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COUNTABLY GENERATED MODULES OVER COMMUTATIVE ARTINIAN RINGS

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A general method is given for constructing countably generated modules with a number of bizarre properties over any commutative Artinian ring which is not a principal ideal ring. The main result shows that if R is a commutative local Artinian ring which is not a principal ideal ring, and the residue class field of R is k, then any pathological property that holds for some $k[t]$-module also holds for a suitable R-module. This method gives easy and uniform proofs of many known results (and some new ones) concerning modules over these rings. A theorem of A. L. S Corner's, concerning countable endomorphism rings of torsion-free Abelian groups, is generalized to algebras over suitable discrete valuation rings, and applied to obtain further pathological results for modules over Artinian rings.

If R is a commutative Artinian principal ideal ring, then all R-modules are direct sums of cyclic modules. If R is a commutative Artinian ring which is not a principal ideal ring, then it is well known that R has indecomposable finitely generated modules requiring arbitrarily many generators. (J. P. Jans assures me that this result was known for algebras in antiquity. For Artinian commutative rings, it is implicit in Colby **[4]**, and explicit (in more general contexts) in Griffith **[8]** and Warfield **[12]**.) Actually, if R is not a principal ideal ring, R has indecomposable modules which are not finitely generated **[8]**, and, even, not countably generated, **[13]**. Recent work on noncommutative Artinian rings suggests that an equally sharp distinction may exist between rings of finite module type and other Artinian rings in general. (R has finite module type if there are only a finite number of isomorphism classes of finitely generated indecomposable modules.) We mention Roiter's theorem **[11]** that a finite dimensional algebra which is not of finite module type has indecomposable finitely generated modules requiring arbitrarily large numbers of generators, and Ringel and Tachikawa's theorem **[10]**, that if R has finite module type then every R-module is a direct sum of finitely generated indecomposable modules.

There is clearly a temptation to hope that even if a ring is not of finite module type, one still might be able to prove some general theorems about the good behavior of its infinitely generated modules. In the commutative case, we give a general method for disillusioning oneself of such hopes. As an application of the method, we prove the following theorem:

THEOREM A. *Let R be a commutative Artinian ring which is not a principal ideal ring. Then*

(i) *R has a countably generated indecomposable module which has a monic endomorphism which is not epic.*

(ii) *R has a countably generated indecomposable module which has an epic endomorphism which is not monic,*

(iii) *R has* $2^{\aleph_0}$ *nonisomorphic indecomposable countably generated modules,*

(iv) *R has a set of three countably generated indecomposable modules, A, B, C, such that B and C are not isomorphic and such that* $A \oplus B \cong A \oplus C$ (i.e. the Krull-Schmidt and cancellation properties fail).

(v) *R has a countably generated module with no indecomposable summands.*

(vi) *For any cardinal n, R has a module which is not a summand of a direct sum of modules which can be generated by n elements.*

We recall that any commutative Artinian ring is a direct sum of local rings, so it is enough to prove our results for a local Artinian commutative ring which is not a principal ideal ring. If R is such a ring, and k is its residue class field, we will show that properties such as the above hold for modules over R if they hold for modules over $k[t]$. Over the ring $k[t]$ of polynomials over a field k, examples of modules satisfying (i), (ii), (iii) and (vi) are quite elementary. This will give easy proofs of (i), (ii), (iii), (vi), and of some other results which are weaker than (iv) and (v). This also gives essentially trivial proofs of the existence of finitely generated indecomposable R-modules requiring arbitrarily many generators, indecomposable R-modules which are countably, but not finitely, generated, and indecomposable R-modules requiring an uncountable number of generators. (There last results were previously known, and are in the above references. (vi) was proved in **[13]**.)

The results (iv), and (v) are more difficult, and require an extension of methods of A.L.S. Corner's, first used to prove similar results for Abelian groups. A somewhat different adaptation of Corner's methods is used by Brenner in **[3]**, to prove these same results for a large class of rings, including many noncommutative rings, but not including all commutative Artinian artinian rings which are not principal ideal rings. In particular, Brenner's results in **[2]**, on group rings, omit the group $(Z/2Z)\oplus(Z/2Z)$, which is not in any way excluded by our methods (i.e. if k is a field of characteristic 2, $k[(Z/2Z)\oplus(Z/2Z)]$ is a ring for which our results hold.)

There are a number of other pathological results known for Abelian groups that one might expect to generalize to modules over the rings $k[t]$, and, therefore, (by our results) to modules over commutative Artinian rings which are not principal ideal rings. We refer to Fuchs [7, §§88–91], and the references there, for some such results, some of which have recently been extended to large cardinals by Shelah (not yet published). The proofs of such pathological results tend to depend on various arithmetic properties of the ring of integers, and, hence, tend not to generalize immediately to the rings $k[t]$. It is probably true, however, that with some ingenuity, most of these results can be extended, if one happens to wish to do so.

1. Translatable modules over an Artinian ring. We recall (e.g. from [13, Lemma 1]) that if R is a commutative Artinian ring which is not a principal ideal ring, then R contains an ideal I such that R/I is a local ring with maximal ideal m, such that $m^2 = 0$ and m has R/m—dimension 2. We may assume, therefore, that R is originally a ring of this sort, and that m is generated by the two elements a and b. An example of such a ring is $k[x, y]/(x^2, xy, y^2)$. We let $k = R/m$.

If M is any R-module, multiplication by a and b respectively induce k-linear transformations α and β, from M/mM to mM. We will say that M is *a-translatable* if α is an isomorphism of M/mM onto mM.

If M is a-translatable, we define a translation (linear transformation) of M/mM as follows: if $x \in M/mM$, there is a unique $y \in M/mM$ such that $\alpha(y) = \beta(x)$. Let $tx = y$. This defines a linear transformation of M/mM into itself, and gives M/mM a natural $k[t]$-module structure.

We note that if M and N are a-translatable R-modules, and $f: M \to N$ is a homomorphism, then f induces a $k[t]$-module homomorphism $M/mM \to N/mN$. ($f(M)$, however, need not be an a-translatable submodule of N.) We have therefore defined a functor from the category of a-translatable R-modules (a full subcategory of the category of all R-modules) to the category of $k[t]$-modules.

THEOREM B. *Let Φ be the functor associating to each a-translatable module M, the $k[t]$-module M/mM. Then for any $k[t]$ module N, there is an a-translatable R-module M such that $\Phi(M) \cong N$. Further, if A and B are $k[t]$-modules, and $f: A \to B$ a homomorphism, and if M and N are a-translatable R-modules such that $\Phi(M) = A$, $\Phi(N) = B$, then there is an R-homomorphism $g: M \to N$ such that $\Phi(g) = f$. If f is monic, so is g, and if f is epic, so is g. If $\Phi(M) = A$,* *then any direct sum decomposition of M gives rise to a decomposition of A, and any decomposition of A arises from a decomposition of M.*

This theorem is proved by a series of lemmas. We first write down an example of an a-translatable module. Let M be the module defined by taking elements $x_i (1 \leqq i < \infty)$ as generators, and imposing the relations $bx_i = ax_{i+1} (1 \leqq i < \infty)$. M is clearly a-translatable, and $\Phi(M) = k[t]$ (regarded as a module over itself). Taking sums of copies of this M, we see that any free $k[t]$-module is a value of the functor Φ. To go further, we naturally consider a-translatable submodules of a-translatable modules.

LEMMA 1. *If M is a-translatable, and K a submodule, then K is a-translatable if and only if for every $x \in M$ such that $ax \in K$, $x \in mM + K$. In this case, $aK = K \cap mM$, K/mK is a $k[t]$-submodule of M/mM, and M/K is a-translatable.*

Proof. If K is a-translatable then $mK = aK$. Also, if $y \in K$ and $ay = 0$, then $y \in mK$. Hence, $mK = aK = K \cap mM$. Also, if $x \in M$, and $ax \in K$, then $ax \in aK$, whence $x \in K + mM$. To show the converse, suppose that K is a submodule of M with the property that if $x \in M$ and $ax \in K$, then $x \in mM + K$. If $z \in K \cap mM$, $z = ax$ for some $x \in M$, so for some $x' \in mM$, $x - x' \in K$. Clearly, $a(x - x') = ax$, so $z \in aK$. This shows that $K \cap mM = aK$. The map $\alpha: K/mK \to mK$ is therefore onto, and it is also one-to-one, since $x \in K$, $x \notin mK$ implies $x \notin mM$ (since $K \cap mM = aK$), so $ax \neq 0$.

It is clear that if K is an a-translatable submodule of M, then K/mK can be regarded as a $k[t]$-submodule of M/mM, so all that remains is to show that M/K is also a-translatable.

If $x \in M/K$, $ax = 0$, and $x = y + K$, then $ay \in K$, so $y \in mM + K$, which implies that $x \in m(M/K)$. To show that M/K is a-translatable, therefore, we need only show that $m(M/K) = a(M/K)$, which follows from the fact that $m(M/K) = (mM + K)/K = (aM + K)/K$.

LEMMA 2. *If M is a-translatable, and L a $k[t]$-submodule of M/mM, then there is an a-translatable submodule K of M such that $K/mK = L$.*

Proof. Choose a set X in M such that the natural map $\phi: M \to M/mM$ takes X bijectively onto a basis for M/mM as a k-vector space, and such that $\phi(X) \cap L$ is a basis for L. Let K be the submodule of M generated by those $x \in X$ such that $\phi(x) \in L$. Clearly $K/(K \cap mM) = L$. Suppose $x \in M$ and $ax \in K$. It is clear by construction that $mK = K \cap mM$, so $ax = ay + bz$, for suitable

y and z in K. If we choose w such that $aw = bz$ ($w \in M$) then $t(z + mM) = w + mM$, and, since L is a $k[t]$-submodule of M/mM, there is a $w' \in K$ and a $w'' \in mM$ such that $w = w' + w''$. Hence $bz = aw = aw' \in aK$, so $ax = a(y + w')$. Since M is a-translatable, this means $x - y - w' \in mM$, so $x \in K + mM$ as required.

LEMMA 3. *If M is a-translatable, and $M/mM = \bigoplus_{i \in I} A_i$, where the decomposition is a $k[t]$-module decomposition, then $M = \bigoplus_{i \in I} B_i$ where the B_i are a-translatable submodules, and $(B_i + mM)/mM = A_i$ for all $i \in I$.*

Proof. Choose a subset X of M such that the natural map $\phi: M \to M/mM$ takes X bijectively onto a basis of M/mM, with the property that $\phi(X) \cap A_i$ is a basis for A_i (as a k-vector-space) for all $i \in I$. Let B_i be the R-submodule of M generated by those $x \in X$ such that $\phi(x) \in A_i$. By the previous lemma, the modules B_i are a-translatable. By construction, $B_j \cap \Sigma_{i \neq j} B_i \subseteq mM$, and, by the argument of the previous lemma, $B_j \cap mM = aB_j = \alpha(A_j)$, so the sum of the B_i is a direct sum. If $B = \bigoplus_{i \in I} B_i$, then $M = B$, since, by construction, $M = B + mM$, and $mM \subseteq B$, since $mM = mB + m^2M = mB$.

LEMMA 4. *If M and N are a-translatable R-modules and $f: M/mM \to N/mN$ a $k[t]$-module homomorphism, then there is an R-homomorphism $g: M \to N$ such that g induces the map f. Further, if f is a monomorphism, then any such g is a monomorphism, and if f is an epimorphism, then any such g is an epimorphism. In particular, if f is an isomorphism, so is g.*

Proof. Choose subsets X and Y of M and N such that if $\phi: M/mM$ and $\psi: N \to N/mN$ are the natural maps, then ϕ and ψ take X and Y bijectively onto k-bases of M/mM and N/mM, $\phi(X) \cap \operatorname{Ker}(f)$ spans $\operatorname{Ker}(f)$ over k, and $f\phi(X) \subseteq \psi(Y)$. If $x \in X$, define $g(x) = y$, where $y \in Y$ and y is chosen so that $\psi(y) = f\phi(x)$.

We would like to extend g to M as a module homomorphism (which can be done in at most one way, since X generates M over R). To show this is possible, we need to show that if $\{x_i: i = 1, \cdots, n\}$ are distinct elements of X, and $y_i = g(x_i)$, and there are elements r_i and s_i in R such that

$$r_1x_1 + \cdots + r_nx_n = s_1x_1 + \cdots + s_nx_n,$$

then, also

$$r_1y_1 + \cdots + r_ny_n = s_1y_1 + \cdots + s_ny_n.$$

It suffices to consider the special case $r_1x_1 + \cdots + r_nx_n = 0$, with the x_i distinct, which clearly implies that $r_i \in m$ for all i. We write $r_i = au_i - bv_i$. The equation then can be written

$$\alpha(u_1\phi(x_1) + \cdots + u_n\phi(x_n)) = \beta(v_1\phi(x_1) + \cdots + v_n\phi(x_n)).$$

We want this to imply that

$$\alpha(u_1\psi(y_1) + \cdots + u_n\psi(y_n)) = \beta(v_1\psi(y_1) + \cdots + v_n\psi(y_n)).$$

Since $\psi(y_i) = f\phi(x_i)$, if $x = v_1x_1 + \cdots + v_nx_n$, and $z = u_1x_1 + \cdots + u_nx_n$, then the first equation says $z = tx$, and the second says $f(z) = tf(x)$. The first, therefore, implies the second, since f is a $k[t]$-homomorphism.

Suppose, now, that f is a monomorphism. Since aM is essential in M, it is enough to show that $\operatorname{Ker}(g) \cap aM = 0$. On aM, an easy computation shows that $g = \alpha f \alpha^{-1}$, a composite of monomorphisms, which proves the result. If f is an epimorphism, then $N = g(M) + mN$, and $mN \subseteq g(M)$, since $mN = mg(M) + m^2N$ and $m^2 = 0$.

These various lemmas prove Theorem B. In particular, we know that for any free module F, there is an a-translatable module M such that $\Phi(M) = F$. By Lemma 2, if L is a submodule of F, there is an a-translatable submodule K of M such that $\Phi(K) = L$. In this case, Lemma 1 implies that M/K is an a-translatable module, and $\Phi(M/K) = F/L$. Hence, every $k[t]$-module is of the form $\Phi(N)$. The remainder of the theorem is contained in Lemmas 3 and 4.

2. Elementary applications of Theorem B. In this section, we write down some completely elementary facts about modules over the ring $k[t]$, and derive examples by using Theorem B. We give elementary proofs of the following facts: If R is a commutative Artinian ring which is not a principal ideal ring, then

(a) R has finitely generated indecomposable modules requiring arbitrarily large numbers of generators,

(b) R has a countably generated indecomposable module which has a monic epimorphism which is not epic,

(c) R has a countably generated indecomposable module which has an epic endomorphism which is not monic,

(d) R has a set of $2^{\aleph_0}$ nonisomorphic indecomposable countably generated modules,

(e) R has a countably generated module with two direct sum decompositions which do not have isomorphic refinements,

(f) R has a countably generated module which is not a direct sum of indecomposable modules,

(g) For any cardinal n, R has a module requiring more than n generators, all of whose direct sum decompositions are finite.

In particular, when this has been done, we will have proved parts (i), (ii), and (iii) of Theorem A. Also, (g) easily implies (vi), since, by the higher cardinal form of Kaplansky's lemma **[9]**, a summand of a direct sum of modules with n generators is again a direct sum of modules with n generators, for any infinite cardinal n.

We first note that $k[t]$ as a module over itself is an indecomposable module of countably infinite k-dimension, which admits a monic endomorphism which is not epic. This example proves (b). (We notice that if $\Phi(M) = A$, then the nunber of generators required for M is the k-dimension of A.) If M_n is an a-translatable R-module such that $\Phi(M_n) = k[t]/(t^n)$, M_n is an indecomposable R-module requiring exactly n generators. This proves (a).

The set N of k-rational functions in t of the form $t^n f$, where $f \in k[t]$ and n is any integer, is a $k[t]$-submodule of the quotient field of $k[t]$. $N/k[t]$ is an indecomposable $k[t]$-module with the descending chain condition which admits an epic endomorphism which is not monic. (This is the analogue over $k[t]$ of the Abelian group $Z(p^\infty)$, **[7**, vol. I, p. 15].) This proves (c).

Choose a countably infinite set C of inequivalent irreducible nonconstant polynomials in $k[t]$. For every subset D of C, let $M(D)$ be the set of rational functions of the form g/f, where $g \in k[t]$ and f is an element of $k[t]$ all of whose irreducible factors are in the set D. The modules $M(D)$ are all indecomposable torsion-free $k[t]$-modules of rank one, and all countably generated. They are pairwise nonisomorphic. (See **[7**, sections 85 and 86] for the Abelian group analogue.) This proves (d).

An indecomposable $k[t]$-module is either torsion or torsion-free, and the indecomposable torsion modules are either divisible or cyclic **[7**, 3.1 and 27.4]. Following Prüfer, (or **[7**, I, p. 150]), we consider the torsion module M given by generators and relations as follows: the generators are x_i, $i = 0, 1, \cdots, n, \cdots$, and the relations $tx_0 = 0$, $t^n x_n = x_0 (n > 0)$. M is clearly not a direct sum of indecomposable modules, which proves (f). In any decomposition of M, $M = B \oplus C$, one of the summands is a direct sum of cyclic modules while the other is not. Using Ulm's theorem, it is easy to show (Baer, **[1]**) that M has two direct sum decompositions which do not have isomorphic refinements. This proves (e).

For any index set I, the $k[t]$-module $\Pi_{i \in I} k[[t]]$ is a module all of whose direct sum decompositions have only a finite number of summands, **[13]**. This proves (g).

3. An adaptation of Corner's theorem. We prove the following theorem:

THEOREM C. *Let R be a discrete valuation ring, R^* its completion, and suppose that R^* has uncountable transcendence degree over R. Then if A is any countably generated R-algebra which is torsion-free and reduced* (no injective summands) *as an R-module, there is a countably generated torsion-free R-module M such that* $\mathrm{End}(M) \cong A$.

The proof is a straightforward application of the method used to prove the corresponding result for Abelian groups in [5]. The details appear later in this section. We first give the applications to modules over commutative Artinian rings. To show that Theorem C is applicable, we need the following lemma.

LEMMA 5. *If k is a field, the power series ring $k[[t]]$ has transcendence degree at least $2^{\aleph_0}$ over k.*

Proof. We first note that if k is a prime field (Z/pZ or the field of rational numbers), then $k[[t]]$ has cardinality $2^{\aleph_0}$, while any countably generated extension of k is countable, so the result is clear for such fields. We now let K be an arbitrary field, and k a prime field in K, and regard $k[[t]]$ as a subring of $K[[t]]$. We claim that if $p_i (i \in I)$ are elements of $k[[t]]$ which are algebraically independent over k, then, as elements of $K[[t]]$, they are algebraically independent over K. Suppose, then, that there were elements $p_i (i = 1, \cdots, n)$ algebraically independent over k, and an equation $\Sigma r_a p^a = 0$, where the index a ranges over a finite set of lattice points $(a_1, \cdots, a_n)$, $a_i \geqq 0$, and $p^a = p_1^{a_1} \cdots p_n^{a_n}$. For every a, p^a is a power series over k with coefficients $q(a, i)$, $0 \leqq i < \infty$, and the equation would say $\Sigma r_a q(a, i) = 0$ for all i. We regard these equations as a set of equations in the variables r_a, with coefficients $q(a, i)$. Our hypothesis says that these equations do not have a simultaneous solution over the field k. An infinite set of linear equations in m variables (m constant) has a solution if and only if every finite subset has a solution, and a finite subset has a solution in K if and only if it has a solution in k, so the insolubility of these equations over k implies their insolubility over K. This implies that the elements $p_i (i = 1, \cdots, n)$ remain algebraically independent over K, as desired.

To prove parts (iv) and (v) of Theorem A, it suffices to prove them for modules over the ring $k[t]$, (by Theorem B). For each of these properties, it is possible to write down a property of a ring E such that if $\mathrm{End}(M) = E$, then M has the indicated property. In [5, pp. 708–9], Corner constructs an algebra E over the ring of integers such that for any nonzero idempotent α of E, there is a nonzero idempotent β of E, distinct from α, such that $\alpha\beta = \beta$. His construction is completely valid over any commutative integral domain. In particular, we let $R = k[t]_{(t)}$, the localization of $k[t]$ at the prime ideal (t), and let E be an R-algebra constructed in this way. By Theorem C, there is a countably generated R-module N with E as its endomorphism ring. If R is countably generated as a $k[t]$-module (which is true if k is countable), then N is already a module of our desired sort, and the proof of (v) is completed by an application of Theorem B. If R is not countably generated over $k[t]$, we can construct a $k[t]$-submodule of N with the desired properties. We note that from Corner's construction, E only has a countable number of idempotents. We let Y_1 be a countable subset of N that generates N over R. Define Y_{n+1} inductively to the set of elements of N of the form αx, where $x \in Y_n$ and α is an idempotent of E. Let Y be the ascending union of the sets Y_n, and let M be the $k[t]$-submodule of N generated by Y. M is clearly countably generated over $k[t]$, and, by construction, it has no indecomposable summands. Applying Theorem B, we now obtain a proof of (v) in complete generality.

The proof of (iv) involves a minor adaptation of a construction given by Corner and Crawley in **[6]**. Again, by Theorem B, it is enough to prove that the pathology can occur for modules over the ring $k[t]$. We first let $R = k[t]_{(t)}$ as before, and construct an algebra S over R. Let S be the ring of matrices of the form

$$\begin{pmatrix} f_{11} & (1-s^2)f_{12} \\ f_{21} & a+(1-s^2)f_{22} \end{pmatrix}$$

where the f_{ij} are elements of the ring $R[s]$, and $a \in R$. S is clearly a countably generated algebra over R. This ring has idempotents α and β such that α and β are equivalent, but $1-\alpha$ and $1-\beta$ are not equivalent, and such that if M is an a R-module with S as endomorphism ring, then αM, βM, $(1-\alpha)M$, and $(1-\beta)M$ are all indecomposable. The proof of these facts is contained in **[6]**, though there is a possibility of confusion arising from the fact that in that paper, S is only a subring of the endomorphism of an Abelian p-group, and there are a number of other things that the authors have to keep track of in the course of their argument. Briefly, one defines

$$\theta = \begin{pmatrix} s & 1-s^2 \\ 0 & 0 \end{pmatrix} \qquad \phi = \begin{pmatrix} s & 0 \\ 1 & 0 \end{pmatrix}$$

and $\alpha = \theta\phi$, $\beta = \phi\theta$. Noting that $\theta\phi\theta = \theta$ and $\phi\theta\phi = \phi$, we see that α and β are equivalent idempotents. If $1-\alpha$ and $1-\beta$ were equivalent, then there would be matrices in S, μ and ν, such that $\mu\nu = 1-\alpha$,

$\nu\mu = 1-\beta$, $\mu\nu\mu = \mu$, and $\nu\mu\nu = \nu$. Ignoring the statement about reducing modulo p in **[6]**, the last paragraph in **[6]** provides a proof of the impossibility of the existence of μ and ν. Hence, if M is an R-module with S as its endormorphism ring, we have $M = \alpha M \oplus (1-\alpha)M = \beta M \oplus (1-\beta)M$, with $\alpha M \cong \beta M$, $(1-\alpha)M \not\cong (1-\beta)M$. The summands appearing here are all indecomposable, since S is a subring of a two-by-two matrix ring over a field, so that there cannot be a set of three orthogonal idempotents in S.

This establishes all that is needed (using Theorem B) to prove (iv), except for the fact that if R is not countably generated over $k[t]$, then M will not be countably generated. In this case, as before, we construct a countably generated $k[t]$-submodule of M which still has the same pathology, using exactly the same technique as before (which is easier in this case, since we do not need to consider all idempotents of S, but only the idempotents 1, α, and β.)

We now prove Theorem C. Let R be a discrete valuation ring, R^* its completion, p a prime element, and we assume that the quotient field of R^* has uncountable transcendence degree over the quotient field of R. We recall that if Q is the quotient field of R, then $Q \otimes_R R^*$ is the quotient field of R^*.

Let A be a countably generated, reduced, torsion-free R-algebra, and let X be a maximal R-independent subset of A, such that $1 \in X$. We must find a countably generated, torsion-free module M whose endomorphism ring $E(M)$ is isomorphic with A.

The p-adic completion A^* of A is a torsion-free R^*-module. We regard A as an R-submodule of A^*, and choose a family $f_i (i \in I)$ of elements of A which is a maximal set of elements which are independent over R^*. For any $x \in X$, there are elements $r(x)$, $r(x,i)$ $(i \in I)$ of R^* such that

$$r(x)x + \sum_{i \in I} r(x,i)f_i = 0,$$

where $r(x,i) = 0$ for all but a finite number of indices i. Multiplying by a suitable unit of R^*, we obtain

$$p^k x = \Sigma\, s(x,i) f_i$$

where the elements $s(x,i)$ are determined uniquely up to a factor of a power of p. Let S be the pure R-subalgebra of R^* generated by the elements $s(x,i)$ $(x \in X, i \in I)$. S is a countably generated R-algebra, from which it follows that the transcendence degree of R^* over S is uncountable.

LEMMA 6. *If $\Sigma_{j \in J} g_j a_j = 0$, where $g_j \in R^*$, the elements g_j are independent over S, $a_j \in A$, and J is a finite set, then the a_j are all zero.*

Proof. For some fixed n, $p^n a_j = \Sigma\, r(x, j)x$ $(x \in X, r(x, j) \in R)$.

We recall that there is a fixed k such that for all x (a finite number) that appear in the above equations,

$$p^k x = \Sigma\, s(i, x) f_i; \quad s(i, x) \in S.$$

The equation becomes

$$\Sigma\, g_j r(x, j) s(i, x) f_i = 0$$

(the summation being over j, x, and i—the Einstein convention). Since the f_i are independent over R^*, we obtain

$$\Sigma\, g_j r(x, j) s(i, x) = 0$$

(summed over j and x.) Since the g_j are independent over S,

$$\Sigma\, r(x, j) s(i, x) = 0$$

for each i and j. Since

$$p^{n+k} a_j = \sum_i \left(\sum_x r(x, j) s(i, x) f_i \right)$$

and A is torsion-free, we conclude that $a_j = 0$, for all j.

We now return to the proof of the theorem. Choose elements $a(x)$, $b(x)$ $(x \in X)$ of R^* that are algebraically independent over S, and define elements $e(x)$ $(x \in X)$ of A^* by setting

$$e(x) = a(x) \cdot 1 + b(x) \cdot x.$$

Let M be the pure R-submodule of A^* generated by A and $e(x)A$, $x \in X$. M is clearly countably generated, reduced, and torsion-free over R.

Since $A^2 \subseteq A$, it is clear that $MA \subseteq M$, so A operates on the right of M. Since $1 \in M$, the map $A \to \operatorname{End}(M)$ obtained in this way is one-to-one. To show that this map is an isomorphism, we must show that every endomorphism of M coincides with right multiplication by some element of A.

If θ is an endomorphism of M (acting on the right), then θ extends to an R^*-endomorphism θ^* of A^*, (since $A^* = M^*$). If $x \in X$, then

$$e(x)\theta = (a(x)) \cdot 1)\theta^* + (b(x) \cdot x)\theta^* = a(x)(1 \cdot \theta) + b(x)(x\theta).$$

Since 1θ, $e(x)\theta$, and $x\theta$ are all in M, we can write

$$p^k(1\theta) = \sum_{y \in X} r_1(y)y + \sum_{\substack{w \in X \\ z \in X}} t_1(z, w)e(z)w \tag{1}$$

$$p^k e(x)\theta = \sum_{y \in X} r_2(y)y + \sum_{\substack{w \in X \\ z \in X}} t_2(z, w)e(z)w \tag{2}$$

$$p^k x\theta = \sum_{y \in X} r_3(y)y + \sum_{\substack{w \in X \\ z \in X}} t_3(z, w)e(z)w \tag{3}$$

where there are only a finite number of nonzero terms, and all of the f's and r's are elements of R.

Substituting all this in the previous equation, we obtain

$$\begin{aligned} \Sigma\, r_2(y)y + \Sigma\, t_2(z, w)e(z)w &= a(x)[\Sigma\, r_1(y)y + \Sigma\, t_1(z, w)e(z)w] \\ &\quad + b(x)[\Sigma\, r_3(y)y + \Sigma\, t_3(z, w)e(z)w]. \end{aligned} \tag{4}$$

(Here, and in all that follows, the element x is fixed, and the summation is over the elements z and w of X. In every case, you sum over z or w if and only if it appears twice in the given expression.) The elements 1, $a(x)$, $b(x)$, $a(x)b(z)$, etc. arc elements of R^* independent over S. By Lemma 6, the coefficients of each of these terms in the above equation m ust be zero. Immediately we see that $\Sigma\, r_2(y)y = 0$, and since the elements y are independent over R, we obtain

$$r_2(y) = 0,\ y \in X. \tag{5}$$

Looking at the coefficient of $a(x)a(z)$ (for all z) we see that $\Sigma\, t_1(z, w)w = 0$, whence

$$t_1(z, w) = 0,\ z \in X, w \in X. \tag{6}$$

Looking at the coefficient of $b(x)a(z)$, $z \neq x$, we obtain

$$t_3(z, w) = 0,\ (z \neq x, z \in X, w \in X). \tag{7}$$

From the coefficient of $a(z)$, $z \neq x$, we obtain

$$t_2(z, w) = 0,\ (z \neq x, z \in X, w \in X), \tag{8}$$

and the coefficient of $a(x)$ gives us

(9) $$r_1(y) = t_2(x, y), \; (y \in X).$$

The equations (1), (2) and (3) now become (using (5)–(9)),

(1′) $$p^k(1\theta) = \Sigma r_1(y)y$$

(2′) $$p^k e(x)\theta = \Sigma r_1(y)e(x)y$$

(3′) $$p^k x\theta = \Sigma r_3(y)y + \Sigma t_3(x, w)e(x)w.$$

Now expand $p^k e(x)\theta$, using the definition and (1′) and (3′), to obtain

$$p^k e(x)\theta = p^k[a(x)(1\theta) + b(x)(x\theta)] = a(x)\Sigma r_1(y)y + b(x)\Sigma r_3(y)y + b(x)a(x)\Sigma t_3(x, w)w + b(x)^2\Sigma t_3(x, w)xw.$$

Setting this equal to (2′), and using Lemma 6 again, we obtain from the coefficient of $b(x)$

(10) $$\Sigma r_1(y)xy = \Sigma r_3(y)y,$$

and from the coefficient of $b(x)a(x)$

(11) $$\Sigma t_3(x, w)w = 0.$$

Substituting these in (3′), and comparing with (1′), we obtain

(12) $$x(1\theta) = x\theta.$$

This shows that the endomorphism θ is given by right multiplication by an element (1θ) of A. Hence, A is exactly the endomorphism ring of M, as required.

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