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PRIME NONASSOCIATIVE ALGEBRAS

THEODORE S. ERICKSON WALLACE S. MARTINDALE, 3RD, AND
J. MARSHALL OSBORN

An arbitrary algebra (not necessarily associative or unital) is said to be prime if the product of any two nonzero ideals is nonzero. The hypothesis that an algebra is prime has now been used in the study of several different varieties of nonassociative algebras, and the need for an understanding of the basic properties of prime nonassociative algebras has become apparent. If Γ is the centroid of a prime algebra A and Λ is the field of fractions of Γ then (under mild hypotheses) $A \otimes_{\Gamma} \Lambda$ is shown to have Λ as its centroid. The extended centroid C of a prime algebra A can be defined, the central closure Q of A can be constructed, and Q is shown to be closed in the sense that it is its own central closure. Tensor products are studied and among other results the following are obtained: (1) if A is a closed prime algebra over Φ and F is an extension field of Φ, then $A \otimes_{\Phi} F$ is a closed prime algebra over F, (2) the tensor product of closed prime algebras is closed. Finally, the results on prime algebras are specialized to obtain results on the tensor products of simple algebras.

We remark that this paper generalizes results proved for the associative case in [1]. Furthermore some of the results of the present paper are needed in [2].

I. The centroid of a prime algebra. Let A be an arbitrary linear nonassociative algebra over Φ, where Φ is a commutative associative ring with 1. Throughout this paper no associativity conditions will be assumed on A: neither is it supposed that A necessarily have an identity element. If $1 \in A$ we shall call A a *unital* algebra. Subrings of Φ are assumed to contain the identity of Φ, and $1a = a$ for all $a \in A$. For emphasis we shall frequently refer to the ideals of A as Φ-ideals.

For $a \in A$ the mapping $a_r: x \to xa$ of A into itself is called the right multiplication of A determined by the element a; similarly one defines the left multiplication $a_l: x \to ax$ of A into itself. A_l and A_r will denote respectively the sets $\{a_l \mid a \in A\}$ and $\{a_r \mid a \in A\}$ of left and right multiplications of A.

The *centroid* Γ of A is by definition the set of all Φ-endomorphisms of A which commute with all the left and right multiplications. Γ is an associative ring with 1. The *center* Z of A is the set of all elements of A which commute and associate with the elements of A. Specifically, $z \in Z$ if and only if $zx = xz$, $z(xy) = (zx)y$, $x(yz) = (xz)y)$ and hence $(xy)z = x(yz))$ for all $x, y \in A$. Z is a commutative associative subalgebra of A and the mapping $z \rightarrow z_l$ is a Φ-algebra homomorphism of Z into Γ. We claim that if $a_l = a_r \in \Gamma$ for some $a \in A$, then $a \in Z$. To see this, we first note that $a_l = a_r$ implies $ax = xa$ for all $x \in A$. Next, for $x, y \in A$, $a(xy) = a_l y_r(x) = y_r a_l(x) = (ax)y$. Finally, $x(ay) = x_l a_l(y) = a_l x_l(y) = a_l(xy) = a_l y_r(x) = y_r a_l(x) = (ax)y$. In case A is unital it is clear that $z \rightarrow z_l$ is an isomorphism of Z onto Γ. Here one makes use of the fact that if $\alpha \in \Gamma$ and $z = \alpha(1)$ then $z_l = \alpha$.

For $\alpha \in \Phi$ define $\bar{\alpha}: A \rightarrow A$ by $\bar{\alpha}: x \rightarrow \alpha x$. $\alpha \rightarrow \bar{\alpha}$ is a ring homomorphism of Φ into Γ and the image $\bar{\Phi}$ is a commutative subring of Γ. We shall call this mapping the canonical homomorphism of Φ into Γ. A will be called *central* over Φ in case $\alpha \rightarrow \bar{\alpha}$ is an isomorphism of Σ onto Γ. The *Φ-multiplication algebra* $\mathcal{M}_\Phi(A)$ is the subring of $\mathrm{End}_\Phi(A)$ generated by $\bar{\Phi}$, A_l, and A_r. For $a \in A$ $\langle a \rangle$ will denote the Φ-ideal of A generated by a; note that $\langle a \rangle = \{p(a) \mid p \in \mathcal{M}_\Phi(A)\}$.

If A and B are Φ-algebras the *tensor product* $A \otimes_\Phi B$ is formed in the usual fashion and is itself a Φ-algebra, with multiplication given by the rule $(a \otimes b)\,(c \otimes d) = ac \otimes bd$ and its extension by linearity. In particular, if A is an algebra over a field Φ and F is an extension field of Φ then the tensor product $A \otimes_\Phi F$ can be formed and becomes an algebra over the field F.

An algebra A over Φ is said to be *prime* if, for any two Φ-ideals U and V of A, $UV = 0$ implies $U = 0$ or $V = 0$. It follows that if $U \cap V = 0$ then $U = 0$ or $V = 0$, since $UV \subseteq U \cap V$.

THEOREM 1.1. *Let A be a prime algebra over Φ with centroid Γ. Then*

(a) *Γ is a commutative integral domain with* 1 *and A is Γ-torsion free.*

(b) *If Ω is a commutative ring with* 1 *and $\alpha \rightarrow \bar{\alpha}$ is a ring homomorphism of Ω into Γ then A is a prime Ω-algebra* (with αx defined to be $\bar{\alpha}x$, $\alpha \in \Omega$, $x \in A$).

(c) *For Ω as in* (b), *Γ coincides with the centroid of A as an Ω-algebra* (i.e., Γ is independent of the ring of scalars).

(d) *A has a characteristic.*

Proof. (a) Suppose $\lambda\mu = 0$ for some $\lambda, \mu \in \Gamma$. λA and μA are ideals of A and $(\lambda A)\,(\mu A) = 0$. Therefore $\lambda A = 0$ or $\mu A = 0$ since A is prime, i.e., $\lambda = 0$ or $\mu = 0$. Next suppose $\lambda a = 0$ and set $U = \lambda A$ and $V = \{x \mid \lambda x = 0\}$. Then $UV = 0$, implying $U = 0$ or $V = 0$ i.e., $\lambda = 0$

or $a = 0$. Thus A is Γ-torsion free. Finally let $\lambda, \mu \in \Gamma$. $A^2 \neq 0$ since A is prime; pick $a, b \in A$ such that $ab \neq 0$. Now $\lambda\mu(ab) = \lambda[(\mu a)b] = (\mu a)(\lambda b) = \mu[a(\lambda b)] = \mu\lambda(ab)$. It follows that $\lambda\mu = \mu\lambda$ since A is Γ-torsion free.

(b) It is clear that A is an Ω-algebra by defining $\alpha x = \bar{\alpha}x$, $\alpha \in \Omega$, $x \in A$. Let U and V be Ω-ideals such that $UV = 0$. Then ΦU and ΦV are Φ-ideals such that $(\Phi U)(\Phi V) = 0$. Hence $\Phi U = 0$ or $\Phi V = 0$ and in particular $U = 0$ or $V = 0$, since $1 \in \Phi$.

(c) Let Γ_0 be the centroid of A as a ring, i.e., as an algebra over the integers, and let Γ_Ω be the centroid of A as an Ω-algebra. Clearly $\Gamma_\Omega \subseteq \Gamma_0$. Let $f \in \Gamma_0$, $\omega \in \Omega$ (and thus $\bar{\omega} \in \Gamma \subseteq \Gamma_0$). Then $f\bar{\omega} = \bar{\omega}f$ by part (a) applied to Γ_0, i.e., $f(\omega x) = \omega f(x)$ for all $x \in A$. Hence $f \in \Gamma_\Omega$ and so $\Gamma_\Omega = \Gamma_0$. In particular, taking $\Omega = \Phi$, we see that $\Gamma = \Gamma_0$.

(d) Regard A as an algebra over the integers Δ. Let $\phi: \alpha \to \bar{\alpha}$ be the ring homomorphism of Δ into Γ. Since $\phi(\Delta)$ is an integral domain, ker $\phi = p\Delta$ for some prime p, or ker $\phi = 0$. In the former case $pA = 0$ (i.e., A is of characteristic p). In the latter case, if $mx = 0$, $0 \neq m \in \Delta$, $x \in A$, then $\bar{m}x = 0$, where $\bar{m} \neq 0$. Since A is Γ-torsion free, $x = 0$.

Let A be a prime algebra over Φ with center Z. Suppose $z_l = 0$ for some $z \in Z$. Then $\langle z \rangle A = 0$ and so $\langle z \rangle$, and hence z, is 0. Thus $z \to z_l$ is an injection of Z into Γ. As remarked earlier, if $1 \in A$ then $z \to z_l$ is in isomorphism of Z onto Γ.

Let A' be the Γ-space $\Gamma \oplus A$. With multiplication in A' defined by

$$(\lambda, a)(\mu, b) = (\lambda\mu, \lambda b + \mu a + ab), \ \lambda, \mu \in \Gamma, \ a, b \in A$$

A' becomes a unital Γ-algebra. Let $T = \{(-z_l, z) \mid z \in Z\}$. From $(\lambda, a)(-z_l, a) = (-\lambda z_l, \lambda z - z_l a + az) = (-(\lambda z)_l, \lambda z)$ and a similar calculation for $(-z_l, z)(\lambda, a)$, it is clear that T is a Γ-ideal of A'.

THEOREM 1.2. *Let A be a prime algebra over its centroid Γ and let $A^* = A'/T$. Then*:

(a) *A^* is a unital central prime algebra over Γ and $\sigma: a \to \overline{(0, a)}$ is a Γ-injection of A into A^*.*

(b) *If B is any unital algebra over Γ and ϕ is a Γ-injection of A into B, then there exists a Γ-homomorphism $\bar{\psi}$ of A^* into B such that $\phi = \bar{\psi}\sigma$.*

Proof. (a) If $(0, a) = (-z_l, z)$ for some $a \in A$, $z \in Z$ then $z_l = 0$ and hence $a = z = 0$. Thus σ is an injection of A into A^*. Denote the image of A under σ by $\tilde{A}$. We claim that if U is a nonzero ideal of A^* then $U \cap \tilde{A} \neq 0$. Suppose $U \cap \tilde{A} = 0$. Pick $\overline{(\lambda, a)} \neq 0 \in U$. Then $\overline{(\lambda, a)}\,\overline{(0, x)} = \overline{(0, \lambda x + ax)} \in U \cap \tilde{A}$ for all $x \in A$. Thus $\lambda x + ax = 0$ for all $x \in A$. Similarly $\lambda x + xa = 0$ for all $x \in A$, and so $a_l = a_r =$

$-\lambda$. Consequently $a \in Z$ and $(\lambda, a) = (-a_l, a) \in T$, a contradiction. The primeness of A^* now follows from the primeness of $\tilde{A}$.

Now let $\lambda \to \bar{\lambda}$ denote the canonical homomorphism of Γ into Ω, the centroid of A^*. If $\bar{\lambda} = 0$, then in particular $\bar{\lambda}\overline{(1,0)} = \overline{(\lambda, 0)} = 0$, i.e., $(\lambda, 0) = (-z_l, z)$ for some $z \in Z$. Thus $z = 0$ and hence $\lambda = -z_l = 0$. $\lambda \to \bar{\lambda}$ is therefore an injection. To show that $\lambda \to \bar{\lambda}$ is surjective it suffices to show that any element $\overline{(\lambda, a)}$ in the center of A^* can be written in the form $\overline{(\beta, 0)}$, $\beta \in \Gamma$. By commuting and associating $\overline{(\lambda, a)}$ with elements $\overline{(0,x)}$ and $\overline{(0,y)}$ in $\tilde{A}$, it follows that $a \in Z$. Then $\overline{(\lambda, a)} = \overline{(\lambda + a_l, 0)} = \overline{(\beta, 0)}$, where $\beta = \lambda + a_l \in \Gamma$.

(b) Define a mapping $\psi\colon A' \to B$ by $\psi\colon (\lambda, a) \to \lambda 1 + \phi(a)$. ψ is clearly a Γ-algebra homomorphism, and we show that its kernel K is contained in T. If $(\lambda, a) \in K$ and $x \in A$ then from $(\lambda, a)\,(0, x) \in K$ one obtains $\phi(\lambda x + ax) = 0$ for all $x \in A$. Since ϕ is an injection we have $a_l = -\lambda \in \Gamma$ and so $a \in Z$ and $(\lambda, a) = (-a_l, a) \in T$. As a result the map $\bar{\psi}\colon \overline{(\lambda, a)} \to \psi(\lambda, a)$ is well-defined and clearly satisfies $\phi = \bar{\psi}\sigma$.

THEOREM 1.3. *Let A be a central prime algebra over Γ and let Λ be the field of fractions of Γ. Then:*

(a) *$A \otimes_\Gamma \Lambda$ is a prime algebra over Λ.*

(b) *If A is finitely generated as an ideal, then $A \otimes_\Gamma \Lambda$ is central over Λ.*

(c) *If the center Z of A is nonzero then $1 \in A \otimes \Lambda$ and $a \otimes \Lambda$ is central over Λ.*

Proof. (a) It is clear that every element of $A \otimes \Lambda$ is of the form $a \otimes \lambda^{-1}$, $\lambda \in \Gamma$, $a \in A$. Let $\tilde{A}$ denote the (isomorphic) image of A in $A \otimes \Lambda$ under the mapping $a \to a \otimes 1$. If U is a nonzero Γ-ideal of $A \otimes \Lambda$ choose $a \otimes \lambda^{-1} \neq 0 \in U$. Then $a \otimes 1 \in U$ and $U \cap \tilde{A}$ is a nonzero ideal of A. The primeness of $A \otimes \Lambda$ then follows from the primeness of A.

(b) Write $A = \langle a_1, a_2, \cdots, a_n \rangle$, the Γ-ideal generated by $a_1, a_2, \cdots, a_n \in A$. If Ω is the centroid of $A \otimes \Lambda$ we must show that the canonical homomorphism $\alpha \to \bar{\alpha}$ of Λ into Ω is surjective. Let $f \in \Omega$, write $f(a_i \otimes 1) = b_i \otimes \lambda_i^{-1}$, $i = 1, 2, \cdots, n$, and set $\lambda = \lambda_1 \lambda_2 \cdots \lambda_n$. For $x \in A$, $x = \Sigma_{i=1}^n p_i(a_i)$, where $p_i \in \mathcal{M}_\Gamma(A)$. Then $\hat{p}_i = p_i \otimes 1 \in \mathcal{M}_\Gamma(A \otimes \Lambda)$, and we have

$$\begin{aligned} f(x \otimes 1) &= f\Big(\sum_i p_i(a_i) \otimes 1\Big) = \sum_i f(p_i(a_i) \otimes 1) \\ &= \sum_i f\hat{p}_i(a_i \otimes 1) = \sum_i \hat{p}_i f(a_i \otimes 1) = \Sigma\, \hat{p}_i(b_i \otimes \lambda_i^{-1}) \\ &= \sum_i p_i(b_i) \otimes \lambda_i^{-1}. \end{aligned}$$

Therefore, if $g = \bar{\lambda} f$, then $g(x \otimes 1) = \lambda f(x \otimes 1) = \lambda(\Sigma_i p_i(b_i) \otimes \lambda_i^{-1}) = y \otimes 1$, since $\lambda_i^{-1} \in \Gamma$. One then defines $\sigma: A \to A$ according to $\sigma(x) = y$, where $g(x \otimes 1) = y \otimes 1$ as indicated above. Since $g \in \Omega$ it is easily seen that $\sigma \in \Gamma$, and therefore $g(x \otimes 1) = \sigma(x) \otimes 1 = \sigma(x \otimes 1)$. Thus $g = \bar{\sigma}$ and so $f = \bar{\gamma}$, where $\gamma = \sigma\lambda^{-1} \in \Lambda$.

(c) If z is a nonzero element of Z, then $z_l = \beta$ for some $\beta \in \Gamma$, since A is central. One easily checks that $z \otimes \beta^{-1} = 1$ in $A \otimes_\Gamma \Lambda$. As in (b) we let Ω be the centroid of $A \otimes \Lambda$ and let $f \in \Omega$. We write $f(1) = a \otimes \lambda^{-1}$, i.e., $\bar{\lambda} f(1) = a \otimes 1$. This implies that $a \in Z$, whence $a_l = \gamma \in \Gamma$, since A is central. As a result $f = \bar{\delta}$, where $\delta = \gamma\lambda^{-1} \in \Lambda$.

Theorem 1.3 suggests the following open question: is $A \otimes_\Gamma \Lambda$ central over Λ without the additional hypothesis of either condition (b) or condition (c)?

Another natural question which arises is the following: if A is central prime over Γ and F is a field containing Γ, does there exist an F-algebra B containing A as a Γ-subalgebra such that AF is prime? By using the notion of extended centroid, which is developed in §II, we settle this question in the affirmative in §III.

II. The extended centroid of a prime algebra. Let A be a prime algebra over Φ and let U be a nonzero Φ-ideal of A. An element $f \in \operatorname{Hom}_\Phi(U, A)$ is said to be *Φ-permissible* if f commutes with all the left and right multiplications of A, i.e., f commutes with the elements of $\mathcal{M}_\Phi(A)$. Such an element will be denoted by (f, U). ker $f = \{u \in U \mid f(u) = 0\}$ and im $f = \{f(u) \mid u \in U\}$ are ideals of A and (ker f) (im f) $= 0$. Hence by the primeness of A either $f = 0$ or f is an injection.

Let $\mathcal{U}_\Phi$ be the set of all nonzero Φ-ideals of A and let $\mathcal{C}_\Phi$ be the set of all Φ-permissible maps (f, U), where $U \in \mathcal{U}_\Phi$. We define $(f, U) \sim (g, V)$ if there exists $W \in \mathcal{U}_\Phi$ such that $W \subseteq U \cap V$ and $f = g$ on W. This is easily shown to be an equivalence relation on $\mathcal{C}_\Phi$. We remark that $(f, U) \sim (gV)$ if and only if there exists $0 \neq x \in U \cap V$ such that $f(x) = g(x)$. This follows from our observation above that a Φ-permissible map is either 0 or an injection. We let $\overline{(f, U)}$ denote the equivalence class determined by (f, U), and we let C_Φ be the set of all equivalence classes. Addition in C_Φ is defined by

$$\overline{(f, U)} + \overline{(g, V)} = \overline{(f + g, U \cap V)}$$

and it is easy to check that this definition is independent of the representatives.

For $(g, V) \in \mathcal{C}_\Phi$ and $U \in \mathcal{U}_\Phi$, let $g^{-1}(U) = \{v \in V \mid g(v) \in U\}$. $g^{-1}(U)$ is clearly an ideal of A and we shall show that it is nonzero. If

$g(V)=0$, then $0 \neq V \subseteq g^{-1}(U)$. If $g(V) \neq 0$ then $g(V) \cap U \neq 0$. Pick $v \in V$ such that $0 \neq g(v) \in U$. Hence $v \neq 0$ and $v \in g^{-1}(U)$. Now define multiplication in C_Φ by

$$\overline{(f, U)}\,\overline{(g, V)} = \overline{(fg, g^{-1}(U))}$$

where fg is the composite of f and g. To see that multiplication is well-defined, suppose $(f_1, U_1) \sim (f_2, U_2)$ and $(g_1, V_1) \sim (g_2, V_2)$. Then $f_1 = f_2$ on $W_1 \subseteq U_1 \cap U_2$ and $g_1 = g_2$ on $W_2 \subseteq V_1 \cap V_2$. Set $W = W_2 \cap g_2^{-1}(W_1)$. For all $x \in W$, $f_1(g_1(x)) = f_1(g_2(x)) = f_2(g_2(x))$ and so multiplication is well-defined. It is then straightforward to verify that C_Φ is an associative ring with 1.

In particular we let $\mathcal{U} = \mathcal{U}_\Gamma$ be the set of all nonzero Γ-ideals of A and call $C = C_\Gamma$ the *extended centroid* of A. We justify this change of scalars from Φ to Γ by showing that $\overline{(f, U)} \to \overline{(f, U)}$ is an isomorphism of C_Γ onto C_Φ. This is easily seen to be a welldefined ring homomorphism which is an injection. To show it is surjective, let $\overline{(f, U)} \in C_\Phi$, set $V = \Gamma U$ and define $\bar{f}\colon V \to A$ according to $\bar{f}(\Sigma_i \lambda_i u_i) = \Sigma \lambda_i f(u_i)$, $\lambda_i \in \Gamma$, $u_i \in U$. We show that $\bar{f}$ is well-defined. Suppose $\Sigma \lambda_i u_i = 0$. For $a \in A$,

$$\left(\sum_i \lambda_i f(u_i)\right) a = \sum_i f(u_i)(\lambda_i a) = \sum_i f[u_i(\lambda_i a)] = f\left(\sum_i u_i(\lambda_i a)\right)$$
$$= f\left(\sum_i (\lambda_i u_i) a\right) = 0.$$

Similarly $a(\Sigma_i \lambda_i f(u_i)) = 0$ for all $a \in A$. It follows that, if $\langle x \rangle$ is the Φ-ideal generated by the element $x = \Sigma_i \lambda_i f(u_i)$, then $\langle x \rangle^2 = 0$. Since A is prime $\langle x \rangle = 0$ and in particular $x = \Sigma \lambda_i f(u_i) = 0$.

THEOREM 2.1. *The extended centroid C of a prime algebra A over Φ is a field.*

Proof. We first show that C is commutative. Let $\lambda = \overline{(f, U)}$ and $\mu = \overline{(g, V)}$ be in C. Set $W = g^{-1}(U) \cap f^{-1}(V)$, note that $W^2 \neq 0$, and pick $x, y \in W$ such that $xy \neq 0$. Then $fg(xy) = f[g(x)y] = g(x)f(y) = g[xf(y)] = gf(xy)$. Thus $(fg, g^{-1}U) \sim (gf, f^{-1}(V))$ and so $\lambda\mu = \mu\lambda$. Next let $\overline{(f, U)} \neq 0$, and note that $f(U) \neq 0$ but $\ker f = 0$. Define $g\colon f(U) \to A$ by $g(f(u)) = u$ for all $u \in U$. g is well-defined since f is an injection and in fact g is a Γ-permissible mapping. Clearly $(g, f(U))$ is the inverse of (f, U).

We have already noted (in §I) that there is a canonical homomorphism $\alpha \to \bar{\alpha}$ of Φ into the centroid Γ. Now, for $\alpha \in \Phi$, define $\hat{\alpha} = \overline{(\bar{\alpha}, A)}$. $\alpha \to \hat{\alpha}$ is clearly a ring homomorphism of Φ into the extended centroid C. We define a prime algebra A over Φ to be *closed* over Φ if the map $\alpha \to \hat{\alpha}$ is an isomorphism of Φ onto C. We remark that if we regard A as an algebra over its centroid Γ then $\lambda \to \hat{\lambda}$ is always an injection of Γ into C. To show that a prime algebra A is closed over Φ it suffices (besides showing that $\alpha \to \hat{\alpha}$ is an injection) to show that, given any Γ-permissible map (f, U), there exists $0 \neq u \in U$ and $\alpha \in \Phi$ such that $f(u) = \alpha u$.

Let A be a prime algebra over Φ with extended centroid C. We form $A \otimes_\Gamma C$ and note that $a \to a \otimes 1$ is a Γ-isomorphism of A onto $\bar{A} = A \otimes 1 \subseteq A \otimes C$. We set

$$I = \left\{ \sum_i \left[u_i \otimes \lambda_i \rho_i - f_i(u_i) \otimes \rho_i \right] \mid \lambda_i, \rho_i \in C, (f_i, U_i) \in \lambda_i,\ u_i \in U_i \right\}.$$

LEMMA 2.2. (a) *I is a Γ-ideal of $A \otimes C$*

(b) $I \cap \bar{A} = 0$.

Proof. (a) Let $x \in A$, $\mu, \lambda \in C$, $\lambda = \overline{(f, U)}$, $u \in U$. From

$$(u \otimes \lambda\rho - f(u) \otimes \rho)(x \otimes \mu) = ux \otimes \lambda\rho\mu - f(ux) \otimes \rho\mu \in I, \text{ etc.,}$$

it follows that I is a Γ-ideal of A.

(b) Suppose

$$0 \neq a \otimes 1 = \sum_i \left[u_i \otimes \lambda_i \rho_i - f_i(u_i) \otimes \rho_i \right] \in I \cap \bar{A}. \tag{1}$$

Letting $p \in \mathcal{M}_\Gamma(A)$ and applying $\hat{p} = p \otimes 1 \in \mathcal{M}_\Gamma(A \otimes C)$ to (1) yields

$$p(a) \otimes 1 = \sum_i \left[p(u_i) \otimes \lambda_i \rho_i - f_i(p(u_i)) \otimes \rho_i \right] \in I \cap \bar{A}. \tag{2}$$

By intersecting a finite number of Γ-ideals there is a $W \in \mathcal{U}_\Gamma$ such that $(g_i, W) \in \rho_i$ and $(f_i g_i, W) \in \lambda_i \rho_i$. We let $C_W = \{\lambda \in C \mid \lambda = \overline{(f, W)}$ for some $f\}$. C_W, and hence $A \otimes C_W$ is a Γ-space, as is also $\mathrm{Hom}_\Gamma(W, A)$. We claim that the map $\underline{(x, \lambda)} \to x_l f$ of $A \times C_W$ into $\mathrm{Hom}_\Gamma(W, A)$ is Γ-bilinear (where $\lambda = \overline{(f, W)}$). It is well-defined since $(f, W) \sim (g, W)$ implies $f = g$ on W. Hence there is a Γ-linear map $\pi: A \otimes C_W \to \mathrm{Hom}_\Gamma(W, A)$ such that $\pi(x \otimes \lambda) = x_l f$. Applying π to (2) yields:

$$p(a)_l = \sum_i \left[p(u_i)_l (f_i g_i) - f_i(p(u_i))_l g_i \right] \quad \text{on} \quad W \tag{3}$$

For $w \in W$ one applies (3) to w to obtain

$$
\begin{aligned}
p(a)w &= \sum_i [p(u_i)\,[(f_i g_i)\,(w)] - [f_i(p(u_i))]g_i(w)] \\
&= \sum_i [p(u_i)f_i(g_i(w)) - p(u_i)f_i(g_i(w))] = 0.
\end{aligned} \tag{4}
$$

From (4) we have $\langle a \rangle W = 0$ and so the ideal $\langle a \rangle = 0$ by the primeness of A. In particular $a = 0$, a contradiction.

In view of Lemma 2.2 there exists a Γ-ideal M of $A \otimes C$ maximal with respect to the property that $I \subseteq M$ and $M \cap \bar{A} = 0$.

LEMMA 2.3. *Let $Q = A \otimes C/M$.*
(a) *Q is a prime algebra over C.*
(b) *$a \to \overline{a \otimes 1}$ is a Γ-injection of A into Q.*

Proof. (a) Let U be the preimage in $A \otimes C$ of a nonzero ideal of Q. Then $U \cap \bar{A} \neq 0$ since U properly contains M. The primeness of Q then follows from the primeness of $\bar{A}$. (b) follows from $M \cap \bar{A} = 0$.

The question of uniqueness in the choice of M has been resolved by McCrimmon and we give his results here. To this end, for $x \in A \otimes C$, define $\langle x \rangle_A = \{\hat{p}(x) \mid p \in \mathcal{M}_\Gamma(A)\}$ and let M_0 denote the set of all elements x in $A \otimes C$ for which there exists $V \in \mathcal{U}$ such that $(V \otimes 1)\langle x \rangle_A \subseteq I$.

LEMMA 2.4 (McCrimmon). $M = M_0$.

Proof. Let $x = \Sigma a_i \otimes \lambda_i \in M$, where $\lambda_j = \overline{(f_i, U_i)}$, and set $V = \cap_i U_i$. Let $v \in V$ and $p \in \mathcal{M}_\Gamma(A)$. Then

$$
\begin{aligned}
(v \otimes 1)\hat{p}(x) &= (v \otimes 1)\sum_i p(a_i) \otimes \lambda_i = \sum_i vp(a_i) \otimes \lambda_i \\
&= \Sigma[vp(a_i) \otimes \lambda_i - f_i(vp(a_i)) \otimes 1] + \Sigma f_i(vp(a_i)) \otimes 1.
\end{aligned}
$$

Therefore $\Sigma f_i(vp(a_i)) \otimes 1 \in M \cap \bar{A} = 0$, since $I \subseteq M$. It follows that $(V \otimes 1)\hat{p}(x) \in I$ and so $(V \otimes 1)\langle x \rangle_A \subseteq I$. Conversely let $x \in M_0$. By definition there exists $V \in \mathcal{U}$ such that $(V \otimes 1)\langle x \rangle_A \subseteq I \subseteq M$. $\overline{V \otimes C}$ and $\overline{\langle x \rangle_A C}$ are ideals of Q with $\overline{V \otimes C} \neq 0$ and $\overline{V \otimes C}\;\overline{\langle x \rangle_A C} = 0$. Therefore $\overline{\langle x \rangle_A C} = 0$ since Q is prime, and in particular $x \in M$.

THEOREM 2.5. *If A is a prime algebra over Φ, then $Q = A \otimes C/M$ is a closed prime algebra over C.*

Proof. It suffices to show that if $(\tilde{f}, \tilde{u})$ is a C-permissable map in Q then there exists $0 \neq x \in \tilde{U}$ such that $\tilde{f}(x) = \lambda x$ for some $\lambda \in C$. Let $V = \{x \in A \mid \overline{x \otimes 1} \in \overline{A \otimes 1} \cap \tilde{U}$ and $\tilde{f}(\overline{x \otimes 1}) \in \overline{(A \otimes 1)}\}$. Since $\tilde{f}$ commutes with the elements of $\mathcal{M}_C(Q)$, and in particular with elements of the form $(\overline{y \otimes 1})_l$ and $(\overline{y \otimes 1})_r$, V is a Γ-ideal of A. We now show that V is nonzero. By the definition of M there is $0 \neq \overline{x \otimes 1} \in \overline{A \otimes 1} \cap \tilde{U}$. We write $\tilde{f}(\overline{x \otimes 1}) = \Sigma_i \overline{a_i \otimes \lambda_i}$ in Q. There exists $W \in \mathcal{U}$ such that $(f_i, W) \in \lambda_i$ for all i. For $w \in W$ we have

$$\tilde{f}(\overline{xw \otimes 1}) = [\tilde{f}(\overline{x \otimes 1})](\overline{w \otimes 1}) = \sum_i (\overline{a_i \otimes \lambda_i})(\overline{w \otimes 1})$$

$$= \sum_i \overline{a_i w \otimes \lambda_i} = \sum_i \overline{f_i(a_i w) \otimes 1}$$

making use of the fact that $I \subseteq M$. Hence $\tilde{f}(\overline{xw \otimes 1}) \in \overline{A \otimes 1}$ for all $w \in W$. Next let $p \in \mathcal{M}_\Gamma(A)$ and let $\tilde{p}$ be the element of $\mathcal{M}_C(Q)$ induced by $\hat{p} \in \mathcal{M}_\Gamma(A \otimes C)$. Then $\tilde{f}(\overline{p(x)w \otimes 1}) = \tilde{f}(\overline{p(xw) \otimes 1}) = \tilde{f}\tilde{p}(\overline{wx \otimes 1}) = \tilde{p}\tilde{f}(\overline{xw \otimes 1}) \in \overline{A \otimes 1}$. $\langle x \rangle W \neq 0$ since A is prime and so there exists $p(x) \in \langle x \rangle$ and $w \in W$ such that $p(x) \neq 0$. Thus $p(x)w \in V$ and so $V \neq 0$.

We now define a mapping $f\colon V \to A$ according to the rule

$$f(v) = y, \quad \text{where} \quad \tilde{f}(\overline{v \otimes 1}) = \overline{y \otimes 1}$$

f is clearly a well-defined Γ-permissible map and so (f, V) is a representative of some $\lambda \in C$. Now pick $v \neq 0 \in V$. Using the fact that $I \subseteq M$ again we see that $\lambda(\overline{v \otimes 1}) = \overline{v \otimes \lambda} = \overline{f(v) \otimes 1} = \tilde{f}(\overline{v \otimes 1})$. It follows that Q is closed over C.

In view of Theorem 2.5 we shall refer to $Q = A \otimes C/M$ as the *central closure* of the prime algebra A.

III. Tensor product of closed prime algebras.

THEOREM 3.1. *Let A be a closed prime algebra over Φ, and let $a_1, a_2, \cdots, a_n$ be Φ-independent elements of A. Then there exists $p \in \mathcal{M}(A)$ such that $p(a_1) \neq 0$ and $p(a_i) = 0$, $i = 2, 3, \cdots, n$.*

Proof. The proof is by induction on n. For $n = 1$ the theorem is trivially true since $a_1 \neq 0$. Now assume that for all $p \in \mathcal{M}(A)$, if $p(a_i) = 0$ for $i = 2, 3, \cdots, n$, then $p(a_1) = 0$. Define $J = \{p \in \mathcal{M}(A) \mid p(a_i) = 0$ for $i = 2, 3, \cdots, n-1\}$. (In case $n = 2$ we define $J = \mathcal{M}(A)$). Since J is a left ideal of $\mathcal{M}(A)$, $Ja_n = \{p(a_n) \mid p \in J\}$ is an ideal of A. Furthermore by the induction assumption Ja_n is nonzero. The mapping $f\colon Ja_n \to A$ given by $p(a_n) \to p(a_1)$, $p \in J$, is well-defined because of our supposition above that $p(a_1) = 0$ for all $p \in \mathcal{M}(A)$ such that $p(a_i) = 0$, $i = 2, 3, \cdots, n$. (f, Ja_n) is clearly a Φ-permissable mapping. Since A is closed there exists $\lambda \in \Phi$ such that $f(p(a_n)) = \lambda p(a_n)$ for all $p \in J$, i.e., $p(a_1 - \lambda a_n) = 0$ for all $p \in J$. By the induction assumption applied to $a_2, a_3, \cdots, a_{n-1}$, $a_1 - \lambda a_n$ there exists $p \in \mathcal{M}(A)$ such that $p(a_i) = 0$, $i = 2, 3, \cdots, n-1$ (i.e., $p \in J$) but $p(a_1 - \lambda a_n) \neq 0$, a contradiction.

LEMMA 3.2. *Let A be a closed prime algebra over Φ and suppose a and b are elements of A such that $p(a)q(b) = p(b)q(a)$ for all $p, q \in \mathcal{M}(A)$. Then a and b are Φ-independent.*

Proof. We may assume that $a \neq 0$. Let $J = \{p \in \mathcal{M}(A) \mid p(a) = 0\}$; Jb is then an ideal of A. Since $0 = p(a)q(b) = p(b)q(a)$ for all $p \in J$ and $q \in \mathcal{M}(A)$ we have $(Jb)\langle a \rangle = 0$. It follows that $Jb = 0$ since A is prime. Hence the mapping $f\colon \langle a \rangle \to A$ given by $f\colon p(a) \to p(b)$, $p \in \mathcal{M}(A)$, is a well-defined Φ-permissable homomorphism. Since A is closed there exists $\lambda \in \Phi$ such that $f(p(a)) = \lambda p(a)$ for all $p \in \mathcal{M}(A)$. In particular $b = f(a) = \lambda a$, and so a and b are Φ-dependent.

THEOREM 3.3. *Let B be an algebra over a commutative ring F, let Φ be a subfield of F, and let A be a closed prime Φ-subalgebra of B Then the mapping $\sigma\colon \Sigma_i a_i \otimes \lambda_i \to \Sigma_i \lambda_i a_i$, $a_i \in A$, $\lambda_i \in F$, is an injection of $A \otimes_\Phi F$ into B.*

Proof. It is clear that σ is a well-defined Φ-algebra homomorphism. If the kernel K of σ is nonzero choose a nonzero element $w = \Sigma_{i=1}^{n} a_i \otimes \lambda_i$ in K of minimal "length" n. Both $\{a_i\}$ and $\{\lambda_i\}$ are then necessarily Φ-independent sets. If $n = 1$ then $\lambda_1 a_1 = 0$ and so $\lambda_1 = 0$ or $a_1 = 0$, a contradiction. Therefore we may assume that $n > 1$. For any $p, q \in \mathcal{M}(A)$ we have $\hat{p}(w) = \Sigma_i p(a_i) \otimes \lambda_i \in K$ and $\hat{q}(w) = \Sigma_i q(a_i) \otimes \lambda_i \in K$. Furthermore $\Sigma_i q(a_1)p(a_i) \otimes \lambda_i \in K$ and $\Sigma_i q(a_i)p(a_i) \otimes \lambda_i \in K$. Subtraction yields

$$\sum_{i=2}^{n} [q(a_1)p(a_i) - q(a_i)p(a_1)] \otimes \lambda_i \in K.$$

By the minimality of n and the Φ-independence of $\{\lambda_i\}$ we have $q(a_1)p(a_i) - q(a_i)p(a_1) = 0$ for all $p, q \in \mathcal{M}(A)$, $i = 2, 3, \cdots, n$. In particular, by Lemma 3.2, a_1 and a_2 are Φ-dependent, a contradiction. Therefore $K = 0$ and σ is an injection.

LEMMA 3.4. *Let A be a closed prime algebra over Φ and let F be an extension field of Φ. Then any nonzero F-ideal of $A\otimes_\Phi F$ has a nonzero intersection with $A\otimes 1$.*

Proof. Suppose N is a nonzero F-ideal of $A\otimes F$ such that $N\cap(A\otimes 1)=0$. Choose a nonzero element $\Sigma_{i=1}^n u_i\otimes\lambda_i$ in N of minimal "length" n; the set $\{u_i\}$ is necessarily Φ-independent. Since N is a F-ideal we may assume without loss of generality that $\lambda_1=1$. By Theorem 3.1 there exists $p\in\mathcal{M}_\Phi(A)$ such that $p(u_1)\neq 0$ but $p(u_i)=0$ for $i>1$. Now $\hat{p}=p\otimes 1\in\mathcal{M}_\Phi(A\otimes F)\subseteq\mathcal{M}_F(A\otimes F)$ and so $\hat{p}(\Sigma_{i=1}^n u_i\otimes\lambda_i)=\Sigma_i p(u_i)\otimes\lambda_i=p(u_1)\otimes 1\in N$. Since $N\cap(A\otimes 1)=0$ we have $p(u_1)=0$, a contradiction.

THEOREM 3.5. *If A is a closed prime algebra over Φ and F is an extension field of Φ, then $A\otimes_\Phi F$ is a closed prime algebra over F.*

Proof. In view of Lemma 3.4 the primeness of $A\otimes F$ follows from the primeness of $A\otimes 1\cong A$. Next let $(\bar{f},\bar{U})$ be an F-permissable map of $\bar{U}$ into $A\otimes F$. By Lemma 3.4 $\bar{U}\cap(A\otimes 1)\neq 0$, and we pick $0\neq u\in A$ such that $u\otimes 1\in\bar{U}\cap(A\otimes 1)$. We write $\bar{f}(u\otimes 1)=\Sigma_{i=1}^n v_i\otimes\lambda_i$, with $\{\lambda_i\}$ Φ-independent. For $p\in\mathcal{M}_\Phi(A)$ we have $\hat{p}=p\otimes 1\in\mathcal{M}_F(A\otimes F)$, so consequently $\bar{f}(p(u)\otimes 1)=\bar{f}(\hat{p}(u\otimes 1))=\hat{p}(\Sigma_i v_i\otimes\lambda_i)=\Sigma_i p(v_i)\otimes\lambda_i$. By the Φ-independence of the $\{\lambda_i\}$, for each i the map $f_i\colon p(u)\to p(v_i)$ is a well-defined map from $\langle u\rangle$ into A. It is evident that each $(f_i,\langle u\rangle)$ is Φ-permissable. Since A is closed there exists $\alpha_i\in\Phi$ such that $f_i(u)=\alpha_i u$, $i=1,2,\cdots,n$. Consequently $\bar{f}(u\otimes 1)=\Sigma_i\alpha_i u\otimes\lambda_i=u\otimes(\Sigma_i\alpha_i\lambda_i)=\beta(u\otimes 1)$, where $\beta=\Sigma_i\alpha_i\lambda_i\in F$. It follows that $A\otimes_\Phi F$ is F-closed.

We are now in a position to settle in the affirmative the question posed at the end of §II.

THEOREM 3.6. *Let A be a central prime algebra over Γ and let F be a field containing Γ. Then there exists an F-algebra B containing a Γ-isomorphic image $\tilde{A}$ of A such that AF is a prime algebra.*

Proof. We recall from Lemma 2.3 that A is Γ-isomorphic to a Γ-subalgebra $\bar{A}$ of the central closure Q of A. From the construction of Q it is clear that $\bar{A}C=Q$ (where C is the extended centroid of A). Let K be the composite field of C and F and form the F-algebra $B=Q\otimes_C K$. $\bar{A}$ is then Γ-isomorphic to the Γ-subalgebra $\tilde{A}=\bar{A}\otimes 1$ of B. Furthermore it follows from $Q=\bar{A}C$ that $B=\tilde{A}K$. We complete the proof by showing that the F-subalgebra $\tilde{A}F$ of B is a prime F-algebra. Let U and B be F-ideals of $\tilde{A}F$ such that $UV=0$. Then UK (and similarly VK) are K-ideals of B since $(UK)B=(UK)$

$(\tilde{A}K) \subseteq (U\tilde{A})K \subseteq UK$. But $(UK)(VK) \subseteq (UV)K = 0$. We conclude that either $UK = 0$ or $VK = 0$, since B is prime by Theorem 3.5. In particular, $U = 0$ or $V = 0$ and so $\tilde{A}F$ is prime.

We now study the structure of the tensor product of a closed prime algebra A over Φ and a unital algebra B over Φ. If W is an ideal of $A \otimes B$ and $v \in B$ we let $U_{(W,v)} = \{u \in A \mid u \otimes v \in W\}$. In the following discussion W remains fixed and so we denote $U_{(W,v)}$ by U_v. U_v is clearly closed under addition. For $u \in U_v$ and $x \in A$ we see that $ux \otimes 1 = (u \otimes v)(x \otimes 1) \in W$, using the fact that B is unital.

Thus $ux \in U_v$ and similarly $xu \in U_v$. Therefore U_v is an ideal of A. We next let $V_w = \{v \in B \mid U_v \neq 0\}$. V_w is nonempty since $U_0 = A$. We claim that V_w is an ideal of B. If $v_1, v_2 \in V_w$, then $U_{v_1+v_2} \subseteq U_{v_1} \cap U_{v_2} \neq 0$ since A is prime. Thus $v_1 + v_2 \in V_w$. Next let $v \in V_w$ and $y \in B$. Again using the primeness of A, we choose $u \in U_v$ and $x \in A$ such that $ux \neq 0$. It follows that $ux \otimes vy = (u \otimes v)(x \otimes y) \in W$, and so $U_{vy} \neq 0$, i.e., $vy \in V_w$. Similarly $yv \in V_w$ and so V_w is an ideal.

LEMMA 3.7. *Let A be a closed prime algebra over Φ, let B be a unital algebra over Φ, and let W be a nonzero ideal of $A \otimes_\Phi B$. Then $V_w \neq 0$.*

Proof. Choose a nonzero element $w = \sum_{i=1}^n a_i \otimes b_i$ in W, with $\{a_i\}$ and $\{b_i\}$ Φ-independent sets. By Theorem 3.1 there exists $p \in \mathcal{M}(A)$ such that $p(a_1) \neq 0$ and $p(a_i) = 0$ for $i > 1$. Since $1 \in B$ we have $\hat{p} = p \otimes 1 \in \mathcal{M}(A \otimes B)$. Thus $\hat{p}(w) = p(a_1) \otimes b_1 \in W$ so b_1 is a nonzero element of V_w.

REMARK. If in Lemma 3.7 A is also assumed to be unital then we claim that W contains a nonzero ideal of the form $U \otimes V$, where U is a nonzero ideal of A and V is a nonzero ideal of B. Indeed, by Lemma 3.7 there exists a nonzero element of the form $u \otimes v$ in W. If $p \in \mathcal{M}(A)$ and $q \in \hat{\mathcal{M}}(B)$ then $p \otimes q = (p \otimes 1)(1 \otimes q) \in \mathcal{M}(A \otimes B)$, since both A and B are unital. Thus $(p \otimes q)(u \otimes v) = p(u) \otimes q(v) \in W$, showing that $\langle u \rangle \otimes \langle v \rangle \subseteq W$.

THEOREM 3.8. *Let A be a closed prime algebra over Φ and let B be a unital algebra over Φ. Then*

(1) *$A \otimes B$ is a prime algebra, if B is a prime algebra.*

(2) *$A \otimes B$ is a closed prime algebra over Φ, if B is a closed prime algebra over Φ.*

Proof. (1) Let W_1 and W_2 be nonzero ideals of $A \otimes B$. By Lemma 3.7 V_{W_1} and V_{W_2} are nonzero ideals of B. By the primeness of B

there exists $v_1 \in V_{W_1}$ and $v_2 \in V_{W_2}$ such that $v_1 v_2 \neq 0$. Since A is prime there exist $u_1 \in U_{(W_1, v_1)}$ and $a_2 \in U_{(W_2, v_2)}$ such that $u_1 u_2 \neq 0$. Therefore $(u_1 \otimes v_1)(u_2 \otimes v_2) = u_1 u_2 \otimes v_1 v_2$ is a nonzero element of $W_1 W_2$, and so $A \otimes B$ is prime.

(2) Let (f, W) be a Φ-permissable map in $A \otimes B$. For $b \in B$ we denote $U_{(W,b)}$ by U_b. By Lemma 3.7 V_W is a nonzero ideal of B. Let $v \in V_W$, choose $u \neq 0 \in U_V$, and write $f(u \otimes v) = \Sigma_{i=1}^n a_i \otimes b_i$, with $\{b_i\}$ Φ-independent. For $p \in \mathcal{M}(A)$ $\hat{p} = p \otimes 1 \in \mathcal{M}(A \otimes B)$ since B is unital. Because f is Φ-permissable we then have $f(p(u) \otimes v) = f\hat{p}(u \otimes v) = \hat{p}(\Sigma_i a_i \otimes b_i) = \Sigma_i p(a_i) \otimes b_i$. The independence of the $\{b_i\}$ shows that the mapping $g_i : \langle u \rangle \to A$ given by $p(u) \to p(a_i)$, $i = 1, 2, \cdots, n$, is well-defined. It is clear that each $(g_i, \langle u \rangle)$ is Φ-permissable, $i = 1, 2, \cdots, n$. Since A is closed over Φ, for each i there exists $\lambda_i \in \Phi$ such that $g_i(u) = \lambda_i u$. As a result $f(u \otimes v) = \Sigma_i \lambda_i u \otimes b_i = u \otimes (\Sigma_i \lambda_i b_i)$. We have thus far shown that for $v \in V_W$ and $u \neq 0 \in U_v$ $f(u \otimes v) = u \otimes y$ for some $y \in B$.

We next show that y is independent of the choice of u. To this end suppose $f(u_1 \otimes v) = u_1 \otimes y_1$ and $f(u_2 \otimes v) = u_2 \otimes y_2$ for u_1, u_2 nonzero elements of U_v. Since A is prime $\langle u_1 \rangle \cap \langle u_2 \rangle \neq 0$ and so $p(u_1) = q(u_2) \neq 0$ for suitable $p, q \in \mathcal{M}(A)$. Again using the fact that f is Φ-permissable and that B is unital we see that $p(u_1) \otimes y_1 = f(p(u_1) \otimes v) = f(q(u_2) \otimes v) = q(u_2) \otimes y_2 = p(u_1) \otimes y_2$. In other words the mapping $g : V_W \to B$ given by $g(v) = y$ (where $f(u \otimes v) = u \otimes y$, $u \neq 0 \in U_v$) is well-defined.

We claim that (g, V_W) is Φ-permissable. Let $v_1, v_2 \in V_W$ and pick $u \neq 0 \in U_{v_1} \cap U_{v_2} \subseteq U_{v_1+v_2}$. Then $u \otimes g(v_1 + v_2) = f(u \otimes (v_1 + v_2)) = f(u \otimes v_1) + f(u \otimes v_2) = u \otimes g(v_1) + u \otimes g(v_2)$, whence g is additive. Next let $v \in V_W$, $y \in B$, and pick $u \in U_v$, $x \in A$ such that $ux \neq 0$. We note that $ux \in U_{vy}$, since $ux \otimes vy = (u \otimes v)(x \otimes y) \in W$. Therefore

$$\begin{aligned} ux \otimes g(vy) &= f(ux \otimes vy) = f[(u \otimes v)(x \otimes y)] = [f(u \otimes v)](x \otimes y) \\ &= (u \otimes g(v))(x \otimes y) = ux \otimes g(v)y. \end{aligned}$$

This shows that g commutes with all right multiplications y_r and it is similarly proved that g commutes with all left multiplications. This completes the verification that (g, V_W) is Φ-permissable.

To complete the proof we note that there exists $\beta \in \Phi$ such that $g(v) = \beta v$ for $v \in V_W$. In particular, for $v \neq 0 \in V_W$ and for $u \neq 0 \in U_v$ we see that $f(u \otimes v) = u \otimes g(v) = u \otimes \beta v = \beta(u \otimes v)$, proving finally that $A \otimes B$ is a closed prime algebra over Φ.

For the remainder of the paper we turn our attention to the study of tensor products of simple algebras. An algebra A over Φ is said to be *simple* if $A^2 \neq 0$ and A contains no ideals other than 0 and A. Clearly any simple algebra is prime. It is also evident that the extended centroid of a simple algebra is isomorphic to its centroid, and so any central simple algebra over Φ is automatically closed. The preceding results on tensor products of prime algebras yield corresponding results on simple algebras which we now give.

THEOREM 3.9. *Let A be a central simple algebra over Φ and let B be a unital algebra over Φ. Then:*

(1) *ϕ: $W \to V_W$ is a lattice isomorphism of the lattice of ideals of $A \otimes_\Phi B$ onto the lattice of ideals of B.*

(2) *If B is simple, then $A \otimes_\Phi B$ is simple.*

(3) *If B is central simple over Φ, then $A \otimes B$ is central simple over Φ.*

Proof. (1) We let ψ by the map $I \to A \otimes I$, I an ideal of B, and show that $\phi\psi = 1$ and $\psi\phi = 1$. This is equivalent to showing (a) $W = A \otimes V_W$ for each ideal W of $A \otimes B$ and (b) $I = V_{A\otimes I}$ for each ideal of B. To prove (a) we first note that $A \otimes V_W \subseteq W$ because of the simplicity of A. Next let $w = \Sigma_{i=1}^n a_i \otimes b_i \in W$ with $\{a_i\}$ Φ-independent. By Theorem 3.1 for each i there exists $p_i \in \mathcal{M}(A)$ such that $p_i(a_i) \neq 0$ and $p_i(a_j) = 0$ for $j \neq i$. Since B is unital $\hat{p}_i = p_i \otimes 1 \in \mathcal{M}(A \otimes B)$ and we have for each i $\hat{p}_i(w) = p_i(\ \ a_i) \otimes b_i \in W$. This puts each b_i in V_W and so $w \in A \otimes V_W$. To establish (b) it is first of all obvious that $I \subseteq V_{A\otimes I}$. Next suppose there exists $b \in V_{A\otimes I}$ such that $b \notin I$. Choose a Φ-basis $\{b_i\}$ of I and pick $a \neq 0 \in A$. Then $a \otimes b \in A \otimes I$ and so $a \otimes b = \Sigma_i a_i \otimes b_i$ for suitable $a_i \in A$. A contradiction results due to the Φ-independence of b, $b_1, b_2, \cdots, \cdots$.

(2) follows immediately from (1) and (3) is implied by Theorem 3.8.

For the case of simple algebras Theorem 3.1 can be sharpened as follows.

THEOREM 3.10. *Let A be a central simple algebra over Φ, let $a_1, a_2, \cdots, a_n$ be Φ-independent elements of A, and let $x_1, x_2, \cdots, x_n$ be arbitrary elements of A. Then there exists $p \in \mathcal{M}(A)$ such that $p(a_i) = x_i$, $i = 1, 2, \cdots, n$.*

Proof. By Theorem 3.1 for each i there exists $p_i \in \mathcal{M}(A)$ such that $p_i(a_i) = b_i \neq 0$ and $p_i(a_j) = 0$ for $j \neq i$. For each i $\langle b_i \rangle = A$ since A is simple. Therefore for each i there exists $q_i \in \mathcal{M}(A)$ such that $q_i(b_i) = x_i$. The element $p = \Sigma_{i=1}^n q_i p_i \in \mathcal{M}(A)$ then has the required property.

Finally we consider the structure of the tensor product of two arbitrary simple unital algebras over Φ.

THEOREM 3.11. *Let A and B be simple unital algebras over Φ, with centers Z and F respectively. Then*

(1) *$A\otimes_\Phi B$ is a free $Z\otimes_\Phi F$ module, with basis $\{a_i\otimes b_j\}$, where $\{a_i\}$ is a Z-basis for A and $\{b_j\}$ is an F-basis of B.*

(2) *$Z\otimes F$ is the center of $A\otimes B$.*

(3) *$\phi: U\to U(A\otimes B)$ is a lattice isomorphism of the ideals of $Z\otimes F$ onto the ideals of $A\otimes B$.*

Proof. (1) It is clear that $\{a_i\otimes b_j\}$ generates $A\otimes B$ as a $Z\otimes F$-module. Suppose now that $\Sigma_{i,j}\gamma_{ij}(a_i\otimes b_j)=0$, $\gamma_{ij}\in Z\otimes F$. For any fixed i and j, by Theorem 3.10 there exists $p\in\mathcal{M}(A)$ such that $p(a_i)=1$, $p(a_k)=0$ for $k\neq i$, and $q\in\mathcal{M}(B)$ such that $q(b_j)=1$, $q(b_l)=0$ for $l\neq j$. Since A and B are unital algebras $p\otimes q\in\mathcal{M}(A\otimes B)$ and thus $p\otimes q$ is a $Z\otimes F$-module mapping. Application of $p\otimes q$ to the above equation thus yields $\gamma_{ij}(1\otimes 1)=0$, or $\gamma_{ij}=0$. Hence $\{a_i\otimes b_j\}$ is $Z\otimes F$-independent and forms a $Z\otimes F$-basis for $A\otimes B$.

(2) By part (1) we may select a $Z\otimes F$-basis $\{a_i\otimes b_j\}$ for $A\otimes B$ in which $a_1=1$ and $b_1=1$. Let $w=\Sigma_{i,j}\gamma_{ij}(a_i\otimes b_j)$, $\gamma_{ij}\in Z\otimes F$, be an element of the center E of $A\otimes B$. Choose $p\in\mathcal{M}(A)$ such that $p(1)=1$, $p(a_i)=0$ for $i\neq 1$ and $q\in\mathcal{M}(B)$ such that $q(1)=1$, $q(b_j)=0$ for $j\neq 1$. As in (1) $p\otimes q\in\mathcal{M}_\Phi(A\otimes B)$. Therefore $w=w(1\otimes 1)=w[(p\otimes q)(1\otimes 1)]=(p\otimes q)(w)=\gamma_{11}\in Z\otimes F$.

(3) For W an ideal of $A\otimes B$ we define $W^\psi=W\cap(Z\otimes F)$. We first show that $W^{\psi\phi}=W$ for any ideal W of $A\otimes B$. It is obvious that $W^{\psi\phi}\subseteq W$. Now let $w\in W$ and, according to (1), write $w=\Sigma_{i,j}\gamma_{ij}(a_i\otimes b_j)$, $\gamma_{ij}\in Z\otimes F$. For any fixed i, j pick $p\in\mathcal{M}(A)$ such that $p(a_i)=1$, $p(a_k)=0$ for $k\neq i$, and $q\in\mathcal{M}(B)$ such that $q(b_j)=1$, $q(b_l)=0$ for $l\neq j$. As before $p\otimes q\in\mathcal{M}(A\otimes B)$ and $(p\otimes q)(w)=\gamma_{ij}\in W$. It follows that $w\in W^\psi(A\otimes B)$. Finally we show that $U^{\phi\psi}=U$ for every ideal U of $Z\otimes F$. Clearly $U\subseteq U^{\phi\psi}$. By (1) there is a $Z\otimes F$ basis $\{a_i\otimes b_j\}$, with $a_1=1$ and $b_1=1$, for $A\otimes B$. If $x\in U^{\phi\psi}$, then it is evident that x can be written in the form $\Sigma_{i,j}\mu_{ij}(a_i\otimes b_j)$, where $\mu_{ij}\in U$. On the other hand $x=\gamma\in Z\otimes F$ and so by the $Z\otimes F$-independence of $\{a_i\otimes b_j\}$ we see that $\mu_{ij}=0$ for $(i,j)\neq(1,1)$ and $x=\mu_{11}\in U$.

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University of Massachusetts
and
University of Wisconsin