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ON SEMIGROUPS IN WHICH $X = XYX = XZX$ IF AND ONLY IF $X = XYZX$

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A semigroup S will be called *quasi-rectangular* if the set of idempotents of S is non-empty and a rectangular band ideal of S. The theorems of this note prove in part that the following are equivalent. (1) S is a semilattice of semigroups each of which is either idempotent free or quasi-rectangular. (2) Every $\mathcal{J}$-class of S is either idempotent free or a rectangular subband of S. (3) Every $\mathcal{D}$-class of S is either idempotent free or a rectangular subband of S. (4) S is a semigroup in which for any $x, y, z \in S$, $x = xyx = xzx$ if and only if $x = xyzx$.

Recently M. S. Putcha and J. Weissglass ([4]) have given a characterization of a semigroup each of whose $\mathcal{D}$-classes has at most one idempotent. Using results in [4], this note gives also a characterization of a semigroup each of whose $\mathcal{D}$-classes is either idempotent free or consists of a single idempotent. Also, $\mathcal{D}$ may be replaced by $\mathcal{J}$ in the above statement.

Throughout this note S will denote a semigroup and $E(S)$ the set of idempotents of S. Let the set-valued functions I and $\bar{I}$ on S be defined by $I(x, S) = \{e \mid e \in E(S),\ e = exe\}$ and $\bar{I}(x, S) = \{y \mid y \in S,\ y = yxy\}$, respectively. We shall write E, $I(x)$ and $\bar{I}(x)$ for $E(S)$, $I(x, S)$ and $\bar{I}(x, S)$, respectively, when there is no possibility of confusion.

PROPOSITION 1. *The following are equivalent.*

(1) $\bar{I}(x) \cap \bar{I}(y) = \bar{I}(xy)$ *for every* $x, y \in S$.

(2) $I(x) \cap I(y) = I(xy)$ *for every* $x, y \in S$.

In this case we have $\bar{I}(x) = I(x)$ *for every* $x \in S$.

Proof. (1) $\Rightarrow$ (2) follows from $\bar{I}(x) \cap E = I(x)$ for every $x \in S$.

(2) $\Rightarrow$ (1). We will prove that $\bar{I}(x) = I(x)$ for every $x \in S$. Let $a \in \bar{I}(x)$. Then $a = axa$. Hence $ax = (ax)(ax) = (ax)(ax)(ax)$. Thus $ax \in I(ax) = I(a) \cap I(x)$. Hence $ax \in I(a)$, i.e., $ax = (ax)a(ax)$. Hence $axa = (axa)(axa)$, i.e., $a = a^2$. Therefore $a \in \bar{I}(x) \cap E = I(x)$. Thus $\bar{I}(x) \subseteq I(x)$. Clearly $I(x) \subseteq \bar{I}(x)$. Hence $\bar{I}(x) = I(x)$ for every $x \in S$.

PROPOSITION 2. *Let N be the set of elements x of S such that $\bar{I}(x) = \varnothing$. If N is nonempty then N is an ideal of S and idempotent free.*

Proof. Suppose that N is nonempty. It is easy to see that N is idempotent free. Let $x \in N$ and $y \in S$. If $xy \notin N$ there exists $a \in S$ such that $a = axya$. Hence $ya = (ya)x(ya)$ and so $ya \in \bar{I}(x)$. This contradicts the fact that $\bar{I}(x) = \varnothing$. Thus $xy \in N$. Similarly $yx \in N$. This completes our proof.

LEMMA 1. *Let N be an idempotent free ideal of S. Then S satisfies $I(x, S) \cap I(y, S) = I(xy, S)$ for every x, $y \in S$ if and only if the Rees factor semigroup S/N satisfies $I(x, S/N) \cap I(y, S/N) = I(xy, S/N)$ for every x, $y \in S/N$.*

Proof. Let 0 denote the equivalence class N in S/N. Since N is idempotent free $E(S/N) = E(S) \cup \{0\}$. If $a, x \notin N$, then $a \in I(x, S)$ if and only if $a \in I(x, S/N)$. Furthermore $I(0, S/N) = 0$ and $I(z, S) = \varnothing$ for $z \in N$, since N is an idempotent free ideal of S. Hence $I(x, S) \cup \{0\} = I(\bar{x}, S/N)$ for every $x \in S$, where $\bar{x} = x$ if $x \notin N$ and $\bar{x} = 0$ if $x \in N$. From this, our result follows easily.

From Proposition 1, Proposition 2 and Lemma 1, we have the following

THEOREM 1. *Let $E(S) \neq \varnothing$. The following are equivalent.*

(1) *$I(x, S) \cap I(y, S) = I(xy, S)$ for every x, $y \in S$.*

(2) *S is an ideal extension of an idempotent free semigroup (possibly empty) by a semigroup T such that $I(x, T) \cap I(y, T) = I(xy, T)$ and $I(x, T) \neq \varnothing$ for every x, $y \in T$.*

Let τ be a congruence on S. If S/τ is a semilattice, τ is called a semilattice congruence on S. In this note, ρ denotes the smallest semilattice congruence on S and σ denotes the relation on S defined by $x \sigma y$ if and only if $I(x) = I(y)$. If $\rho = S \times S$, then S is called s-indecomposable. Furthermore, for any congruence τ on a semigroup S we denote by $\tau \mid E$ the restriction of τ to E and by $x\tau$ the equivalence class mod τ containing an element x.

Now we note that S is quasi-rectangular if and only if $E(S)$ is nonempty and $e = exe$ for every $e \in E(S)$ and $x \in S$.

THEOREM 2. *The following are equivalent.*

(1) *$I(x) \cap I(y) = I(xy)$ for every x, $y \in S$.*

(2) (i) *σ is a semilattice congruence on S,*

(ii) *each σ-class is either idempotent free or a quasi-rectangular semigroup.*

(3) *S is a semilatice of s-indecomposable semigroups each of which is either idempotent free or quasi-rectangular.*

(4) *S is a semillatice of semigroups each of which is either idempotent free or quasi-rectangular.*

In this case, for a semilattice congruence τ on S induced by the decomposition in (4) we have $\rho \subseteq \tau \subseteq \sigma$ and $\rho \mid E = \tau \mid E = \sigma \mid E$. Moreover, for any $a, b \in E$ we have $a\ \sigma\ b$ if and only if $a = aba$ and $b = bab$.

roof. (1) $\Rightarrow$ (2) follows from easy calculations.

(1) $\Rightarrow$ (3). S is a semilattice of s-indecomposable semigroups ([5]). On the other hand, since S satisfies $I(x) \cap I(y) = I(xy)$ for every $x, y \in S$, any subsemigroup of S satisfies also the same. Therefore, if we consider the congruence σ on each component of S, it follows from (2) (ii) above that any component is idempotent free or quasi-rectangular. Thus (3) holds.

(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4) a fortiori.

(4) $\Rightarrow$ (1). Let τ be the congruence induced by the decomposition in (4) and let $x, y \in S$. If $a \in I(x) \cap I(y)$, we have $a = axa = aya$. Since τ is a semilattice congruence on S, we have $a\ \tau\ ax\ \tau\ ay$. Hence $axy \in a\tau$. On the other hand, $a \in a\tau \cap E$. Hence $a = a(axy)a = axya$. Thus $a \in I(xy)$. Conversely, if $a \in I(xy)$ we have $a = axya$. Hence $a\ \tau\ axy$. Thus $ay\ \tau\ axy^2\ \tau\ axy$. Hence $ay \in a\tau$. Since $a \in a\tau \cap E$, $a = a(ay)a = aya$. Hence $a \in I(y)$. Similarly, $a \in I(x)$. Hence $a \in I(x) \cap I(y)$. Therefore $I(x) \cap I(y) = I(xy)$, i.e., (1) holds.

Now let $x, y \in S$ such that $x\ \tau\ y$. Let $a \in I(x)$. Then $a = axa$. Hence $ax \in ax\tau \cap E$ and $ay \in ax\tau$. Since $ax\tau$ is quasi-rectangular, $ax = (ax)(ay)(ax)$. Hence $a = axa = (ax)(ay)(ax)a = (axa)y(axa) = aya$, i.e., $a \in I(y)$. Thus $I(x) \subseteq I(y)$. By symmetry, $I(y) \subseteq I(x)$. Hence $I(x) = I(y)$. Thus $x\ \sigma\ y$. This shows that $\tau \subseteq \sigma$. On the other hand, clearly $\rho \subseteq \tau$. Now let $a, b \in E$. If $a\ \sigma \mid E\ b$, then $a, b \in I(a) = I(b)$. Hence $a = aba$ and $b = bab$. Conversely, if $a = aba$ and $b = bab$ we have $a\ \rho \mid E\ b$ since ρ is a semilattice congruence on S. On the other hand, $\rho \subseteq \tau \subseteq \sigma$. Hence $\rho \mid E = \tau \mid E = \sigma \mid E$.

COROLLARY. *Let S be a semigroup such that $I(x) \cap I(y) = I(xy)$ and $x\rho \cap E \neq \varnothing$ for every $x, y \in S$. Then:*

(1) *$\rho = \tau = \sigma$, where τ is a congruence induced by the decomposition in Theorem* 2 (4).

(2) *S is s-indecomposable if and only if E is a rectangular band. In this case,. S is quasi-rectangular.*

Proof. (1) Let $x, y \in S$ such that $x \sigma y$. Let $a \in x\rho \cap E$ and $b \in y\rho \cap E$. Since $\rho \subseteq \tau \subseteq \sigma$, $a \sigma x \sigma y \sigma b$, that is, $a \sigma | E b$. Hence $a \rho | E b$ by Theorem 2. Therefore $x \rho a \rho b \rho y$, i.e., $s \rho y$. Since $\rho \subseteq \tau \subseteq \sigma$, this shows that $\rho = \tau = \sigma$.

(2) Let S be s-indecomposable. From Theorem 2 (3), S is quasi-rectangular and so E is a rectangular band. Conversely, let E be a rectangular band. Let $x, y \in S$. Then there exist $a \in x\rho \cap E$ and $b \in y\rho \cap E$. Since $a = aba$ and $b = bab$, $a \rho b$ and so $x \rho y$. Hence S is s-indecomposable.

We shall say that S has the *decomposition* (D) if S satisfies the following condition (D).

(D) $E(S)$ is nonempty and $E(S)$ is a disjoint union of maximal rectangular subbands $E_\alpha (\alpha \in \Gamma)$ of S, that is, if M is a rectangular subband of S and $M \cap E_\alpha \neq \varnothing$ for $\alpha \in \Gamma$, then $M \subseteq E_\alpha$.

In this case, each $E_\alpha (\alpha \in \Gamma)$ will be called a (D)-*component* of E.

Proposition 3. *Let S be a semigroup such that E is nonempty. Then the following are equivalent.*

(1) $I(x) \cap I(y) \subseteq I(xy)$ *for every* $x, y \in S$.

(2) *S has the decomposition* (D).

Proof. (1) $\Rightarrow$ (2). Let τ be the relation on E defined by $u \tau v$ if and only if $u = uvu$ and $v = vuv$. We shall prove that if (1) holds then τ is an equivalence relation on E. The reflexive law and the symmetric law hold evidently. We prove that the transitive law holds. Let $u \tau v$ and $v \tau w$. Then $v = vuv = vwv$. Since $v \in I(u) \cap I(w) \subseteq I(uwu)$, $v = v(uwu)v$. Therefore $u = uvu = u\{v(uwu)v\}u = (uvu)w(uvu) = uwu$. Similarly $w = wuw$. Hence $u \tau w$. The decomposition of E by τ shows that S has the decomposition (D).

(2) $\Rightarrow$ (1). Let $e \in E$ and $x, y \in S$ such that $e \in I(x) \cap I(y)$. Since $e = exe = eye$, $\{e, ex\}$ and $\{e, ye\}$ are rectangular subbands of S. On the other hand, there exists a (D)-component E_α such that $e \in E_\alpha$. Then $\{e, ex\} \cap E_\alpha \neq \varnothing$ and $\{e, ye\} \cap E_\alpha \neq \varnothing$. Hence $e, ex, ye \in E_\alpha$ by (2). Thus $exye = (ex)(ye) \in E_\alpha$. Hence $e = e(exye)e = exye$. This shows that (1) holds.

Remark. It is well known that any band has the decomposition (D) where the set Γ of suffixes is a semilattice and $E_\alpha E_\beta \subseteq E_\gamma$ if $\alpha\beta = \gamma$ for $\alpha, \beta, \gamma \in \Gamma$ ([2] and [3]). But, even if a semigroup S satisfies $I(x) \cap I(y) = I(xy)$ for every $x, y \in S$ and E is nonempty, E need not be a subsemigroup of S. The following example shows it.

	x	y	z	u
x	x	z	z	u
y	u	y	u	u
z	u	z	u	u
u	u	u	u	u

, where $I(x) = \{x, u\}$, $I(y) = \{y, u\}$, $I(z) = \{u\}$ and $I(u) = \{u\}$.

PROPOSITION 4. *The following are equivalent.*

(1) $I(xy) \subseteq I(x) \cap I(y)$ *for every* $x, y \in S$.

(2) *For any* $x, y \in S$, $e = exe$ *whenever* $e = xy \in E$.

(3) *For any* $x, y \in S$, $e = eye$ *whenever* $e = xy \in E$.

(4) (i) *For any* $x, y \in S$, *if* $xy, yx \in E$ *then* $\{xy, yx\}$ *is contained in a rectangular subband of* S,

(ii) *for any* $x, y \in S$, $e = ex = ey$ *whenever* $e = xy = yx \in E$.

Proof. (1) $\Rightarrow$ (2) follows from $e \in I(e) = I(xy) \subseteq I(x) \cap I(y) \subseteq I(x)$.

(2) $\Rightarrow$ (1). Let $e \in I(xy)$. Then $e = exye$ and hence $e = e(ex)e = exe$. Set $u = yex$. Then $u \in E$ and hence $u = uyeu$ by (2). Thus $yex = yexyeyex$. Hence $ex(yex)e = ex(yexyeyex)e$. Therefore $e = (exye)(exe) = exyexe = exyexyeyexe = (exyexye)y(exe) = eye$. Hence $e \in I(x) \cap I(y)$. Thus $I(xy) \subseteq I(x) \cap I(y)$.

(1) $\Leftrightarrow$ (3) is proved by the same way as used in the proof of (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (4) (i). Since $xy \in I(xy) = I(xyxy) \subseteq I(yx)$ by (1), $xy = xy(yx)xy$. Similarly $yx = yx(xy)yx$. Hence (4) (i) holds.

(1) $\Rightarrow$ (4) (ii). If $e = xy = yx \in E$ for $x, y \in S$, then $e \in I(xy) \subseteq I(x) \cap I(y)$. Hence $e = exe = eye$ and so $e = ex = ey$.

(4) $\Rightarrow$ (1). Let $e \in I(xy)$. Then $e = exye = (ex)(ye)$. Set $u = (ye)(ex)$. Then $u \in E$. Hence $e = eue$ and $u = ueu$ by (4) (i), that is, $e = eyexe$ and $yex = yexeyex$. Hence $ex(yex)ye = ex(yexeyex)ye$, that is, $(exye)(exye) = (exye)xey(exye)$. Hence $e = exeye$. Thus $e = (eye)(exe) = (exe)(eye)$. Hence, by (4) (ii), $e = e(exe) = exe$ and $e = e(eye) = eye$. This shows that $I(xy) \subseteq I(x) \cap I(y)$.

PROPOSITION 5. *Let S be a semigroup such that E is a left* (*right*) *ideal of S. Then* $I(x) \cap I(y) = I(xy)$ *for every* $x, y \in S$.

Proof. Let E be a left ideal of S. Since E is a band, S has the decomposition (D). Now let $e = xy \in E$. Then $ye \in E$. Hence $ye = yeye$ and so $xye = xyeye$. Thus $e = eye$. Therefore the condition (3) in Proposition 4 holds. Hence, by Proposition 3 and Proposition 4,

$I(x) \cap I(y) = I(xy)$ for every $x, y \in S$. In the case that E is a right ideal of S, we can prove it by the same way.

THEOREM 3. *The following are equivalent.*

(1) $I(x) \cap I(y) = I(xy)$ *for every* $x, y \in S$.

(2) *Every* $\mathcal{J}$*-class of* S *is either idempotent free or a rectangular subband of* S.

(3) *Every* $\mathcal{D}$*-class of* S *is either idempotent free or a rectangular subband of* S.

Proof. (1) $\Rightarrow$ (2). Let $x, y \in S$. Suppose that $x \mathcal{J} y$. Then there exist $a, b, c, d \in S^1$ such that $x = ayb$ and $y = cxd$. Hence $I(x) = I(ayb) \subseteq I(y)$ and $I(y) = I(cxd) \subseteq I(x)$. Thus $I(x) = I(y)$. This shows that $x \mathcal{J} y$ implies $x \sigma y$. Now suppose that $e \mathcal{J} x$ and $e \mathcal{J} y$ for $e \in E(S)$ and $x, y \in S$. We shall prove $e \mathcal{J} xy$. Since $I(e) = I(x) = I(y)$, $e \in I(e) = I(x) \cap I(y) = I(xy)$. Hence $e = exye$ and so $S^1eS^1 \subseteq S^1xyS^1$. On the other hand, $S^1xyS^1 \subseteq S^1xS^1 = S^1eS^1$. Therefore $S^1eS^1 = S^1xyS^1$. Hence $e \mathcal{J} xy$. This shows that any $\mathcal{J}$-class containing an idempotent is a subsemigroup of S. Next let $e \in E(S)$ and $x \in S$ such that $e \mathcal{J} x$. Then $I(e) = I(x)$. Hence $e = exe$. Therefore any $\mathcal{J}$-class containing an idempotent is a quasi-rectangular subsemigroup of S. Now let J be a $\mathcal{J}$-class of S and a quasi-rectangular subsemigroup of S. Then $E(J)$ is an ideal of J, so $J = E(J)$. For, by **[1**, Lemma 2.39], $J \cup \{0\}$ is 0-simple, so J is simple. Hence (2) holds.

(2) $\Rightarrow$ (3). Let D and J be a $\mathcal{D}$-class and a $\mathcal{J}$-class containing the same idempotent, respectively. Since a rectangular subband of S is contained in a $\mathcal{D}$-class of S, $J \subseteq D$. Thus, $D = J$ follows from $\mathcal{D} \subseteq \mathcal{J}$. Hence (3) holds.

(3) $\Rightarrow$ (1). Suppose $e \in I(x) \cap I(y)$. Then $e = exe = eye$. Hence $ex \mathcal{D} e \mathcal{D} ye$ and so $e \mathcal{D} (ex)(ye)$. Therefore $e = e\{(ex)(ye)\}e = exye$, that is, $e \in I(xy)$. Conversely, suppose $e \in I(xy)$. Then $e = exye$ and so $ex = (ex)y(ex)$. Hence $(ex)y \in E$ and $ex \mathcal{D} (ex)y$. Thus $ex \in E$. Hence $e = exye = (exex)ye = ex(exye) = exe$. Similarly $e = eye$. Hence $e \in I(x) \cap I(y)$. Thus (1) holds.

PROPOSITION 6. *The following are equivalent.*

(1) (i) S *is a regular semigroup,*

(ii) $I(x) \cap I(y) = I(xy)$ *for every* $x, y \in S$.

(2) S *is a band.*

Proof. (1) $\Rightarrow$ (2). Let $x \in S$. Then there exists $y \in S$ such that $x = xyx$, i.e., $x \in \bar{I}(y)$. On the other hand, $\bar{I}(y) = I(y)$ by Proposition 1. Hence $x \in I(y)$ and so x is an idempotent. Thus S is a band.

(2) $\Rightarrow$ (1) (i) Obvious.

(2) $\Rightarrow$ (1) (ii) follows from Proposition 5.

A semigroup is called *viable* if for any $x, y \in S$, $xy = yx$ whenever $xy, yx \in E$. The following lemma is due to M. S. Putcha and J. Weissglass ([4]).

LEMMA 2. *The following are equivalent.*

(1) *S is viable.*

(2) *S is a semilattice of semigroups having at most one idempotent.*

(3) *S is a semilattice of s-indecomposable semigroups having at most one idempotent.*

(4) *Every $\mathcal{J}$-class of S has at most one idempotent.*

(5) *Every $\mathcal{D}$-class of S has at most one idempotent.*

Now let N be the set-valued function on S defined by $N(x) = \{e \mid e \in E, e = ex = xe\}$.

THEOREM 4. *The following are equivalent.*

(1) *S is viable and $I(x) \cap I(y) = I(xy)$ for every $x, y \in S$.*

(2) *$N(x) \cap N(y) = N(xy)$ for every $x, y \in S$.*

(3) *S is a semilattice of semigroups each of which is either idempotent free or contains only one idempotent as zero element.*

(4) *S is a semilattice of s-indecomposable semigroups each of which is either idempotent free or contains only one idempotent as zero element.*

(5) *For any $x, y \in S$, $xy = yxy = xyx$ whenever $xy \in E$.*

(6) *For any $x, y \in S$, $xy = yx = x^2y = y^2x$ whenever $xy, yx \in E$.*

(7) *Every $\mathcal{J}$-class of S is either idempotent free or consists of a single idempotent.*

(8) *Every $\mathcal{D}$-class of S is either idempotent free or consists of a single idempotent.*

Proof. (1) $\Rightarrow$ (2). Clearly $N(x) \cap N(y) \subseteq N(xy)$. Let $e \in N(xy)$. Then $e = exy = exye$. Hence $e \in I(xy) = I(x) \cap I(y)$. Therefore $e = exe$ and $e = eye$. Hence $e(ex)$, $(ex)e$, $e(ey)$, $(ey)e \in E$. Since S is viable, $e = ex = ey$. Similarly $e = xe = ye$ and hence $e \in N(x) \cap N(y)$. Thus $N(xy) \subseteq N(x) \cap N(y)$ and so (2) holds.

(2) $\Rightarrow$ (3). Let τ be the relation on S defined by $x \tau y$ if and only if $N(x) = N(y)$. Then τ is a semilattice congruence on S. If we consider the decomposition of S by τ then it is easy to see that (3) holds.

(3) $\Rightarrow$ (1). follows from Theorem 2 and Lemma 2.

(1) $\Leftrightarrow$ (4). For any semigroup, there exists the smallest semilattice congruence and every component in the decomposition by this congruence is s-indecomposable ([5]). Hence it follows from Theorem 2 and Lemma 2 that (1) and (4) are mutually equivalent.

(2) $\Rightarrow$ (5). Let x and y be elements of S such that $xy \in E$. Then $xy \in N(xy) = N(x) \cap N(y)$. Hence $xy \in N(x)$ and $xy \in N(y)$. Therefore $xy = xyx = yxy$.

(5) $\Rightarrow$ (6) obvious.

(6) $\Rightarrow$ (1). In this case, S is viable. Since any rectangular subband of a viable semigroup consists of a single element, S has the decomposition (D). Moreover, Proposition 4 (4) (i) and (ii) hold. Hence (1) follows from Proposition 3 and Proposition 4.

(1) $\Leftrightarrow$ (7) and (1) $\Leftrightarrow$ (8) follow from Theorem 3 and Lemma 2.

PROPOSITION 7. *The following are equivalent.*

(1) (i) *S is a regular semigroup,*

(ii) *$N(x) \cap N(y) = N(xy)$ for every $x, y \in S$.*

(2) *S is a semilattice.*

Proof. This follows from Proposition 6 and Theorem 4.

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