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COMPACTNESS-LIKE PROPERTIES FOR GENERALIZED WEAK TOPOLOGICAL SUMS

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It is shown that under suitable conditions (involving cardinal numbers) on a family of spaces $\{X_i: i \in I\}$ with $p_i \in X_i$ for $i \in I$, their γ-weak sum $\{x \in \Pi_{i\in I} X_i: |\{i \in I: x_i \neq p_i\}| < \gamma\}$ is α-compact in the κ-box topology. For example, there is Corollary 2.5: If α is regular and uncountable and $|X_i| < \alpha$ for all $i \in I$, then the ω-weak sum (= direct sum) is α-compact in the α-box topology; in particular, the direct sum of any set of finite spaces is α-compact in the α-complete topology for regular $\alpha > \omega$.

1. Notation and definitions. We denote the smallest infinite cardinal by ω, and for a cardinal α we denote by α^+ the smallest cardinal β such that $\beta > \alpha$. For $\alpha \geqq \omega$ we denote by cf(α) the smallest cardinal β for which there is a family $\{\alpha_\xi: \xi < \beta\}$ of cardinals such that

$$\alpha_\xi < \alpha \quad \text{for all} \quad \xi < \beta \quad \text{and}$$

$$\sum_{\xi<\beta} \alpha_\xi = \alpha.$$

The (infinite) cardinal α is *regular* if $\alpha = \text{cf}(\alpha)$, *singular* otherwise.

For cardinals α and γ we set

$$\alpha^{\underline{\gamma}} = \sum \{\alpha^\beta: \beta \text{ is a cardinal and } \beta < \gamma\}.$$

and if I is a set we define

$$\mathcal{P}(I) = \{J: J \subset I\} \quad \text{and}$$

$$\mathcal{P}_\gamma(I) = \{J \in \mathcal{P}(I): |J| < \gamma\}.$$

It is well-known and easy to prove (see for example §1 of [5]) that

$$\gamma \leqq \alpha^{\underline{\gamma}} \quad \text{if} \quad 2 \leqq \alpha, \quad \text{and}$$

$$|\mathcal{P}_\gamma(\alpha)| = \alpha^{\underline{\gamma}} \quad \text{if} \quad \omega + \gamma \leqq \alpha^+.$$

Throughout this paper we denote by $\{X_i: i \in I\}$ or by $\{X_\xi: \xi < \alpha\}$ a set of nonempty topological spaces (not assumed to satisfy any special separation axioms), we fix $p_i \in X_i$, and for every infinite cardinal γ we set

$$\gamma \prod_{i\in I} X_i = \left\{x \in \prod_{i\in I} X_i : |\{i \in I: x_i \neq p_i\}| < \gamma\right\}.$$

When $\gamma = \omega$, the space $\gamma\Pi_{i\in I} X_i$ is called the *weak topological sum* of $\{X_i: i \in I\}$.

If $U_i \subset X_i$ and $U = \Pi_{i\in I} U_i$, we set

$$R(U) = \{i \in I: U_i \neq X_i\};$$

for a family $\mathcal{U}$ of such sets U we set $R(\mathcal{U}) = \cup\{R(U): U \in \mathcal{U}\}$. For an infinite cardinal κ, the *κ-box topology* on $\Pi_{i\in I} X_i$ is the topology generated by sets of the form $U = \Pi_{i\in I} U_i$ for which

U_i is open in X_i for all i, and

$|R(U)| < \kappa.$

The space $\Pi_{i\in I} X_i$ with the κ-box topology is denoted $(\Pi_{i\in I} X_i)_\kappa$, and for $Y \subset \Pi_{i\in I} X_i$ we denote by Y_κ the set Y with the topology inherited from $(\Pi_{i\in I} X_i)_\kappa$. We note that $(\Pi_{i\in I} X_i)_\omega$ is $\Pi_{i\in I} X_i$ with the usual product topology, and that if $|I| < \kappa$ then $(\Pi_{i\in I} X_i)_\kappa$ is $\Pi_{i\in I} X_i$ with the usual box topology.

The sets $U = \Pi_{i\in I} U_i$ as above are elements of the *canonical basis* for $(\Pi_{i\in I} X_i)_\kappa$ and if $Y \subset \Pi_{i\in I} X_i$ then the *canonical basis for Y_κ* consists of all sets of the form $U \cap Y$ (with U as above).

Let $\alpha \leqq \beta$. A space X is said to be *$[\alpha, \beta]$-compact* if for every open cover $\mathcal{U}$ of X such that $|\mathcal{U}| < \beta$ there is $\mathcal{V} \in \mathcal{P}_\alpha(\mathcal{U})$ such that $X = \cup \mathcal{V}$; and X is *α-compact* if X is $[\alpha, |\mathcal{P}(X)|^+]$-compact, *i.e.*, if for every open cover $\mathcal{U}$ of X there is $\mathcal{V} \in \mathcal{P}_\alpha(\mathcal{U})$ such that $X = \cup \mathcal{V}$. If $\mathcal{B}$ is a basis for X then X is *$[\alpha, \beta]$-compact with respect to $\mathcal{B}$* if for every cover $\mathcal{U} \in \mathcal{P}_\beta(\mathcal{B})$ there is $\mathcal{V} \in \mathcal{P}_\alpha(\mathcal{U})$ such that $X = \cup \mathcal{V}$.

We note that in this terminology a space is compact if and only if it is ω-compact, and Lindelöf if and only if it is ω^+-compact.

For a survey of properties related to $[\alpha, \beta]$-compactness see Vaughan **[14]**, **[15]**; for connections with topological groups, see Comfort and Saks **[8]**; and for product-space theorems relating to $[\alpha, \beta]$-compact spaces, see Vaughan **[13]** and Greene **[12]**. In these, notation and terminology differ slightly from that above.

There is at least one assertion concerning α-compactness of the space $X = \omega\, \Pi_{i\in I} X_i$ already available in the literature: It is pointed out by Corson **[9]** (Proposition 4) that if each of the spaces X_i is σ-compact, then X is σ-compact (and hence is a Lindelöf space).

It is not pleasing to be forced to introduce into the picture the bizarre concept "$[\alpha, \beta]$-compact with respect to $\mathscr{B}$". As partial justification, and as an explanation of its appearance in Theorem 2.3 below, we note that for every $\beta \geqq \omega$ there are a completely regular Hausdorff space X and a basis $\mathscr{B}$ for X such that X is $[\alpha, \beta]$-compact with respect to $\mathscr{B}$ for all cardinals α such that $2 \leqq \alpha \leqq \beta$, and X is not $[\alpha, \beta]$-compact for any cardinal $\alpha < \beta$. Indeed, we may take for X the discrete space β and for $\mathscr{B}$ the basis

$$\mathscr{B} = \{X\} \cup \{\{\xi\}: \xi < \beta\}.$$

2. Compactness for weak topological sums.

LEMMA 2.1. *Let $\omega \leqq \kappa$, and let I be a set and α, β and γ cardinals such that*

$$\omega \leqq \gamma \leqq |I|^{\underset{\smile}{\gamma}} < \mathrm{cf}(\alpha) \leqq \alpha \leqq \beta.$$

If $(\Pi_{i \in J} X_i)_\kappa$ is $[\alpha, \beta]$-compact for all $J \in \mathscr{P}_\gamma(I)$, then $(\gamma\, \Pi_{i \in I} X_i)_\kappa$ is $[\alpha, \beta]$-compact.

Proof. If $|I| < \gamma$ then with $J = I$ we have

$$\left(\gamma \prod_{i \in I} X_i\right)_\kappa = \left(\prod_{i \in I} X_i\right)_\kappa = \left(\prod_{i \in J} X_i\right)_\kappa$$

and the statement is obvious. We assume therefore that $|I| \geqq \gamma$.

For $J \in \mathscr{P}_\gamma(I)$ we set

$$Y(J) = \left\{x \in \gamma \prod_{i \in I} X_i : x_i = p_i \text{ for all } i \in I \backslash J\right\}.$$

Clearly $\gamma\, \Pi_{i \in I} X_i = \cup \{Y(J): J \in \mathscr{P}_\gamma(I)\}$, and for $J \in \mathscr{P}_\gamma(I)$ the space $(Y(J))_\kappa$ is homeomorphic to $(\Pi_{i \in J} X_i)_\kappa$. Now let $\mathscr{U}$ be an open cover of $(\gamma\, \Pi_{i \in I} X_i)_\kappa$ such that $|\mathscr{U}| < \beta$. For every $J \in \mathscr{P}_\gamma(I)$ there is $\mathscr{V}(J) \in \mathscr{P}_\alpha(\mathscr{U})$ such that $Y(J) \subset \cup \mathscr{V}(J)$. Then $\cup \{\mathscr{V}(J): J \in \mathscr{P}_\gamma(I)\}$ is a cover of $\gamma\, \Pi_{i \in I} X_i$ by elements of $\mathscr{U}$, and since $|\mathscr{P}_\gamma(I)| = |I|^{\underset{\smile}{\gamma}} < \mathrm{cf}(\alpha)$ and $|\mathscr{V}(J)| < \alpha$ for all $J \in \mathscr{P}_\gamma(I)$ we have

$$|\cup \{\mathscr{V}(J): J \in \mathscr{P}_\gamma(I)\}| \leqq \sum \{|\mathscr{V}(J)|: J \in \mathscr{P}_\gamma(I)\} < \alpha.$$

The proof is complete.

We note that if in Lemma 2.1 it is assumed simply that $(\Pi_{i\in J} X_i)_\kappa$ is $[\alpha, \beta]$-compact with respect to its canonical basis for all $J \in \mathcal{P}_\gamma(I)$, then (as the proof shows) the space $(\gamma \Pi_{i\in I} X_i)_\kappa$ is $[\alpha, \beta]$-compact with respect to its canonical basis.

REMARK 2.2. We note that if for every $i \in I$ the space X_i is a T_1-space such that $|X_i| \geqq 2$, and if further $\gamma \leqq \kappa$, $\gamma \leqq |I|$ and $\beta \geqq (2^\gamma)^+$, *then we have* $2^\lambda < \alpha$ *for all* $\lambda < \gamma$. *For under these (additional)* assumptions the space $(2^J)_\kappa$ is a closed, discrete subspace of $(\Pi_{i\in J} X_i)_\kappa$ (and is therefore α-compact) for all $J \in \mathcal{P}_\gamma(I)$, and hence $2^{|J|} < \alpha$ for all $J \in \mathcal{P}_\gamma(I)$.

DEFINITION. Let α and γ be cardinals such that $\gamma \leqq \alpha$. Then α is *strongly* γ*-inaccessible* if $\beta^\lambda < \alpha$ whenever $\beta < \alpha$ and $\lambda < \gamma$.

We note that if $\omega \leqq \gamma < \alpha$ and α is regular then the following three conditions are equivalent:

(i) α is strongly γ-inaccessible;

(ii) $\Pi_{\xi<\lambda} \alpha_\xi < \alpha$ whenever $\lambda < \gamma$ and $\alpha_\xi < \alpha$ for all $\xi < \lambda$;

(iii) $\beta^{\underline{\gamma}} < \alpha$ *for all* $\beta < \alpha$.

It has been shown by Erdős and Rado **[10]** that for α a regular infinite cardinal and $\gamma < \alpha$, the following combinatorial condition is equivalent to (i)–(iii).

(iv) if $\{S_\xi : \xi < \alpha\}$ is a family of sets such that $|S_\xi| < \gamma$ for all $\xi < \alpha$, then there is $A \subset \alpha$ and a set B such that $|A| = \alpha$ and $S_\xi \cap S_\eta = B$ whenever $\xi, \eta \in A$ and $\xi \neq \eta$.

For another proof of the equivalence (i) $\Leftrightarrow$ (iv) and several topological equivalences, see **[3]**; and for applications to generalized Σ-spaces see **[4]** and **[6]**. The notion of pairs γ, α such that $\omega \leqq \gamma < \alpha$ and α is strongly γ-inaccessible has been introduced independently and used by Fuhrken **[11]** in connection with a generalization of Łoś' ultraproduct theorem.

THEOREM 2.3. *Let* $\omega \leqq \kappa \leqq \alpha$ *and* $\omega \leqq \gamma \leqq \alpha \leqq \beta$ *with* α *regular and strongly* γ*-inaccessible, let* $\{X_i : i \in I\}$ *be a family of spaces such that* $(\Pi_{i\in J} X_i)_\kappa$ *is* $[\alpha, \beta]$*-compact for all* $J \in \mathcal{P}_\gamma(\alpha)$*, and define* $X = (\gamma \Pi_{i\in I} X_i)_\kappa$. *Then* X *is* $[\alpha, \beta]$*-compact with respect to the canonical basis.*

Proof. Let $\mathcal{U}$ be a cover of X such that $|\mathcal{U}| < \beta$ and the elements of $\mathcal{U}$ are basic open sets, *i.e.*, they are sets of the form $U \cap X$ where $U = \Pi_{i\in I} U_i$ with U_i open in X_i for all $i \in I$, and $|R(U)| < \kappa$ for all $i \in I$. We show there is $\mathcal{V} \in \mathcal{P}_\alpha(\mathcal{U})$ such that $\cup \mathcal{V} = X$.

For $J \subset I$ we set

$$Y(J) = \{x \in X : x_i = p_i \text{ for all } i \in I \backslash J\}$$

and we note that if $J \in \mathcal{P}_\alpha(I)$ then since $|J| < \alpha = \mathrm{cf}(\alpha)$ and $(Y(J))_\kappa$ is homeomorphic to $(\gamma \prod_{i\in J} X_i)_\kappa$ the space $(Y(J))_\kappa$ is $[\alpha, \beta]$-compact by Lemma 2.1.

We set

$$\bar{\gamma} = \gamma \text{ if } \gamma \text{ is regular}$$
$$= \gamma^+ \text{ if } \gamma \text{ is singular}$$

and we note that $\bar{\gamma} < \alpha$ (since if γ is singular then $\bar{\gamma} = \gamma^+ \leqq \gamma^{\mathrm{cf}(\gamma)} < \alpha$).

We observe now that there are families $\{J_\xi: \xi < \bar{\gamma}\} \subset \mathcal{P}(I)$ and $\{\mathcal{V}_\xi: \xi < \bar{\gamma}\} \subset \mathcal{P}(\mathcal{U})$ such that

(i) $J_0 = \emptyset$;

(ii) $\mathcal{V}_0 = \emptyset$;

(iii) $J_{\xi+1} = J_\xi \cup \bigcup\{R(V): V \in \mathcal{V}_\xi\}$ for $\xi < \bar{\gamma}$;

(iv) $\mathcal{V}_{\xi+1}$ is a cover of $Y(J_{\xi+1})$ and $|\mathcal{V}_{\xi+1}| < \alpha$ for $\xi < \bar{\gamma}$; and

(v) $J_\xi = \bigcup_{\eta<\xi} J_\eta$ and $\mathcal{V}_\xi = \bigcup_{\eta<\xi} \mathcal{V}_\eta$ for nonzero limit ordinals $\xi < \bar{\gamma}$.

Indeed, these families may be defined recursively, it being noted inductively that $|J_{\xi+1}| < \alpha$ for $\xi < \bar{\gamma}$ (so that $(Y(J_{\xi+1}))_\kappa$ is $[\alpha, \beta]$-compact and $\mathcal{V}_{\xi+1}$ may be defined as required in (iv)).

The definition of the families $\{J_\xi: \xi < \bar{\gamma}\}$ and $\{\mathcal{V}_\xi: \xi < \bar{\gamma}\}$ being complete, we set

$$J = \bigcup_{\xi<\bar{\gamma}} J_\xi \quad \text{and} \quad \mathcal{V} = \bigcup_{\xi<\bar{\gamma}} \mathcal{V}_\xi.$$

Since $|J_\xi| < \alpha$ and $|\mathcal{V}_\xi| < \alpha$ for all $\xi < \bar{\gamma}$ (by (iii), (iv) and (v) above) we have $|J| < \alpha$ and $|\mathcal{V}| < \alpha$.

We note that if $y \in Y(J)$ and $y_i \neq p_i$ then $i \in J$ and hence there is $\xi(i) < \bar{\gamma}$ such that $i \in J_{\xi(i)}$. Since $|\{i \in I: y_i \neq p_i\}| < \gamma \leqq \bar{\gamma}$ and $\bar{\gamma}$ is regular there is $\bar{\xi} < \bar{\gamma}$ such that $\xi(i) < \bar{\xi}$ for all i with $y_i \neq p_i$; it follows that

$$y \in Y(J_{\bar{\xi}}) \subset \bigcup \mathcal{V}_{\bar{\xi}} \subset \bigcup \mathcal{V}.$$

We conclude that $Y(J) \subset \bigcup \mathcal{V}$.

We claim that in fact $X \subset \bigcup \mathcal{V}$. Indeed let $x \in X$ and define

$$y_i = x_i \quad \text{if} \quad i \in J$$
$$= p_i \quad \text{if} \quad i \in I \backslash J.$$

It is clear that ($y \in X$ and) $y \in Y(J)$. From the preceding paragraph there is $V = \prod_{i\in I} V_i \in \mathcal{V}$ such that $y \in V$, and since $R(V) \subset J$ we have $V_i = X_i$ for all $i \in I \backslash J$ and hence $x \in V$. The proof is complete.

COROLLARY 2.4. *Let $\omega \leqq \kappa \leqq \alpha$ and $\omega \leqq \gamma < \alpha$ with α regular and strongly γ-inaccessible, let $\{X_i: i \in I\}$ be a family of spaces such that $|X_i| < \alpha$ for all $i \in I$, and set $X = (\gamma \Pi_{i \in I} X_\xi)_\kappa$. Then X is α-compact.*

Proof. From the equivalence (i) $\Leftrightarrow$ (ii) remarked above we have $|\Pi_{i \in J} X_i| < \alpha$ whenever $J \in \mathcal{P}_\gamma(I)$, so that $(\Pi_{i \in J} X_i)_\kappa$ is $[\alpha, \beta]$-compact for all $J \in \mathcal{P}_\gamma(I)$ and $\beta \geqq \alpha$. Thus the result follows from Theorem 2.3.

We remark that in Corollary 2.4 (and earlier) the hypothesis that α is strongly γ-inaccessible cannot be omitted (or even substantially modified). Indeed suppose that $\omega \leqq \gamma < \alpha$ and that there are cardinals $\beta < \alpha$ and $\lambda < \gamma$ such that $\beta^\lambda \geqq \alpha$; then the power space $(\beta^\lambda)_\alpha$, which is $(\gamma\beta^\lambda)_\alpha$, is a discrete space of cardinality at least α and is not α-compact.

In the positive direction, every infinite cardinal is strongly ω-inaccessible. Thus from Corollary 2.4 we have the following further specialization of Theorem 2.3.

COROLLARY 2.5. *Let α be an uncountable, regular cardinal, let $\{X_i: i \in I\}$ be a family of spaces such that $|X_i| < \alpha$ for all $i \in I$ and let X be the "weak sum in the α-box topology"*, i.e., $X = (\omega \Pi_{i \in I} X_i)_\alpha$. *Then X is α-compact.*

3. An application. For a nondiscrete space X we denote by pX the smallest possible number of open subsets of X with nonopen intersection. Now the space 2 is an (additive) topological group with identity element 0, and it is clear that for regular $\alpha > \omega$ and with the choice $p_\xi = 0_\xi = 0$ for all $\xi < \beta$, the space $G(\alpha, \beta) = (\omega \Pi_{\xi < \beta} 2_\xi)_\alpha$ is a topological group such that $|G| = \beta$ and $pG = \alpha$. It is shown in **[1]** that the groups $G(\alpha, \alpha)$ are *fine, i.e.*, every continuous real-valued function on $G(\alpha, \alpha)$ is uniformly continuous, and the question is raised there whether for regular $\alpha > \omega$ there are fine groups G of arbitrarily large cardinality such that $pG = \alpha$. It follows from Theorem 2.3. (iii) of **[1]** and the fact that $G(\alpha, \beta)$ is α-compact for all $\beta \geqq \alpha$ (a special case of Corollary 2.5 above) that the answer to this question is "Yes". The present paper sheds no light on the several other questions posed in **[1]**.

4. Concluding remarks. My understanding of the full generality under which the conclusions of Theorem 2.3 and Corollary 2.4 can be established has been achieved slowly. over an extend period of time, and with the substantial help of other mathematicians. It is a pleasure to cite in particular my indebtedness to:

(a) K. A. Ross. The case of Corollary 2.4 in which $\gamma = \omega < \omega^+ = \alpha = \kappa = I$ and each $X_i = 2$ is Example 3.2 of **[7]**;

(b) A. W. Hager. Portions of the argument required for 2.4 above appear in 5.4 and 5.7 of **[1]**, where we considered however only the case in which $\gamma = \omega$ and there are no limit cardinals between α and $|I|$.

(c) A. Hajnal and I. Juhász. Conversations following my presentation of a colloquium with this title at the Bolyai János Mathematical Society resulted in a proof **[2]** with combinatorial emphasis of the case $X_i = 2$ of Corollary 2.4.

(d) K. Kunen. As referee of the present paper, he has strengthened the statement of Theorem 2.3 and shortened its proof.

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