ps+2}(\Omega X; Zp))$. By the formula $(*)$ in section §4 x must be primitive. Since β_p acts trivially on B^* it follows from 3.4 that $x\theta = 0$ for any $\theta \in A^*(p)$.

For odd primes we conclude that B^* must be a polynomial algebra. For, by 4.1, the properties possessed by x in 5.2 are incompatible.

This concludes our proof of Theorem 5.1.

We now prove Theorem 1.2. For p odd 1.2 follows immediately from 5.1. For $p = 2$ we begin by repeating the proof of 5.1 up to the end of Lemma 5.2. We now show that under the stronger hypothesis of 1.2 the properties possessed by x in 5.2 are incompatible for $p = 2$ as well. First of all x in 5.2 can be shown to possess one more property, namely $x \in H_{-n}(\Omega X; Z_2)$ where $n = 2^{q+1}Q + 2^q - 2$ and $q, Q > 0$. For, by 2.5, x has degree $-2^{a+1}b + 2$ where $Q(H^b(X; Z_2)) \neq 0$ and $E_2^{**}(X)$ has an element of bidegree $(-1, b)$. If b is even this element cannot survive to $E_4^{**}(X) = E_\infty^{**}(X)$ because of 4.3. But it would then follow from the definition of B^* that B^* has an element of higher degree than x such $y \neq 0$ and $y^2 = 0$. This contradicts the manner in which we choose x in 5.2. Hence b is odd. In particular, since X is 1-connected, $b > 1$.

Let $P_2(X)$ be the projective plane of an H-space (X, m). We have long exact sequences:

$$\text{(5.3)} \qquad \cdots \xleftarrow{\lambda^*} \tilde{H}^i(X \wedge X; Z_2) \xleftarrow{\phi^*} \tilde{H}^i(X; Z_2) \xleftarrow{\iota^*} \tilde{H}^{i+1}(P_2(X); Z_2)$$
$$\xleftarrow{\lambda^*} \tilde{H}^{i-1}(X \wedge X; Z_2) \xleftarrow{\phi^*} \cdots$$

(5.4)
$$\cdots \xrightarrow{\lambda_*} \tilde{H}_i(X \wedge X; Z_2) \xrightarrow{\phi_*} \tilde{H}_i(X; Z_2) \xrightarrow{\iota_*} \tilde{H}_{i+1}(P_2(X); Z_2)$$
$$\xrightarrow{\lambda_*} \tilde{H}_{i-1}(X \wedge X; Z_2) \xrightarrow{\phi_*} \cdots$$

The sequences are dual to each other and all maps respect the Steenrod operations. Moreover:

(5.5) Image $\iota^* = P(H^*(X; Z_2)$

(5.6) ϕ_* agrees with the Pontryagin product if we equate $\tilde{H}_*(X \wedge X; Z_2)$ with $\tilde{H}_*(X; Z_2) \otimes \tilde{H}_*(X; Z_2)$

(5.7) Given $a \in P(\tilde{H}^*(X; Z_2))$ and $b \in \tilde{H}^*(P_2(X); Z_2)$ such that $\iota^*(b) = a$ then $b^2 = \lambda^*(a \otimes a)$.

As a reference for the projective plane and its properties consult **[2]**.

Now since $x^2 = 0$ in $\tilde{H}_*(\Omega X; Z_2)$ we can find $y \in \tilde{H}_*(P_2(\Omega X); Z_2)$ such that $\lambda_*(y) = x \otimes x$. Pick $a \in P(H^n(\Omega X; Z_2)$ such that $\langle x, a\rangle \neq 0$. Pick $b \in H_{n+1}(P_2(\Omega X); Z_2)$ such that $\iota^*(b) = a$.

Then

(5.8)
$$\langle y, Sq^{n+1}(b)\rangle = \langle y, b^2\rangle = \langle y, \lambda^*(a \otimes a)\rangle = \langle \lambda_*(y), a \otimes a\rangle$$
$$= \langle x \otimes x, a \otimes a\rangle \neq 0.$$

But it is also true that

LEMMA 5.9. *$ySq^{2i} = 0$ for $i > 0$.*

Proof. In even degrees $\operatorname{Ker} \lambda_* = 0$ by 4.3. Hence $ySq^{2i} \neq 0$ implies
$$(x \otimes x)Sq^{2i} = \lambda_*(y)Sq^{2i} = \lambda_*(ySq^{2i}) \neq 0.$$

But by 5.2 and the Cartan formula $(x \otimes x)Sq^{2i} = 0$.

We can now apply a reduction argument to conclude that x in 5.2 cannot exist.

Given $m = 2^{r+1}R + 2^r - 1$ where $r, R > 0$ we have an Adem relation of the form

(5.10)
$$Sq^{2^r}Sq^{m-2^r} = Sq^m + \sum_{0 \leq t \leq 2^{r-1}} \alpha_t Sq^{m-t}Sq^t$$

where $\alpha_t \in Z_2$.

We use 5.10 in proving:

LEMMA 5.11. *Given $c \in H^*(P_2(\Omega X); Z_2)$ such that*

$$\langle y, Sq^{2^{r+1}R+2^r} - 1\,(c)\rangle \neq 0 \text{ where } R > 0 \text{ and } r > 1$$

we can find $d \in H^*(P_2(\Omega X); Z_2)$ such that

$$\langle y, Sq^{2^{s+1}S+2^s-1}(d) > \neq 0 \text{ where } S > 0 \text{ and } 0 < s < r.$$

Proof. Letting $m = 2^{r+1}R + 2^r - 1$ we have, by 5.10, that

$$Sq^m(c) = Sq^{2^r}Sq^{m-2^r}(c) + \sum_{0 \leqq t \leqq 2^{r-1}} \alpha_t Sq^{m-t}Sq^t(c).$$

By 5.9

$$\langle y, Sq^{2^r}Sq^{m-2^r}(c)\rangle = \langle ySq^{2^r}, Sq^{m-2^r}(c)\rangle = 0$$

Hence there exists $0 \leqq t \leqq 2^{r-1}$ such that $\alpha_t = 1$ and

$$\langle y, Sq^{m-t}Sq^t(c)\rangle \neq 0$$

Also, t is even since for t odd

$$\langle y, Sq^{m-t}Sq^t(c)\rangle = \langle ySq^{m-t}, Sq^t(c)\rangle = 0$$

Pick any such t and let $d = Sq^t(c)$. Since $t \leqq 2^{r-1}$ it follows that $m - t = 2^{s+1}S + 2^s - 1$ where $S > 0$ and $0 < s < r$.

We apply 5.11 in turn to prove:

LEMMA 5.12. *There exists $T > 0$ and $e \in H^*(P_2(\Omega X); Z_2)$ such that $\langle y, Sq^{4T+1}(e)\rangle \neq 0$.*

Proof. Let 0 be the collection of ordered pairs (s, S) of positive integers such that $\langle y, Sq^{2^{s+1}S+2^s-1}(e)\rangle \neq 0$ for some $e \in H^*(P_2(\Omega X); Z_2)$. By 5.8 0 is nonempty. Let $\bar{s}$ be the smallest integer which appears in the first factor. By 5.11 $\bar{s} = 1$.

But now consider the Adem relation

$$Sq^2Sq^{4T-1} = Sq^{4T+1} + Sq^{4T}Sq^1. \tag{5.13}$$

Again, using 5.9, we conclude

$$\langle y, Sq^2 Sq^{4T-1}(e)\rangle = 0$$

and

$$\langle y, Sq^{4T} Sq^1(e)\rangle = 0.$$

Hence 5.12 and 5.13 are incompatible.

This concludes our proof of 1.2.

6. Proof of main theorems.

Proof of 1.1. First consider the case $p = 2$. Suppose $H^*(X; Z_2)$ has no even degree indecomposables. By 2.5 $E_2^{**}(X)$ has no elements whose total degree is odd. The spectral sequence collapses and, by 4.3, ΩX has no 2 torsion. Conversely, suppose ΩX has no 2 torsion. By 1.2 and 4.3 $E_2^{**}(X)$ has no elements whose total degree is odd. By 2.5 $H^*(X; Z_2)$ has no indecomposables with even degree.

For odd primes we make an analogous argument. The one extra fact used is the characterization of the differential d_{p-1} for odd primes in terms of the Steenrod powers $\beta_p \mathcal{P}^m$. For details consult Theorem 14 of **[5]** and Theorem 2.3 of **[6]**.

Proof of 1.4. This is an application of the Bockstein spectral sequence $\{B_r^*(X)\}_{r \geqq 1}$ ((see **[1]**). From 1.1 we conclude that $B_2^*(X)$ is an exterior algebra on generators of odd degree. Hence $B_2^*(X) = B_\infty^*(X)$ and there is no higher torsion.

Proof of 1.5. Pick a Borel decomposition of $H^*(X; Z_p)$ with generators $a_1, a_2, \cdots a_n$. Since $H^*(X; Z_p)$ is commutative and associative we need only prove the theorem for the above generators. We need the following results of Browder.

LEMMA 6.1. *Given $x \in H^{2s}(X; Z_p)$ then $\beta_p(x) = \beta_p(y)$ where y is decomposable.*

See Lemma 4.5 of **[1]** for a proof of this.

LEMMA 6.2. *Any Borel decomposition of $H^*(X; Z_p)$ has exactly r generators of odd degree and at most r generators of even degree.*

Proof. The restriction on the generators of odd degree comes from Corollary 3.12 of **[1]**. The restriction on even degree generators then follows from 1.1.

We will now prove 1.5 in two parts.

(i) *Case* $p = 2$. We apply Lemma 6.1 to obtain the following

LEMMA 6.3. *If* $a \in H^{2n+1}(X; Z_2)$ *is a generator of the decomposition and* $a^2 \neq 0$ *then we can find another generator* $b \in H^{4n+1}(X; Z_2)$ *such that* $a^2 \otimes \xi_1$ *appears nontrivially in* $\lambda(b)$.

We now prove the theorem for the case $p = 2$ by contradiction. Let a_1 be a generator such that $a_1^{2^{r+1}} \neq 0$. By 1.1 a_1 is of odd degree. In particular $a_1^2 \neq 0$ and, by 6.3, we can find another odd degree generator a_2 such that $a_1^2 \otimes \xi_1$ appears non trivially in $\lambda(a_2)$. By 3.4 $a_2^{2^r} \neq 0$. Since X is 1-connected a_1 and a_2 are distinct.

Assume by induction that for $1 \leqq k < r + 1$ we have found distinct generators $a_1, a_2, \cdots a_k$ of odd degree such that for $2 \leqq i \leqq k$ $a_i^2 \otimes \xi_1$ appears nontrivially in $\lambda(a_{i+1})$ and $a_k^{2^{r-k+1}} \neq 0$. As above we can find a_{k+1} such that $a_k^2 \otimes \xi_1$ appears nontrivially in $\lambda(a_{k+1})$ and $a_{k+1}^{2^{r-k}} \neq 0$.

It follows that we can produce $r + 1$ odd degree generators of a Borel decomposition of $H^*(X; Z_2)$ in contradiction to 6.2.

(ii) *Case p odd.* We argue in an analogous manner with the exception that, this time, we consider only generators of even degree.

First, from 1.1, we see that $Q(H^{2s}(X; Z_p)) = 0$ unless $s \equiv 1 \pmod p$. From this we deduce:

LEMMA 6.4. *If* $a \in H^{2mp+2}(X; Z_p)$ *is a generator of the decomposition and* $a^p \neq 0$ *then we can find another generator* $b \in H^{2mp^2+2}(X; Z_p)$ *such that* $a^p \otimes \xi_1$ *appears nontrivially in* $\lambda(b)$.

As in the case $p = 2$ we can use this lemma to show that $a^{p^{r+1}} \neq 0$ will produce a contradiction to 6.2.

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