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ON REES LOCALITIES AND H_i-LOCAL RINGS

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The main theorem gives a necessary and sufficient condition for each Rees locality $\mathscr{L} = R[tb, u]_{(M,tb,u)}$ of a local ring (R, M) with respect to a principal ideal bR in R to be either an H_i-ring (that is, for all prime ideals p in $\mathscr{L}$ such that height $p = i$, depth p = altitude $\mathscr{L} - i$) or a homogeneously H_i-ring (same condition holds for homogeneous p). Numerous corollaries follow concerning the cases: R is complete; R is Henselian; and, $\mathscr{L}$ is H_i, for all $i \geqq 0$. A generalization to ideals generated by more than one element is given, and we relate the results to two of the chain conjectures on prime ideals.

1. Introduction. All rings in this paper are assumed to be commutative rings with identity, and the undefined terminology is, in general, the same as that in [5].

The results in this paper are related to problems concerning saturated chains of prime ideals in a Noetherian ring (for example, the Catenary Chain Conjecture and the H-Conjecture (see (3.22)-(3.23))). These and other chain conjectures on prime ideals have remained unsettled for quite some time. In the hope of shedding new light on these conjectures, the concept of H_i-local rings was introduced in [11], and studied in [12], [6], [7], and [15], where a number of characterizations of H_i-local rings were given. These results are important, since the condition of being an H_i-local ring is more general than, for example, to satisfy the first chain condition for prime ideals (f.c.c.), so results on H_i-local rings imply results on local rings which satisfy the f.c.c.

In the present paper, we use some of the results and characterizations of H_i-local rings given in the above mentioned papers to determine necessary and sufficient conditions on a local ring R for certain Rees localities of R to be H_i-rings (or, homogeneously H_i). (See (2.1) and (2.3) for the definitions, and see (2.10) for the theorem.) In studying other properties of a local ring, Rees rings have, in the past, provided either valuable auxiliary rings, or rings of interest in their own right. (For example, we mention [16], [17], [18], [8], and [10, Section 3] among many possible.) This is again true in this paper, as will now be explained.

In Section 2 the main theorem is proved (2.10). The theorem is too technical to state here, but, as already noted, it gives a necessary and sufficient condition for certain Rees localities $\mathscr{L} = \mathscr{L}(R, bR) = R[tb, u]_{(M,tb,u)}$ of a local ring (R, M) to be H_i (resp., homogeneously

H_i). The proof of (2.10) is quite long and deep, and it requires considerable preliminary information (given in (2.1)–(2.9)). But, once the theorem is proved, many corollaries (and closely related results) follow, and these show some interesting things. For example, if R is Henselian, then the rings $\mathscr{L}$ are H_i if and only if they are homogeneously H_i (2.12); and, if R is complete, then the rings $\mathscr{L}$ are H_i if and only if they are $H_i, \cdots, H_{a+1}$ (a = altitude R) (2.13). Also, a number of other rings related to the rings $\mathscr{L}$ are easily shown to be H_i (2.14).

In Section 3, a variation of (2.10) is first considered in (3.1)–(3.3). Then, in (3.5)–(3.13), it is shown that from just knowing that at least one of the rings $\mathscr{L}$ is H_i, considerable information about all the other such rings can be proved. For example, if R is Henselian and one $\mathscr{L}(R, bR)$ is H_i, then every $\mathscr{L}(R, cR)$ is H_i (3.8). Next we consider the case that R or one of the rings $\mathscr{L}$ satisfies the f.c.c. in (3.17)–(3.21). Finally, Section 3 is closed by asking two questions, showing that an affirmative answer to either is equivalent to the fact that one of the chain conjectures (previously studied in the literature) holds, and then showing that the results in this paper lend a good deal of support for affirmative answers.

In Section 4, a generalization of (2.10.2) to ideals generated by more than one element is given in (4.2), and then some corollaries of (4.2) are proved. However, since the condition needed to generalize (2.10.2) is quite strong, and since the main interest is in the principal ideal case (as is partly indicated by (3.22)–(3.23)), Section 4 is kept fairly short.

Throughout the paper, a number of examples and/or remarks are given to indicate that certain hypotheses are necessary, and a number of open problems are mentioned.

Professor M. E. Pettit, Jr. has communicated to me that he has also done some work on the subject of this paper.

2. Main theorem.

In this section we prove the main theorem concerning H_i-rings and Rees rings of principal ideals. The proof is quite lengthy and deep, and requires a number of preliminary definitions and lemmas. We begin with the following definition.

DEFINITION 2.1. Let $B = (b_1, \cdots, b_k)R$ be an ideal in a local ring (R, M), let t be an indeterminate, and let $u = 1/t$. The *Rees ring* $\mathscr{R} = \mathscr{R}(R, B)$ *of* R *with respect to* B is defined to be the subring $\mathscr{R} = R[tb_1, \cdots, tb_k, u]$ of $R[t, u]$. (In particular, $\mathscr{R}(R, (0)) = R[u]$.) The *Rees locality* $\mathscr{L} = \mathscr{L}(R, B)$ *of* R *with respect to* B is defined to be the ring $\mathscr{L} = \mathscr{R}_{\mathscr{M}}$, where $\mathscr{M} = (M, tb_1, \cdots, tb_k, u)\mathscr{R}$. (In particular, $\mathscr{L}(R, (0)) = R[u]_{(M,u)}$.)

The known properties of $\mathscr{R}$ and of $\mathscr{L}$ which are needed in this paper are summarized in the following remark.

REMARK 2.2. Let (R, M), B, $\mathcal{R}$, $\mathcal{M}$, and $\mathcal{L}$ be as in (2.1).

(2.2.1) The elements in $\mathcal{R}$ are finite sums $\Sigma_{-m}^{n} c_i t^i$, where $c_i \in B^i$ (with the convention that $B^i = R$, if $i \leqq 0$). Therefore $\mathcal{R}$ is a graded Noetherian ring. Also, u isn't a zero-divisor in $\mathcal{R}$ and $u^i\mathcal{R} \cap R = B^i$, for all $i \geqq 0$ [**16**, p. 229].

(2.2.2) $\mathcal{M}$ is the (unique) maximal homogeneous (irrelevant) ideal in $\mathcal{R}$, so every homogeneous ideal in $\mathcal{R}$ is contained in $\mathcal{M}$ [**18**, Theorem 3.1 (step (ii))]. Also, altitude $\mathcal{R}$ = altitude $R + 1$ = height $\mathcal{M}$ = altitude $\mathcal{L}$ [**10**, Remark 3.7].

(2.2.3) For an ideal I in R let $I^* = IR[t, u] \cap \mathcal{R}$. Then I^* is a homogeneous ideal in $\mathcal{R}$ and $\mathcal{R}/I^* \cong \mathcal{R}(R/I, (B + I)/I)$ [**17**, Lemma 1.1], hence $\mathcal{L}/I^*\mathcal{L} \cong \mathcal{L}(R/I, (B + I)/I)$. Moreover, height I^* = height I and depth $I + 1$ = depth I^* = (by (2.2.2) and the isomorphism) height $\mathcal{M}/I^*$ [**10**, Remark 3.7]; and I^* is prime (primary) if I is prime (primary) [**17**, Theorem 1.5].

(2.2.4) $\mathcal{R}/u\mathcal{R} \cong \mathcal{F}(R, B)$, where $\mathcal{F}(R, B)$ is the *form ring of R with respect to B* [**17**, Theorem 2.1].

(2.2.5) Let P be a prime ideal in $\mathcal{R}$.

(i) Assume $u \notin P$. Then $P = (P \cap R[u])R[t, u] \cap \mathcal{R}$, height P = height $P \cap R[u]$, and $(P \cap R)^* \subseteq P$ (see (2.2.3)). If P is homogeneous, then $P = (P \cap R)^*$, so height P = height $P \cap R$. If P isn't homogeneous, then $(P \cap R)^* \subset P$ and height P = height $P \cap R + 1$.

(ii) Assume $u \in P$. Then $B \subseteq P \cap R$ and $P \cap R[u] = (P \cap R, u)R[u]$. If P is homogeneous, then $P \subseteq ((P \cap R)^*, u)\mathcal{R}$.

(2.2.6) Let p be a prime ideal in R.

(i) Depth $(p^*, u)\mathcal{R}$ = height $\mathcal{M}/(p^*, u)\mathcal{R}$ = depth p.

(ii) If $B \subseteq p$, then $\mathcal{R}_{(R-p)} \cong \mathcal{R}(R_p, BR_p)$, $(p^*, u)\mathcal{R}$ is prime, and height $(p^*, u)\mathcal{R}$ = height $p + 1$.

Proof. (2.2.1)–(2.2.4) are proved in the cited references.

(2.2.5) (i) Since $R[t, u]$ is a quotient ring of $R[u]$ and of $\mathcal{R}$ and $u \notin P$, then $P = (P \cap R[u])R[t, u] \cap \mathcal{R}$ and height P = height $P \cap R[u]$. Also, $(P \cap R)R[u] \subseteq P \cap R[u]$, so $(P \cap R)^* \subseteq (P \cap R[u])R(t, u] \cap \mathcal{R} = P$. Now if P is homogeneous, then $P \subseteq ((P \cap R)^*, u)\mathcal{R}$, since if $ct^i \in P$ and $i \geqq 0$, then $c = u^i(ct^i) \in P \cap R$, so $ct^i \in (P \cap R)^*$; and if $i < 0$, then $ct^i \in u\mathcal{R}$. Therefore $(P \cap R)^* \subseteq P \subseteq ((P \cap R)^*, u)\mathcal{R}$ implies that $P = (P \cap R)^*$ (since $u \notin P$), hence height P = height $P \cap R$ (2.2.3). If P isn't homogeneous, then $(P \cap R)^* \subset P$ (since $(P \cap R)^* \subseteq P$ and $(P \cap R)^*$ is homogeneous (2.2.3)), and height P = height $P \cap R[u]$ = height $P \cap R + 1$ (since $(P \cap R)R[u] \subset P \cap R[u]$ and both prime ideals lie over $P \cap R$).

(2.2.5) (ii) $B = u\mathcal{R} \cap R \subseteq P \cap R$ (since $u \in P$), so

$$(P \cap R)R[u] \subset ((P \cap R), u)R[u] \subseteq P \cap R[u],$$

hence, since all three of these prime ideals lie over $P \cap R$, $P \cap R[u] = (P \cap R, u)R[u]$ [**2**, Theorem 37]. If P is homogeneous, then $P \subseteq ((P \cap R)^*, u)\mathcal{R}$ as in the proof of (i).

(2.2.6) (i) By (2.2.3), height $\mathcal{M}/p^* = \text{depth } p^* = \text{depth } p + 1$, and since $\mathcal{R}/p^* \cong \mathcal{R}(R/p, (B+p)/p)$ (2.2.3), height $\mathcal{M}/p^* = \text{altitude } \mathcal{R}/p^*$ (2.2.2). Therefore depth $(p^*, u)\mathcal{R} = \text{altitude } \mathcal{R}/(p^*, u)\mathcal{R} = \text{altitude } (\mathcal{R}/p^*)/((p^*, u)\mathcal{R}/p^*) = \text{altitude } \mathcal{R}/p^* - 1 = \text{height } \mathcal{M}/p^* - 1 = \text{depth } p$; and, likewise, height $\mathcal{M}/(p^*, u)\mathcal{R} = \text{height } \mathcal{M}/p^* - 1 = \text{depth } p$.

(2.2.6) (ii) The map sending $(\Sigma c_i t^i)/s$ into $\Sigma (c_i/s)t^i$ ($c_i \in B^i$ and $s \in R$, $\notin p$) is readily seen to be an isomorphism from $\mathcal{R}_{(R-p)}$ onto $\mathcal{R}(R_p, BR_p)$. Therefore altitude $\mathcal{R}(R_p, BR_p) = \text{height } p + 1 = \text{height } (p^*, u)\mathcal{R}$, by (2.2.2), since $(p^*, u)\mathcal{R}_{(R-p)}$ corresponds to the maximal (irrelevant) homogeneous ideal in $\mathcal{R}(R_p, BR_p)$. Finally, $\mathcal{R}/(p^*, u)\mathcal{R} = R/p$, hence $(p^*, u)\mathcal{R}$ is prime, q.e.d.

We next define H_i-rings and C_i-rings and list some of their basic properties.

DEFINITION 2.3. Let i be a non-negative integer. A ring R is said to be an *H_i-ring* (or, R is said to be H_i) in case, for every height i prime ideal p in R, depth $p = \text{altitude } R - i$ (that is, height $p + \text{depth } p = \text{altitude } R$). If R is a graded ring and P is a homogeneous prime ideal in R, then it will be said that R_P is *homogeneously* H_i in case, for every height i homogeneous prime ideal p in R such that $p \subseteq P$, height $P/p = \text{height } P - i$ (equivalently, depth $pR_P = \text{altitude } R_P - i$).

A number of properties of H_1-local domains are given in [**11**] and [**12**]. These have been generalized to H_i-local domains and further properties of H_i-local domains are given in [**6**] and [**7**]. Most of these latter results have, in turn, been generalized to local rings in [**15**]. The reason these rings are of interest was mentioned in the introduction.

The properties of H_i-local rings which are most frequently used in the remainder of this paper are summarized in the following remark.

REMARK 2.4. Let (R, M) be a local ring, and let $a = \text{altitude } R$.

(2.4.1) Clearly, R is H_{a-1} and H_i, for all $i \geqq a$; and R is H_0, if R is an integral domain.

(2.4.2) Fix j $(0 \leqq j \leqq i)$. Then R is H_i if and only if, for all height j prime ideals p in R, R/p is H_{i-j} and either depth $p = a - j$ or depth $p \leqq i - j$ [**15**, (2.4)].

(2.4.3) R is H_i if and only if $R(X) = R[X]_{MR[X]}$ is H_i [**15**, (2.7)].

(2.4.4) Let S be a local ring which contains and is integral over R such that minimal prime ideals in S lie over minimal prime ideals in R. Then R is H_i if and only if S is H_i [**15**, (2.17)].

In regard to (2.4.2), an example is given in [**15**, (2.5.1)] with R H_i, height $p = j < i$, and altitude $R/p \leqq i - j$.

DEFINITION 2.5. Let i be a non-negative integer. A ring R is said to be a *C_i-ring* (or, R is said to be C_i) in case, R is H_i, H_{i+1}, and, for all height i prime ideals p in R and for all maximal ideals N in the integral closure of R/p, height $N =$ altitude R/p ($=$ altitude $R - i$).

Properties of C_i-local domains were first given in **[6]** and in **[7]**. These results were generalized to C_i-local rings in **[15]**, and some additional properties of such rings are given there.

The properties of C_i-local rings which will be most frequently used in this paper are summarized in the following remark.

REMARK 2.6. Let (R, M) be a local ring, and let $a =$ altitude R.

(2.6.1) Clearly, R is C_{a-1} and C_i, for all $i \geqq a$.

(2.6.2) Fix j $(0 \leqq j \leqq i)$. Then R is C_i if and only if, for each height j prime ideal p in R, R/p is C_{i-j} and either depth $p = a - j$ or depth $p \leqq i - j$ **[15**, (3.3)].

(2.6.3) R is C_i if and only if $R[X]_{(M,X)}$ is H_{i+1} **[15**, (3.7)].

(2.6.4) R is C_i if and only if, for each height i prime ideal p in R, $D = (R/p)[X]_{(M/p,X)}$ is H_1 and altitude $D =$ altitude $R - i + 1$ ($=$ depth $p + 1$) (by (2.6.2) and (2.6.3)).

In regard to (2.6.2), an example is given in **[15**, (2.5.1)] with R C_i, height $p = j < i$, and depth $p \leqq i - j$.

We now give two lemmas needed for the proof of (2.10). The lemmas are of some interest in themselves, and should be useful in other investigations. The first of these is similar to **[10**, Lemma 4.3], but that result doesn't give the information that we need.

LEMMA 2.7. (cf. **[10**, Lemma 4.3].) *Let p be a prime ideal in a Noetherian ring R, and let $b_0, \cdots, b_k$ be elements in p such that* (0): $b_0R =$ (0) *and such that the b_j are a subset of a system of parameters in R_p. Then, for each prime ideal P in R such that $p \subseteq P$, and for each $i = 1, \cdots, k$, (and with $A_i = R[b_1/b_0, \cdots, b_i/b_0]$), the residue classes modulo PA_i of the b_j/b_0 are algebraically independent over R/P and PA_i is a prime ideal such that depth $PA_i =$ depth $P + i$ and height $PA_i \leqq$ height $P - i$. Moreover, if the b_i are a subset of a system of parameters in R_P, then height $PA_i =$ height $P - i$.*

Proof. Let c_j be the image of b_j in R_p $(j = 0, \cdots, k)$. Then, for each $i = 1, \cdots, k$ (and with $C_i = R_p[c_1/c_0, \cdots, c_i/c_0]$), pC_i is a prime ideal such that depth $pC_i = i$, height $pC_i =$ height $p - i$, and the residue classes modulo pC_i of the c_j/c_0 are algebraically independent over R_p/pR_p **[10**, Lemma 4.3]. Let $U(p) = R - (p \cup z_1 \cup \cdots \cup z_d)$, where the z_h are the maximal prime divisors of zero in R. Then $pR_{U(p)}$ is a maximal ideal (since p contains the non-zero-divisor b_0), and C_i is a quotient ring of $B_i = R_{U(p)}[b_1/b_0, \cdots, b_i/b_0]$. Therefore $p^* = pC_i \cap B_i$ is a prime ideal such that height $p^* = \text{height } p - i$ and depth $p^* = i$ (since, with K denoting the quotient field of R/p, $C_i/pC_i = K[X_1, \cdots, X_i] = B_i/p^*$ (since $p^* \cap R_{U(p)} = pR_{U(p)}$ is maximal)). Now $pB_i \subseteq p^*$ and $B_i/pB_i = K[y_1, \cdots, y_i]$, where y_j is the residue class modulo pB_i of b_j/b_0. But since $pB_i \subseteq p^*$, B_i/p^* is a homomorphic image of B_i/pB_i, so $i \geqq \text{altitude } B_i/pB_i \geqq \text{altitude } B_i/p^* = i$, hence altitude $B_i/pB_i = i$, so the y_i are algebraically independent over K, hence $B_i/pB_i = B_i/p^*$, and so $pB_i = p^*$ is prime. Therefore, since $R \subseteq B_i$ and the residue classes modulo pB_i of the b_j/b_0 are algebraically independent over R/p, [**10**, Lemma 4.2] says that, with $A_i = R[b_1/b_0, \cdots, b_i/b_0]$, $pA_i = pB_i \cap A_i$ is a prime ideal such that depth $pA_i = \text{depth } p + i$; and height $pA_i = \text{height } p - i$, since B_i is a quotient ring of A_i. Therefore, if P is a prime ideal in R such that $p \subseteq P$, then, since $A_i/pA_i = (R/p)[X_1, \cdots, X_i]$ and $PA_i/pA_i = (P/p)(A_i/pA_i)$, the residue classes modulo PA_i of the b_j/b_0 are algebraically independent over R/P and PA_i is a prime ideal such that depth $PA_i = \text{depth } P + i$. To see that height $PA_i \leqq \text{height } P - i$, let z be a minimal prime ideal in A_i such that $z \subseteq PA_i$ and height $PA_i = \text{height } PA_i/z$. Let $w = z \cap R$. Then, by the altitude inequality for PA_i/z over P/w [**19**, (5), p. 326], height $PA_i/z + \operatorname{trd}(A_i/PA_i)/(R/P) \leqq \text{height } P/w + \operatorname{trd}(A_i/z)/(R/w)$, so height $PA_i + i \leqq \text{height } P/w \leqq \text{height } P$. The last statement follows as in the proof that height $pA_i = \text{height } p - i$, q.e.d.

The following corollary to (2.7) gives somewhat more general information than we need for (2.10). We give it in this form, since its proof is essentially the same as the proof of the more specific result we need.

COROLLARY 2.8. *Let (R, M) be a local ring, and let $b_1, \cdots, b_k$ be elements in M such that, with $B = (b_1, \cdots, b_k)R$, height $B = k$. Let $\mathcal{R} = \mathcal{R}(R, B)$. Then, for each prime ideal P in R such that $B \subseteq P$, $P' = (P, u)\mathcal{R}$ is a prime ideal such that height $P' = \text{height } P + 1 - k$, depth $P' = \text{depth } P + k$, and the residue classes modulo P' of the tb_i are algebraically independent over R/P. In particular, the minimal prime divisors of $u\mathcal{R}$ are the ideals $(p, u)\mathcal{R}$ with p a minimal prime divisor of B.*

Proof. Let P be a prime ideal in R such that $B \subseteq P$. Then $u, b_1, \cdots, b_k$ are a subset of a system of parameters in $R[u]_{(P,u)}$. Therefore, by (2.7) and since $tb_i = b_i/u$, $P' = (P, u)\mathcal{R}$ is a prime ideal such that height $P' = \text{height } P + 1 - k$ and depth $P' = \text{depth } P + k$ (since height $(P, u)R[u] = \text{height } P + 1$ and depth $(P, u)R[u] = \text{depth } P$), and the residue classes modulo P' of the tb_i are algebraically independent over R/P. In particular, for each minimal prime divisor P of B, $(P, u)\mathcal{R}$ is a minimal prime divisor of $u\mathcal{R}$; and if Q is a minimal prime divisor of $u\mathcal{R}$, then $B \subseteq Q \cap R$, so there exists a minimal prime divisor q of B such that

$q \subseteq Q \cap R$, and then $(q, u)\mathcal{R} \subseteq Q$, so by what has already been shown, $Q = (q, u)\mathcal{R}$, q.e.d.

In the proof of the following lemma we need to use a result of E. G. Evans, Jr. concerning Zariski's Main Theorem **[1]**.

LEMMA 2.9. *Let $A = R[c_1, \cdots, c_n]$ be a finitely generated ring over a local ring (R, M) such that $\mathcal{M} = (M, c_1, \cdots, c_n)A$ is a proper (hence maximal) ideal. Let P be a prime ideal in A such that $P \subseteq \mathcal{M}$, and assume that $P \cap R[c_i] \not\subseteq MR[c_i]$ $(i = 1, \cdots, n)$. Then $A_{\mathcal{M}}/PA_{\mathcal{M}} = S_N$, where S is the integral closure of $R/(P \cap R)$ in A/P and N is a maximal ideal in S.*

Proof. Since $p_i = P \cap R[c_i] \not\subseteq \mathrm{MR}[c_i]$ $(i = 1, \cdots, n)$, there exists a polynomial $f_i(X) \in R[X]$ such that $f_i(c_i) \in p_i$ and such that some coefficient r_{ij} of $f_i(X)$ is a unit in R $(j > 0$, since $p_i \subseteq \mathcal{M} \cap R[c_i] = (M, c_i)R[c_i])$. Let $a_i = c_i + P \in A/P$. Then the a_i are algebraic over $R/(P \cap R) =$ (say) D, so there exists a non-zero $s_i \in D$ such that $s_i a_i$ is integral over D, so if $0 \neq m \in M/(P \cap R)$, then $U = D[ms_1a_1, \cdots, ms_ka_k]$ is a local domain which is integral over D, $U \subseteq A/P$, and U and A/P have the same quotient field. Therefore, since the a_i are roots of polynomials with coefficients in U such that some coefficient is a unit in U, **[19**, Lemma, p. 19] says that, for $i = 1, \cdots, k$ and for each maximal ideal Q in the integral closure U' of U, a_i or $1/a_i \in U'_Q$. Now $A/P = D[a_1, \cdots, a_k]$, so $A/P = U[a_1, \cdots, a_k] \subseteq U'[a_1, \cdots, a_k] =$ (say) B. Hence, since B is integral over A/P, there exists a maximal ideal Q' in B such that $Q' \cap (A/P) = \mathcal{M}/P$. Now $N' = Q' \cap U'$ is maximal, since $(\mathcal{M}/P) \cap D$ is maximal and U' is integral over D. Thus, since, for $i = 1, \cdots, k$, a_i or $1/a_i \in U'_{N'}$, it follows that each $a_i \in N'U'_{N'}$, so $U'_{N'} = B_{Q'} \supseteq A_{\mathcal{M}}/PA_{\mathcal{M}} =$ (say) L. Therefore Q' is isolated over the maximal ideal M' in U (that is, Q' is maximal and minimal in the set of prime ideals in B which lie over M'), hence, since B is integral over A/P and Q' was an arbitrary maximal ideal in B lying over $\mathcal{M}/P$, $\mathcal{M}/P$ is isolated over M' (by the Going-Up Theorem). Therefore, by **[1]** $L = S_N$, where S is the integral closure of $D = R/(P \cap R)$ in A/P and N is a maximal ideal in S, q.e.d.

We are now able to prove the main theorem in this paper. Even with the information we now have, its proof is quite lengthy.

It will be shown in (2.11.2) below that R is H_0 if and only if $\mathcal{L}(R, bR)$ is H_0. For this reason, we restrict attention to the case $i > 0$ in the theorem.

THEOREM 2.10. *Let (R, M) be a local ring, let altitude $R = a$, and let $E = \{b \in M;$ height $bR = 1\} \cup \{0\}$. Then the following statements hold for $i > 0$:*

(2.10.1) *R is H_{i-1} and H_i if and only if, for all $b \in E$, $\mathscr{L}(R, bR)$ is homogeneously H_i.*

(2.10.2) *R is C_{i-1} if and only if, for all $b \in E$, $\mathscr{L}(R, bR)$ is H_i.*

Proof. (2.10.1) Assume that R is H_{i-1} and H_i, let $b \in E$, let $\mathscr{R} = \mathscr{R}(R, bR)$, and let $\mathscr{L} = \mathscr{L}(R, bR)$. Let P be a height i homogeneous prime ideal in $\mathscr{R}$, so $P \subseteq \mathscr{M}$. Then to show that $\mathscr{L}$ is homogeneously H_i, it suffices to show that height $\mathscr{M}/P =$ height $\mathscr{M} - i$ $(=(2.2.2)$ $a+1-i)$. For this, let $p = P \cap R$. We now consider the two cases: $u \notin P$; and, $u \in P$.

If $u \notin P$, then $P = p^*$ (2.2.5)(i), hence $i =$ height $P =$ height p (2.2.3), so $a - i =$ depth p (since R is H_i), and so, by (2.2.3), height $\mathscr{M}/P =$ depth $P =$ depth $p + 1 = a - i + 1$.

If $u \in P$, then $b \in p$. If $b = 0$, then $P = (p, u)R[u]$, so $i =$ height $P =$ height $p + 1$ and $\mathscr{M}/P = (M, u)R[u]/(p, u)R[u] = M/p$, hence height $\mathscr{M}/P =$ height $M/p = a - i + 1$ (since R is H_{i-1}). Therefore assume $b \neq 0$. Then since $0 \neq b \in p \cap E$, (2.8) says that $p' = (p, u)\mathscr{R}$ is prime, depth $p' =$ depth $p + 1$, and height $p' =$ height p. Also, $p' \subseteq P \subseteq$ (2.2.5)(ii) $(p^*, u)\mathscr{R}$, $(p^*, u)\mathscr{R}$ is prime (2.2.6)(ii), and all three prime ideals lie over $(p, u)R[u]$, so either $P = p'$ or $P = (p^*, u)\mathscr{R}$ [**2**, Theorem 37]. If $P = p'$, then $i =$ height $P =$ height $p' =$ height p, so depth $p = a - i$ (since R is H_i), hence depth $P =$ depth $p' =$ depth $p + 1 = a - i + 1$, and depth $p' =$ height $\mathscr{M}/p'$ (since $\mathscr{R}/p' \cong (R/p)[X]$ (2.8)). If $P = (p^*, u)\mathscr{R}$, then $i =$ height $P =$ (2.2.6)(ii) height $p + 1$, so height $\mathscr{M}/P =$ (2.2.6)(i) depth $p = a - i + 1$, since R is H_{i-1}.

Thus, in both cases, height $\mathscr{M}/P = a - i + 1$, so $\mathscr{L}$ is homogeneously H_i.

For the converse, since $0 \in E$, $D = R[u]_{(M,u)}$ is homogeneously H_i. Therefore, if p is a height i prime ideal in R, then $p' = pD$ is height i, hence depth $p =$ depth $p' - 1 = a - i$, so R is H_i. And, if q is a height $i - 1$ prime ideal in R, then $q' = (q, u)D$ is height i, so depth $q =$ depth $q' = a + 1 - i$, and so R is H_{i-1}.

(2.10.2) Assume that R is C_{i-1}, let $b \in E$, let $\mathscr{R} = \mathscr{R}(R, bR)$, and let $\mathscr{L} = \mathscr{L}(R, bR)$. Let P be a height i prime ideal in $\mathscr{L}$. Then, to prove that $\mathscr{L}$ is H_i, it must be shown that depth $P = a + 1 - i$, and for this it may be assumed, by (2.10.1), that $P' = P \cap \mathscr{R}$ isn't homogeneous. Also, it may be assumed that $b \neq 0$, since if $b = 0$, then $\mathscr{L} = R[u]_{(M,u)}$ is H_i (2.6.3). We now consider the two cases: $u \in P$; and, $u \notin P$.

If $u \in P$, then $p' = P \cap \mathscr{R}$ contains a minimal prime divisor q of $u\mathscr{R}$ such that height $P'/q = i - 1$ (since height $P'/u\mathscr{R} = i - 1$), and $q = (p, u)\mathscr{R}$, for some minimal prime divisor p of bR (2.8). Then $\mathscr{R}/q \cong (R/p)[X]$ (2.8), and, by (2.6.2), R/p is C_{i-2} and either altitude $R/p = a - 1$ or $\leqq i - 2$. Therefore $\mathscr{L}/q\mathscr{L} \cong (R/p)[X]_{(M/p,X)}$ is H_{i-1} (2.6.3), and either altitude $\mathscr{L}/q\mathscr{L} = a$ or $\leqq i-1$. Now it may clearly be assumed that $i < a$ (2.4.1), so $P' \neq \mathscr{M}$, and so altitude $\mathscr{L}/q\mathscr{L} = a$ (since height $P/q\mathscr{L} =$ height $P'/q = i-1$). Therefore, since $\mathscr{L}/q\mathscr{L}$ is H_{i-1}, depth $P =$ depth $P/q\mathscr{L} = a - i + 1$, as desired.

Therefore, assume $u \notin P$. If $p = P \cap R[u] \subseteq MR[u]$, then since $R[u]_{MR[u]}$ is H_i (2.4.3), and since height $p = i$ (2.2.5)(i), height $MR[u]/p = a - i$. Therefore there exists a chain of prime ideals $p = p_0 \subset \cdots \subset p_{a-i} = MR[u]$ in $R[u]$ of length $a-i$, so $P' =$ (2.2.5)(i) $pR[t,u] \cap \mathscr{R} \subset \cdots \subset MR[t,u] \cap \mathscr{R} \subset \mathscr{M}$, and so height $\mathscr{M}/P' \geqq a - i + 1$, hence depth $P =$ height $\mathscr{M}/P' = a - i + 1$. Likewise, if $P \cap R[tb] \subseteq MR[tb]$, then height $\mathscr{M}/P' = a - i + 1$. Therefore, it may be assumed that $P \cap R[tb] \not\subseteq MR[tb]$ and $P \cap R[u] \not\subseteq MR[u]$. Then, by (2.9), $\mathscr{L}/P = S_N$, where S is the integral closure of $D = R/(P \cap R)$ in $\mathscr{R}/P'$ and N is a maximal ideal in S. Now height $P \cap R = i - 1$ (2.2.5)(i), so every maximal ideal in the integral closure D' of D has height $a - i + 1$ (by hypothesis), hence, since S is integral over D, height $N = a - i + 1$. (For, let N' be a maximal ideal in the integral closure of S such that $N' \cap S = N$ and height $N' =$ height N. Then, by [**5**, (10.14)] height $N' =$ height $N' \cap D' = a - i + 1$.) Therefore depth $P =$ altitude $\mathscr{L}/P =$ height $N = a - i + 1$.

Hence in both cases, depth $P = a - i + 1$, so $\mathscr{L}$ is H_i.

For the converse, since $0 \in E$, $R[u]_{(M,u)}$ is H_i (by hypothesis), hence R is C_{i-1} (2.6.3), q.e.d.

Before giving some corollaries to (2.10), we note that it will be shown in (3.1) below that a strengthened form of the converses of (2.10.1) and (2.10.2) holds (omitting the case $b = 0$).

Also, in (3.5) and its corollaries, it will be seen that if at least one $\mathscr{L}(R, bR)$ is known to be H_i, then quite a lot can be said about the other $\mathscr{L}(R, cR)$.

We now make two brief remarks about (2.10) before giving some of its corollaries.

REMARK 2.11. Let the notation be as in (2.10). Then the following statements hold:

(2.11.1) (2.10) holds for $i \in \{a, a+1\}$.

(2.11.2) The following statements are equivalent: R is H_0; there exists $b \in E$ such that $\mathscr{L}(R, bR)$ is H_0; for all $b \in E$, $\mathscr{L}(R, bR)$ is H_0.

Proof. (2.11.1) follows from (2.4.1) and (2.6.1).

(2.11.2) Let $b \in E$. Then the minimal prime ideals in $\mathscr{L} = \mathscr{L}(R, bR)$ are the ideals $z^*\mathscr{L}$, where z is a minimal prime ideal in R, by (2.2.5)(i) and (2.2.3), and depth $z^*\mathscr{L} =$ depth $z + 1$ (2.2.3). (2.11.2) follows from this, q.e.d.

The first corollary to (2.10) shows that (2.10.1) and (2.10.2) are equivalent for Henselian local rings.

COROLLARY 2.12. *With the notation of* (2.10), *assume that* R *is Henselian. Then the following statements are equivalent*:

(2.12.1) R *is* H_{i-1} *and* H_i.

(2.12.2) R *is* C_{i-1}.

(2.12.3) *For all* $b \in E$, $\mathscr{L}(R, bR)$ *is homogeneously* H_i.

(2.12.4) *For all* $b \in E$, $\mathscr{L}(R, bR)$ *is* H_i.

Proof. If p is a prime ideal in R, then the integral closure of R/p is quasi-local (since R/p is Henselian), hence (2.12.1) implies (2.12.2). Therefore, since clearly (2.12.4) implies (2.12.3), all four statements are equivalent by (2.10), q.e.d.

One reason (2.12) is of interest is that, to prove the Chain Conjecture (that is, a Henselian local domain satisfies the f.c.c. (see (3.14) for the definition)), it suffices to prove that every Henselian local domain is H_1 [**12**, (2.4)].

Even more than (2.12) can be said when R is complete, as will now be shown.

COROLLARY 2.13. *With the notation of* (2.10), *assume that* R *is complete. Then* R *is* H_i *if and only if, for all* $b \in E$, $\mathscr{L}(R, bR)$ *is* $H_{i+1}, H_{i+2}, \cdots, H_{a+1}$.

Proof. Assume that R is H_i and let p be a height j prime ideal in R with $i < j \leqq a$. Then there exists a height i prime ideal q in R such that $q \subset p$ and height $p/q = j - i$. Therefore, since R/q is a complete local domain, R/q satisfies the f.c.c. (3.14), so height p/q + depth p/q = altitude R/q; that is, depth p = depth $q - j + i = a - j$. Hence R is H_j. Also, R is Henselian, so R is $C_i, \cdots, C_a$ (2.12), hence every $\mathscr{L}(R, bR)$ with $b \in E$ is $H_{i+1}, H_{i+2}, \cdots, H_{a+1}$ (2.10.2).

The converse follows from (2.10.2), q.e.d.

The next corollary shows that from knowing that certain Rees localities are H_i, a number of other rings can be shown to be H_i (or, H_{i-1}).

It should be mentioned that (2.14.4) is known [**15**, (3.14)]. Also, in regard to (2.14.2), if $B = (b_1, \cdots, b_k)R$ is an ideal in R, then the ring $R[tb_1, \cdots, tb_k]$ is called the *restricted Rees ring of R with respect to B.*

COROLLARY 2.14. *With the notation of* (2.10), *assume that* R *is* C_{i-1}. *Then the following statements hold, for all* $b \in E$:

(2.14.1) *For all maximal ideals* N *in* $\mathscr{R} = \mathscr{R}(R, bR)$ *such that* $N \cap R = M$, $\mathscr{R}_N$ *is* H_i *and altitude* $\mathscr{R}_N = a + 1$.

(2.14.2) *For all maximal ideals N in $S = R[tb]$ such that $N \cap R = M$, S_N is H_i and altitude $S_N = a + 1$.*

(2.14.3) *For all maximal ideals N in $\mathcal{F} = \mathcal{F}(R, bR)$ such that $N \cap (R/bR) = M/bR$, $\mathcal{F}_N$ is H_{i-1} and either altitude $\mathcal{F}_N = a$ or altitude $\mathcal{F}_N \leqq i - 1$.*

(2.14.4) *For all non-zero-divisors $c \in R$ and for all maximal ideals N in $A = R[b/c]$ such that $N \cap R = M$, A_N is H_{i-1} and either altitude $A_N = a$ or altitude $A_N \leqq i - 1$.*

Proof. (2.14.1) By (2.10.2), it may be assumed that $N \neq \mathcal{M}$, so either $u \notin N$ or $tb \notin N$. If $u \notin N$, then $\mathcal{R}_N = R[u]_Q$, where $Q = N \cap R[u]$. Then $MR[u] \subset Q$ (since $N \neq M^*$), so $Q = (M, f)R[u]$, for some monic polynomial $f = f(u)$. Therefore $f(u)$ is transcendental over R, so $D = R[f]_{(M,f)} \cong R[X]_{(M,X)}$, hence, by hypothesis and the isomorphism (and (2.6.3)), D is H_i and altitude $D = a + 1$. Further, $R[u]_Q$ is integral over D (since $R[u]$ is integral over $R[f]$ and $PR[u] = Q$, where $P = (M, f)R[f] = Q \cap R[f]$). Therefore $\mathcal{R}_N = R[u]_Q$ is H_i (2.4.4) and altitude $\mathcal{R}_N = a + 1$. A similar proof holds if $tb \notin N$.

(2.14.2) $N = (M, f)S$, for some monic polynomial $f = f(tb)$, so, since $f(tb)$ is transcendental over R, the proof of (2.14.2) is similar to the proof of (2.14.1).

(2.14.3) By (2.2.4), $\mathcal{F} \cong \mathcal{R}/u\mathcal{R}$, where $\mathcal{R} = \mathcal{R}(R, bR)$, so $\mathcal{F}_N \cong \mathcal{R}_Q/u\mathcal{R}_Q$, where Q is the pre-image in $\mathcal{R}$ of N. Also, the minimal prime divisors of $u\mathcal{R}_Q$ have height one, so it follows from (2.4.2) and (2.14.1) that $\mathcal{F}_N$ is H_{i-1} and either altitude $\mathcal{F}_N = a$ or $\leqq i - 1$.

(2.14.4) If $1 \in MA$, then no such N exists, so the conclusion is vacuously true. Therefore assume that MA is proper. Then $A = \mathcal{R}/I$, where $\mathcal{R} = \mathcal{R}(R, bR)$ and $I = (u - c)R[t, u] \cap \mathcal{R}$, so $A_N = \mathcal{R}_Q/I\mathcal{R}_Q$, where Q is the pre-image of N in $\mathcal{R}$. Therefore, since $R[t, u]$ is a quotient ring of $\mathcal{R}$, the minimal prime divisors of I have height one. Hence the conclusion follows from (2.14.1) and (2.4.2), q.e.d.

By [15, (3.14)], the converse of (2.14.4) is true, if Rad $R = (0)$. And, of course, the converse of (2.14.2) is true (by (2.6.3)), and the converse of (2.14.1) is true (by (2.10.2)). It will be shown in (2.15) below that the converse of (2.14.3) is also true.

Using [5, Example 2, pp. 203–205], an example can be given to show that altitude $A_N \leqq i - 1$ is possible in (2.14.4), and that altitude $\mathcal{F}_N \leqq i - 1$ is possible in (2.14.3).

As a final comment on (2.14), it should be noted that the proof of (2.14.4) shows that if a given $\mathcal{L}(R, bR)$ is H_i, then, for all non-zero-divisors c in R, $R[b/c]_{(M,b/c)}$ is H_{i-1} (if $(M, b/c)$ is proper).

We next show that a strong converse of (2.14.3) is true. In proving (2.15), we will identify the form ring of R with respect to bR with $\mathcal{R}(R, bR)/u\mathcal{R}(R, bR)$ via the isomorphism given in (2.2.4).

PROPOSITION 2.15. *Let* (R, M), a, E, *and* $i > 0$ *be as in* (2.10). *Assume that, for each* $b \in E - \{0\}$, $\mathcal{F}_N$ *is* H_{i-1} *and either altitude* $\mathcal{F}_N = a$ *or* $\leqq i-1$, *where* $\mathcal{F} = \mathcal{F}(R, bR)$ *and* $N = \mathcal{M}/u\mathcal{R}(R, bR)$. *Then* R *is* C_{i-1}.

Proof. Let p be a height one prime ideal in R. Let $0 \neq b \in p \cap E$. Then $p' = (p, u)\mathcal{L}$ is a height one prime divisor of $u\mathcal{L}$ (2.8), where $\mathcal{L} = \mathcal{L}(R, bR)$. Also, $\mathcal{L}/p' \cong$ (2.8) $(R/p)[X]_{(M/p, X)} \cong$ (2.2.4) $\mathcal{F}_N/(p'/u\mathcal{L})$ is, by (2.4.2), H_{i-1} and either altitude $\mathcal{L}/p' =$ altitude $\mathcal{F}_N = a$ or $\leqq i-1$. Therefore, by (2.6.3), R/p is C_{i-2} and either depth $p = a-1$ or $\leqq i-2$. Hence, by (2.6.2), R is C_{i-1}, q.e.d.

In the proof of (2.15), it may happen that altitude $\mathcal{F}_N = a$ and altitude $\mathcal{F}_N/(p'/u\mathcal{L}) \leqq i-1$.

The next result gives some information related to (2.10.2). One of the problems on H_i-local rings is what can be said about R_p, if R is H_i. (2.16) shows that at least some information about this can be given for Rees rings. To prove (2.16), we need the following known result: If a local ring (R, M) is H_i and b, c are analytically independent elements in R such that b isn't a zero-divisor, then, with $B = R[c/b]$, MB is prime and B_{MB} is H_{i-1} [**15**, (2.11)].

PROPOSITION 2.16. *Let* (R, M), a, E, *and* $i > 0$ *be as in* (2.10), *and assume that* R *is* C_{i-1}. *Let* $b \in E$, *and let* $\mathcal{R} = \mathcal{R}(R, bR)$. *Then, for all non-maximal prime ideals* Q *in* $\mathcal{R}$ *such that* $Q \cap R = M$, $\mathcal{R}_Q$ *is* H_{i-1}.

Proof. Let Q be a non-maximal prime ideal in $\mathcal{R}$ such that $Q \cap R = M$. If $u \in Q$, then $Q \cap R[u] = (M, u)R[u]$, so $Q = (M, u)\mathcal{R}$ (since Q isn't maximal and $(M, u)\mathcal{R}$ is prime (2.8)). Therefore $\mathcal{R}_Q = A_{QA}$, where $A = R[u]_{(M,u)}[tb]$. Hence, since $R[u]_{(M,u)}$ is H_i (2.6.3), $\mathcal{R}_Q$ is H_{i-1}, by the comment preceding this proposition. If $u \notin Q$, then $Q \cap R[u] = MR[u]$ and $Q = M^*$ (since Q isn't maximal and by (2.2.5) (i)), so $\mathcal{R}_Q = R[u]_{MR[u]}$ is H_{i-1} (2.4.3), q.e.d.

It should be noted that both the prime ideals M^* and $(M, u)\mathcal{R}$ in (2.16) have height $= a$. For M^*, this follows from (2.2.3); and for $(M, u)\mathcal{R}$, it follows from (2.8).

We close this section with a result which shows that if there exists $0 \neq b \in E$ such that $\mathcal{R}(R, bR)_{(M,u)\mathcal{R}(R,bR)}$ is H_i, then R is H_i. A related (and more important) result will be considered in (3.5) below.

It should be noted, for (2.17), that height $(M, u)\mathcal{R} = a$, by (2.8).

PROPOSITION 2.17. *Let* (R, M), a, E, *and* $i > 0$ *be as in* (2.10). *Let* $0 \neq b \in E$, *let* $\mathcal{R} = \mathcal{R}(R, bR)$, *and let* $A = \mathcal{R}_{(M,u)\mathcal{R}}$. *If* A *is* H_i, *then* R *is* H_i.

Proof. Let p be a height i prime ideal in R. If $b \in p$, then $(p, u)\mathscr{R}$ is a prime ideal of height i (2.8), so depth $(p, u)A = a - i$. Also, $\mathscr{R}/(p, u)\mathscr{R} \cong (R/p)[X]$ (2.8), so depth $p =$ depth $(p, u)A = a - i$. If $b \notin p$, then $p\mathscr{R} = p^*$; for, clearly $p\mathscr{R} \subseteq p^*$, and if $t^k c \in p^*$, then $c \in p \cap b^k R = b^k(p : b^k R) = b^k p$, so there exists $d \in p$ such that $t^k c = (tb)^k d \in p\mathscr{R}$. Therefore $p^* = p\mathscr{R} \subseteq (M, u)\mathscr{R}$ and height $p^* = i$, so height $(M, u)\mathscr{R}/p^* = a - i$. Thus depth $p =$ depth $p^* - 1 \geqq a - i$, and so depth $p = a - i$. Therefore R is H_i, q.e.d.

3. Related results. In this section we do four things related to (2.10). First, in (3.1) we show that most of (2.10) holds using $E - \{0\}$. Then we consider what can be said when it is known that at least one $\mathscr{L}(R, bR)$ is H_i in (3.5)–(3.13). Next, some results on local rings which are H_i, for all $i > 0$, are given in (3.17)–(3.21). Then we end this section with two questions and some comments on them.

We begin with the following result. It will be shown in (3.3) below that $i > 1$ (instead of $i > 0$) in (3.1.1) is necessary.

PROPOSITION 3.1. *Let (R, M) be a local ring, let a = altitude R, and let $E' = \{b \in M;$ height $bR = 1\}$. Then the following statements hold:*

(3.1.1) *Let $i > 1$. Then R is H_{i-1} and H_i if and only if, for all $b \in E'$, $\mathscr{L}(R, bR)$ is homogeneously H_i.*

(3.1.2) *Let $i > 0$. Then R is C_{i-1} if and only if, for all $b \in E'$, $\mathscr{L}(R, bR)$ is H_i.*

Proof. (3.1.1) Assume that, for each $b \in E'$, $\mathscr{L} = \mathscr{L}(R, bR)$ is homogeneously H_i, and let p be a prime ideal in R. If height $p = i$, then p^* is a height i homogeneous prime ideal in $\mathscr{R} = \mathscr{R}(R, bR)$, so depth $p =$ (2.2.3) height $\mathscr{M}/p^* - 1 = a - i$; hence R is H_i. If height $p = i - 1$, then let $b \in p \cap E'$ (since $i > 1$), and let $\mathscr{L} = \mathscr{L}(R, bR)$. Then $\mathscr{L}$ is homogeneously H_i (by hypothesis) and height $(p^*, u)\mathscr{L} = i$ (2.2.6) (ii), so depth $p =$ (2.2.6) (i) depth $(p^*, u)\mathscr{L} = a + 1 - i$; hence R is H_{i-1}.

The converse was proved in (2.10.1).

(3.1.2) Assume that, for each $b \in E'$, $\mathscr{L} = \mathscr{L}(R, bR)$ is H_i. Assume temporarily that $i = 1$. Then R is H_1, as in the proof of (3.1.1). Also, R is H_0, since if z is a minimal prime ideal in R, then $z^* \subset M^* \subset \mathscr{M}$, so depth $z^*\mathscr{L} > 1$ and $\mathscr{L}$ is H_1, hence depth $z^*\mathscr{L} = a + 1$ (2.4.2). Therefore depth $z =$ (2.2.3) depth $z^*\mathscr{L} - 1 = a$.

Now let i be arbitrary ($i > 0$). Then to prove that R is C_{i-1}, it suffices, by (3.1.1) and the preceding paragraph, to prove that, for all height $i - 1$ prime ideals p in R, every maximal ideal in the integral closure of R/p has height equal to altitude R/p. For this, fix a height $i - 1$ prime ideal p in R, and let N be a maximal ideal in the integral closure S of R/p. Let $y = c'/b'$ $(c', b' \in M/p)$ be an element in N such that y isn't in any other maximal ideal in S, so the integral closure of $D = (R/p)[y]_{(M/p,y)}$ is S_N. Let b, c be pre-images in M of b', c' such that height cR = height $bR = 1$, and let $\mathscr{L} = \mathscr{L}(R, cR)$. Then $\mathscr{L}$ is H_i (by hypothesis), so $\mathscr{L}/p^*\mathscr{L} \cong (2.2.3)$ $\mathscr{L}(R/p, (cR+p)/p) =$ (say $\mathscr{L}'$ is H_1 (2.4.2), and altitude $\mathscr{L}' =$ depth $p + 1 = a - i + 2$ (since R is H_{i-1}). Also, $q = (u - b')(R/p)[t, u] \cap \mathscr{R}(R/p, (cR+p)/p)$ is a height one prime ideal such that $q \cap (R/p) = (0)$ and $\mathscr{L}'/q\mathscr{L}' \cong D$. Therefore height $N =$ altitude D = depth $q\mathscr{L}' = (\mathscr{L}'$ is $H_1)$ $a - i + 1$ = depth p = altitude R/p, as desired.

The converse was proved in (2.10.2), q.e.d.

The condition $i > 1$ (instead of $i > 0$) in (3.1.1) is necessary, as will be shown in (3.3) below. However, if R is a local domain, then the case $i = 1$ also holds (by the proof of (3.1.1) and since R is H_0).

REMARK 3.2. Let (R, M) and E' be as in (3.1), assume that R is Henselian, and let $i > 1$. Then, by the same proof as (2.12) (only using (3.1)), the following statements are equivalent: R is H_{i-1} and H_i; R is C_{i-1}; for all $b \in E'$, $\mathscr{L}(R, bR)$ is homogeneously H_i; for all $b \in E'$, $\mathscr{L}(R, bR)$ is H_i.

The following example shows that the condition $i > 1$ is necessary in (3.1) (that is, all $\mathscr{L} = \mathscr{L}(R, bR)$ (with $b \in E'$) homogeneously H_1 does not imply that R is H_0) and in (3.2) (that is, for R Henselian, all $\mathscr{L}$ (as above) homogeneously H_1 does not imply that all such $\mathscr{L}$ are H_1).

EXAMPLE 3.3. There exists a complete local ring (L, N) which is H_i if and only if $i > 0$ such that, for all $b \in E' = \{b \in N;$ height $bL = 1\}$, $\mathscr{L} = \mathscr{L}(L, bL)$ is homogeneously H_i if and only if $i > 0$ and such that $\mathscr{L}$ is H_i if and only if $i > 1$.

Proof. Let (R, I) be as in [**5**, Example 2, pp. 203–205] in the case $m = 0$, so the completion (L, N) of (R, I) has exactly two minimal prime ideals, say z and w, such that depth $z = 1 <$ depth $w = a$ = altitude L. (Since the integral closure R' of R is a finite R-algebra and is a regular domain with two maximal ideals MR' and NR' such that height $MR' = 1$ and height $NR' = r + 1 =$ (say) a, L is as described by [**9**, Proposition 3.5].) Then clearly L isn't H_0. Also, L is H_i $(0 < i < a)$, since if p is a height i prime ideal in L, then w is the only minimal prime ideal in L which is contained in p, so since altitude $L/w = a$ and L/w satisfies the f.c.c. (3.14), depth $p = a - i$. Further L is H_a. Now let $b \in E'$ and let $\mathscr{L} = \mathscr{L}(L, bL)$. Then $\mathscr{L}$ isn't H_0 (since L isn't). Further, $\mathscr{L}$ is $H_2, \cdots, H_{a+1}$ (2.13). Moreover, $\mathscr{L}$ isn't H_1, since depth $z^*\mathscr{L} = 2 <$ altitude $\mathscr{L}$, so, by [**3**, Theorem 1], there exists a height one prime ideal p in $\mathscr{L}$ such that depth $p = 1$. Thus it remains to show that $\mathscr{L}$ is homogeneously H_1.

For this, let p be a height one homogeneous prime ideal in $\mathscr{L}$, and suppose $z^*\mathscr{L} \subseteq p$. If $u \notin p$, then $p = (p \cap L)^*\mathscr{L}$ (2.2.5)(i), so height $p \cap L = 1$ and $z \subset p \cap L$. But this contradicts the fact that depth $z = 1 < a$. Therefore $u \in p$, so $(z, b)L \subseteq p \cap L$. Hence, since depth $z = 1$ and $b \in E'$, $p \cap L = N$. Therefore $(N, u)\mathscr{L} \subseteq p$, and $(N, u)\mathscr{L}$ is a prime ideal such that height $(N, u)\mathscr{L} = a$ (2.8). But this contradicts $a > 1$. Therefore no height one homogeneous prime ideal in $\mathscr{L}$ contains $z^*\mathscr{L}$, so each height one homogeneous prime ideal p in $\mathscr{L}$ contains $w^*\mathscr{L}$, hence $p/w^*\mathscr{L}$ is a height one prime ideal in $\mathscr{L}/w^*\mathscr{L}$. Therefore, since $\mathscr{L}/w^*\mathscr{L} \cong \mathscr{L}(L/w, (bL + w)/w)$ and L/w is a complete local domain of altitude $= a$, depth $p =$ depth $p/w^*\mathscr{L} =$ (2.13) a. Therefore $\mathscr{L}$ is homogeneously H_1, q.e.d.

We now begin to consider what can be said if it is known that some $\mathscr{L}(R, bR)$ is either H_i or homogeneously H_i.

By (2.10.1) together with the last paragraph of its proof, for each non-zero $b \in E$, $\mathscr{L}(R, bR)$ is homogeneously H_i, if $\mathscr{L}(R, (0))$ is. The following remark and (2.10.1) show that if some $\mathscr{L}(R, bR)$ is homogeneously H_i and $b \neq 0$ has a certain property, then all $\mathscr{L}(R, cR)$ (with $c \in E$) are homogeneously H_i.

REMARK 3.4. With the notation of (3.1), assume that there exists an element $b \in E'$ such that height $(p, b)R =$ height $p + 1$, for all height $i - 1$ prime ideals p in R such that $b \notin p$. Then R is H_{i-1} and H_i if and only if $\mathscr{L} = \mathscr{L}(R, bR)$ is homogeneously H_i.

Proof. If $\mathscr{L}$ is homogeneously H_i, then R is H_i, as in the proof of (3.1.1), so let p be a height $i - 1$ prime ideal in R. If $b \in p$, then depth $p = a - i + 1$, as in the proof of (3.1.1). If $b \notin p$, then there exists a height i prime ideal q in R such that $(p, b)R \subseteq q$ (by hypothesis), so $a - i + 1 \geqq$ depth $p \geqq$ depth $q + 1 = a - i + 1$ (since R is H_i); hence R is H_{i-1}.

The converse was proved in (2.10.1), q.e.d.

Concerning (3.4), the author conjectures that, with no condition on b other than $b \in E'$, if $\mathscr{L}(R, bR)$ is homogeneously H_i and R is a local domain, then R is H_{i-1} and H_i. Nagata's examples [**5**, Example 2, pp. 203–205] support the conjecture. We haven't been able to prove the conjecture, but the next result shows that if some $\mathscr{L}(R, bR)$ is H_i, then R is H_{i-1} and H_i.

PROPOSITION 3.5. *Let (R, M), a, E, and $i > 0$ be as in* (2.10). *If there exists $b \in E$ such that $\mathscr{L}(R, bR)$ is H_i, then R is H_{i-1} and H_i, hence, for all $c \in E$, $\mathscr{L}(R, cR)$ is homogeneously H_i.*

Proof. Assume that $\mathscr{L} = \mathscr{L}(R, bR)$ is H_i, let $\mathscr{R} = \mathscr{R}(R, bR)$, and let p be a prime ideal in R. If height $p = i$, then depth $p = a - i$, as in the proof of (3.1.1), so R is H_i. If height $p = i - 1$, then height $p^* = i - 1$. Also, we may assume that $i \leqq a$ (2.11.1), so $p^* \subset M^* \subset \mathcal{M}$, hence depth $p^*\mathcal{L} > 1$, and so depth $p^*\mathcal{L} = a - i + 2$ (2.4.2). Therefore, by (2.2.3), depth p = height $\mathcal{M}/p^* - 1$ = depth $p^* - 1 = a - i + 1$, hence R is H_{i-1}. Therefore, for all $c \in E$, $\mathcal{L}(R, cR)$ is homogeneously H_i (2.10.1), q.e.d.

It follows from (3.5) that if R is H_i and isn't H_{i-1}, then there does not exist $b \in E$ such that $\mathcal{L}(R, bR)$ is H_i.

The author doesn't know if the hypothesis of (3.5) implies that R is C_{i-1}. (Of course, this is true if $b = 0$ (2.6.3).) However, this is true if R is Henselian, as is shown in (3.8) below.

(3.3) shows that if some $\mathcal{L}$ is homogeneously H_i, then R need not be H_{i-1} (for $i = 1$).

We now give some corollaries of (3.5) (and (2.10)).

COROLLARY 3.6. *With the notation of* (2.10), *assume that R is a local domain such that all maximal ideals in the integral closure of R have the same height. If there exists $b \in E$ such that $\mathcal{L}(R, bR)$ is homogeneously H_1, then, for all $c \in E$, $\mathcal{L}(R, cR)$ is H_1.*

Proof. As in the proof of (3.1.1), R is H_1. Therefore, by hypothesis, R is C_0, so the conclusion follows from (2.10.2), q.e.d.

If R is Henselian in (3.6), then the hypothesis can be simplified, as will now be shown.

COROLLARY 3.7. *With the notation of* (2.10), *assume that R is a Henselian local domain. If there exists $b \in E$ such that $\mathcal{L}(R, bR)$ is homogeneously H_1, then, for all $c \in E$, $\mathcal{L}(R, cR)$ is H_1.*

Proof. Since R is Henselian, the hypothesis of (3.6) is satisfied, so the conclusion follows from (3.6), q.e.d.

The next corollary shows that if, in (3.5), R is Henselian, then R is C_{i-1}.

COROLLARY 3.8. *With the notation of* (2.10), *assume that R is Henselian. If there exists $b \in E$ such that $\mathcal{L}(R, bR)$ is H_i, then R is C_{i-1} and, for all $c \in E$, $\mathcal{L}(R, cR)$ is H_i.*

Proof. By (3.5), R is H_{i-1} and H_i, so the conclusion follows from (2.12), q.e.d.

The next corollary shows that if R is complete in (3.5), then considerably more can be said.

COROLLARY 3.9. *With the notation of* (2.10), *assume that R is complete. If there exists $b \in E$ such that $\mathcal{L}(R, bR)$ is homogeneously H_i, then, for all $c \in E$, $\mathcal{L}(R, cR)$ is $H_{i+1}, \cdots, H_{a+1}$.*

Proof. If $\mathscr{L}(R, bR)$ is homogeneously H_i, then R is H_i (as in the proof of (3.1.1)). Therefore the conclusion follows from (2.13), q.e.d.

REMARK 3.10. If, in (3.9), there exists $b \in E$ such that $\mathscr{L}(R, bR)$ is H_i, then, for all $c \in E$, $\mathscr{L}(R, cR)$ is $H_i, \cdots, H_{a+1}$.

Proof. By (3.5), R is H_{i-1}, so the conclusion follows from (2.13), q.e.d.

To prove some further corollaries of (3.5) and (2.10), we need the following lemma.

LEMMA 3.11. *Let R and S be local rings such that R is a dense subspace of S. If S is H_i, then R is H_i.*

Proof. Let p be a height i prime ideal in R. Then every minimal prime divisor of pS has height i [**5**, (22.9)] and, since R/p is a dense subspace of S/pS, there exists a minimal prime divisor q of pS such that depth q = depth p. Hence, if S is H_i, then depth p = depth q = altitude $S - i$ = altitude $R - i$, so R is H_i, q.e.d.

Combining (3.8) and (3.11), we have the following result.

COROLLARY 3.12. *With the notation of* (2.10), *let R^H be the Henselization of R. If there exists $b \in E$ such that $\mathscr{L}(R^H, bR^H)$ is H_i, then, for all $c \in F$, $\mathscr{L}(R, cR)$ is H_i, and R is C_{i-1}.*

Proof. If $\mathscr{L}(R^H, bR^H)$ is H_i, then, for each $c \in E$, $\mathscr{L}' = \mathscr{L}(R^H, cR^H)$ is H_i (3.8). Also, $\mathscr{L}(R, cR)$ is a dense subspace of $\mathscr{L}'$ (since $\mathscr{L}(R, cR)$ and $\mathscr{L}(R^H, cR^H)$ are dense subspaces of $\mathscr{L}(R^*, cR^*)$, by [**10**, Lemma 3.2], where R^* is the completion of R). Therefore, for all $c \in E$, $\mathscr{L}(R, cR)$ is H_i (3.11), hence R is C_{i-1} (2.10.2), q.e.d.

Of course, the conclusion of (3.12) holds if there exists an element $b \in R^H$ such that either $b = 0$ or height $bR^H = 1$ and $\mathscr{L}(R^H, bR^H)$ is H_i (by the proof of (3.12)).

Combining (3.9) and (3.11), we have the following corollary to (3.5).

COROLLARY 3.13. *With the notation of* (2.10), *let R^* be the completion of R. If there exists $b \in E$ such that $\mathscr{L}(R^*, bR^*)$ is homogeneously H_i, then, for all $c \in E$, $\mathscr{L}(R, cR)$ and $\mathscr{L}(R^H, cR^H)$ are $H_{i+1}, \cdots, H_{a+1}$.*

Proof. If $\mathscr{L}(R^*, bR^*)$ is homogeneously H_i, then, for each $c \in E$, $\mathscr{L}'' = \mathscr{L}(R^*, cR^*)$ is $H_{i+1}, \cdots, H_{a+1}$ (3.9). Also, by [**10**, Lemma 3.2], $\mathscr{L}(R, cR)$ and $\mathscr{L}(R^H, cR^H)$ are dense subspaces of $\mathscr{L}''$, so the conclusion follows from (3.11), q.e.d.

Again, the conclusion of (3.13) holds if there exists $b \in R^*$ such that either $b = 0$ or height $bR^* = 1$ and $\mathscr{L}(R^*, bR^*)$ is homogeneously H_i (by the proof of (3.13)). And, if $\mathscr{L}(R^*, bR^*)$ is H_i, then, for all $c \in E$, $\mathscr{L}(R, cR)$ and $\mathscr{L}(R^H, cR^H)$ are $H_i, \cdots, H_{a+1}$ (by (3.10) and the proof of (3.13)).

It follows from (3.13) and (2.10.2), that if there exists $b \in E$ such that $\mathscr{L}(R^*, bR^*)$ is homogeneously H_i, then R and R^H are $C_i, \cdots, C_a$.

To derive some further corollaries to (2.10), we need the following definitions.

DEFINITION 3.14. A ring R satisfies the *first chain condition for prime ideals* (*f.c.c.*) in case every maximal chain of prime ideals in R has length equal to the altitude of R.

DEFINITION 3.15. A ring R satisfies the *second chain condition for prime ideals* (*s.c.c.*) in case, for each minimal prime ideal z in R, depth $z =$ altitude R and every integral extension domain of R/z satisfies the f.c.c.

DEFINITION 3.16. A local ring R is said to be *taut* (resp., *taut level*) in case R is H_i, for all $i = 1, \cdots, a$ (resp., $i = 0, 1, \cdots, a$), where $a =$ altitude R. If P is a homogeneous prime ideal in a graded ring R, then R_P is said to be *homogeneously taut* (resp. *homogeneously taut level*), in case R_P is homogeneously H_i, for all $i = 1, \cdots, a$ (resp., $i = 0, 1, \cdots, a$), where $a =$ height P.

Numerous properties of rings which satisfy the f.c.c. or the s.c.c. are known. A summary of the basic properties is given in [**11**, Remarks 2.22–2.25]. Also, a number of properties of taut semi-local rings are given in [**4**] and [**13**]. We mention only that taut level local rings are the same as local rings which satisfy the f.c.c. [**4**, Proposition 7].

With the above definitions, we will now give some additional corollaries of (2.10).

COROLLARY 3.17. *With the notation of* (2.10), *R is taut level if and only if, for each $b \in E$, $\mathscr{L}(R, bR)$ is homogeneously taut level.*

Proof. By (2.10.1), R is taut level if and only if all $\mathscr{L} = \mathscr{L}(R, bR)$ are homogeneously $H_1, \cdots, H_a$. Finally, $\mathscr{L}$ is H_{a+1} (2.4.1); and $\mathscr{L}$ is H_0, if R is H_0 (2.11.2), q.e.d.

We now give a number of results related to (2.10), to the above definitions, and to (3.17).

REMARK 3.18. Let (R, M), a, and E be as in (2.10). Then the following statements hold:

(3.18.1) R is taut if and only if, for all $b \in E$, $\mathscr{L}(R, bR)$ is homogeneously $H_2, \cdots, H_{a+1}$.

(3.18.2) If, for all non-zero $b \in E$, $\mathscr{L}(R, bR)$ is homogeneously taut level, then R is taut level.

(3.18.3) For each $b \in E$, $\mathscr{L}(R, bR)$ is taut if and only if $\mathscr{L}(R, bR)$ satisfies the f.c.c.

(3.18.4) For all $b \in E$, $\mathscr{L}(R, bR)$ may be homogeneously taut, but not homogeneously taut level.

(3.18.5) $\mathscr{L}(R, bR)$ may be homogeneously taut level but not taut level.

(3.18.6) All $\mathscr{L}(R, bR)$ (with $b \in E$) are taut if and only if R satisfies the s.c.c.

(3.18.7) If there exists $b \in E$ such that $\mathscr{L}(R, bR)$ is taut, then R satisfies the f.c.c. and, for all $c \in E$, $\mathscr{L}(R, cR)$ is homogeneously taut level.

Proof. The proof of (3.18.1) is similar to the proof of (3.17).

(3.18.2) The hypothesis implies that R is $H_1, \cdots, H_a$, by (3.1.1). Also, R is H_0 by (2.11.2).

(3.18.3) Assume that $\mathscr{L} = \mathscr{L}(R, bR)$ is taut and let z be a minimal prime ideal in $\mathscr{L}$. It may clearly be assumed that $a > 0$. Then $z = (z \cap R)^*\mathscr{L} \subset M^*\mathscr{L} \subset \mathscr{M}\mathscr{L}$, hence, since $\mathscr{L}$ is H_1, depth $z = a + 1$ (2.4.2). Therefore $\mathscr{L}$ is taut level, hence $\mathscr{L}$ satisfies the f.c.c. [**4**, Proposition 7]. The converse is clear.

(3.18.4) follows from (3.3).

(3.18.5) For an example, let (R, M) be a local domain such that altitude $R = 2$ and R isn't C_0 (that is, there exists a height one maximal ideal in the integral closure of R) (for example, [**5**, Example 2, pp. 203–205] in the case $m = 0$ and $r = 1$). Then, for each $b \in M$, $\mathscr{L} = \mathscr{L}(R, bR)$ is homogeneously taut level (by (3.17)), but $\mathscr{L}$ isn't H_1, since otherwise R would be C_0 (2.10.2).

(3.18.6) If all $\mathscr{L}$ are taut, then, in particular, by (3.18.3), $R[u]_{(M,u)}$ satisfies the f.c.c., hence R satisfies the s.c.c. [**10**, Theorem 2.21]. Conversely, if R satisfies the s.c.c., then $D = R[X, Y]_{(M,X,Y)}$ satisfies the f.c.c., by [**10**, Theorem 2.21], hence, since each $\mathscr{L}$ is a homomorphic image of D and each $\mathscr{L}$ is H_0 (since R is H_0), all $\mathscr{L}$ satisfy the f.c.c., and so all $\mathscr{L}$ are taut.

(3.18.7) If $\mathscr{L} = \mathscr{L}(R, bR)$ is taut, then $\mathscr{L}$ is $H_0, \cdots, H_{a+1}$ (3.18.3), so R is $H_0, \cdots, H_a$ (3.5), hence R satisfies the f.c.c. [**4**, Proposition 7]. Therefore, for all $c \in E$, $\mathscr{L}(R, cR)$ is homogeneously taut level (3.17), q.e.d.

We now give two more corollaries relating the above definitions and (2.10).

COROLLARY 3.19. *With the notation of* (3.12), *if there exists* $b \in E$ *such that* $\mathscr{L}(R^H, bR^H)$ *is taut, then* R *and* R^H *satisfy the s.c.c.*

Proof. If $\mathscr{L}(R^H, bR^H)$ is taut, then R^H satisfies the f.c.c. (3.18.7). Therefore R and R^H satisfy the s.c.c. [**10**, Theorem 2.21], q.e.d.

COROLLARY 3.20. *With the notation of* (3.13), *if there exists* $b \in E$ *such that* $\mathscr{L}(R^*, bR^*)$ *is homogeneously taut level, then* R *and* R^H *satisfy the s.c.c.*

Proof. If $\mathscr{L}(R^*, bR^*)$ is homogeneously taut level, then R^* is H_0 (2.11.2), hence R is quasi-unmixed (by definition), and so R and R^H satisfy the s.c.c. [**10**, Corollary 2.8], q.e.d.

To prove (3.21.2), we need the following fact [**15**, (3.13)]: If a local ring R is $C_1, \cdots, C_{a-2}$ ($a =$ altitude R), then R is taut and, for each minimal prime ideal z in R and for each maximal ideal N in the integral closure $(R/z)'$ of R/z, $(R/z)'_N$ satisfies the s.c.c. and height $N \in \{1, a\}$.

REMARK 3.21. With the notation of (3.20), the following statements hold:

(3.21.1) If there exists $b \in E$ such that $\mathscr{L}(R^*, bR^*)$ is taut, then R and R^H satisfy the s.c.c.

(3.21.2) If there exists $b \in E$ such that $\mathscr{L}(R^*, bR^*)$ is homogeneously taut, then R is taut and, for each minimal prime ideal z in R and for each maximal ideal N in the integral closure $(R/z)'$, of R/z, $(R/z)'_N$ satisfies the s.c.c. and height $N \in \{1, a\}$.

Proof. (3.21.1) follows from (3.18.3) and (3.20).

(3.21.2) If there exists such $b \in E$, then $\mathscr{L}(R^*, bR^*)$ is homogeneously H_1, so, by the paragraph preceding (3.14), R is $C_1, \cdots, C_a$. Therefore the conclusion follows from [**15**, (3.13)], q.e.d.

This section will be closed with two questions and some comments on why they are important, and why the results in the first two sections of this paper lend support for an affirmative answer to each. For the first question we note that it is known that a local domain (R, M) satisfies the s.c.c. if and only if $R[X]_{(M,X)} = \mathscr{L}(R, (0))$ satisfies the *f.c.c.* [**10**, Theorem 2.21], so in (3.22) we restrict attention to $0 \neq b \in M$.

QUESTION 3.22. If R is a local domain which satisfies the f.c.c. and is C_0, is it true that, for each $0 \neq b \in M$, $\mathscr{L}(R, bR)$ satisfies the f.c.c.?

If the answer to (3.22) is yes, then the Catenary Chain Conjecture holds (that is, if R is a C_0-local domain which satisfies the f.c.c., then R satisfies the s.c.c.) For, if R is C_0 and satisfies the f.c.c., then all rings $\mathscr{L}(R, bR)$ $(0 \neq b \in M)$ satisfy the f.c.c., hence R satisfies the s.c.c. (3.18.6).

Also, the Catenary Chain Conjecture implies that the answer to (3.22) is yes. For, if R is C_0 and satisfies the f.c.c., then, by the Catenary Chain Conjecture, R satisfies the s.c.c., hence for all $b \in M$, $\mathscr{L}(R, bR)$ satisfies the f.c.c., by (3.18.6) and (3.18.3).

By (3.18.3), (3.22) is equivalent to: If R satisfies the f.c.c. and is C_0, does it hold that, for all $0 \neq b \in E$, $\mathscr{L}(R, bR)$ is taut? (2.10) lends, in the author's opinion, much support for an affirmative answer to this version of (3.22). That is, by (2.10.2), all $\mathscr{L} = \mathscr{L}(R, bR)$ are H_1; and, by (2.10.1), all $\mathscr{L}$ are homogeneously taut. Now, by (3.18.5), $\mathscr{L}$ may be homogeneously taut and not taut. However, the only examples the author knows where $\mathscr{L}$ is homogeneously taut and not taut are those for which $\mathscr{L}$ is H_i, for all $i > 1$, but not H_1 (as in (3.18.5)), and in this case, there exists a height one maximal ideal in the integral closure of R (so R isn't C_0).

Before stating the second question, we note that it is known [**12**, (4.3)] that if the Catenary Chain Conjecture holds and R is a local domain which satisfies the f.c.c., then, for all height one prime ideals p in R, R/p is C_0. We will use this below in showing that an affirmative answer to (3.23) is equivalent to one of the chain conjectures holding.

QUESTION 3.23. If (R, M) is a local domain which is C_{i-1}, is $D = R[u]_{(M,u)}$ C_i?

The H-Conjecture (that is, a H_1-local domain satisfies the f.c.c.) implies that the answer to (3.23) is yes. That is, if R is C_{i-1}, then D is H_i (2.6.3). Therefore, for each height $i-1$ prime ideal p in D, D/p is H_1 (2.4.2), hence satisfies the f.c.c. (by the H-Conjecture). Now the H-Conjecture implies the Catenary Chain Conjecture [**12**, (4.5)]. Therefore, it follows from [**12**, (4.3)] that, for each height i prime ideal q in D, $D/q(=(D/p)/(q/p)$ with $p \subset q$ and height $p = i-1)$ is C_0. Therefore, since D is H_i, D is C_i (2.6.2).

Also, if the answer to (3.23) is yes, then the H-Conjecture holds. For, if R is an H_1-local domain, then to prove that R satisfies the f.c.c., it may be assumed that R is C_0 [**14**, (2.12)]. Then D is C_1 (by (3.23)). Now it clearly follows from (2.6.3) that if D is C_i, then R is C_i. Therefore R is C_1, hence D is C_2 (by (3.23)). Repeating, R is $C_0, \cdots, C_a$, hence R is $H_0, \cdots, H_a$, and so R satisfies the f.c.c. [**4**, Proposition 7].

(2.10.2) gives some support for an affirmative answer to (3.23). Namely, it is known [**6**, (4.7)] that R is C_i if and only if, for all x in the quotient field F of R such that $(M, x)R[x]$ is proper, $R(x)_{(M,x)}$ is H_i. In fact, [**6**, (4.7)] shows that to prove that R is C_i, it suffices to consider only certain subsets of such $x \in F$. (For example, those $x = c/b$ with height $(b, c)R = 2$.) Thus, to prove that D is C_i, it suffices to prove that all $D[e/d]_{(N,e/d)}$ are H_i, where height $(d, e)D = 2$ and N is the maximal ideal in D. Now (2.10.2) shows that many of these rings are

H_i, if R is C_{i-1}. Namely, for all $0 \neq b \in M$, height $(b, u)D = 2$ and $\mathscr{L}(R, bR) = D[b/u]_{(N,b/u)}$ is H_i (2.10.2).

4. A generalization.

In this section we give a generalization of (2.10.2) in (4.2), and then derive some corollaries of (4.2).

To prove (4.2), the following corollary of (2.10) will be helpful.

COROLLARY 4.1. *Let (R, M) be a local ring, let i and k be positive integers, and let $P_j = R[X_1, \cdots, X_j]_{(M,X_1,\cdots,X_j)}$ $(j = 1, 2, \cdots)$. Then the following statements are equivalent*:

(4.1.1) *P_k is H_{i+k}.*

(4.1.2) *P_{k-1} is C_{i+k-1}.*

(4.1.3) *$\mathscr{L}(P_{k-1}, (b))$ is H_{i+k}, for all $b \in E(P_{k-1}) = \{c \in P_{k-1};$ height $cP_{k-1} = 1\} \cup \{0\}$.*

(4.1.4) *$\mathscr{L}(P_{k-1}, (b))$ is H_{i+k}, for all $b \in E'(P_{k-1}) = \{c \in P_{k-1};$ height $cP_{k-1} = 1\}$.*

Proof. This follows from (2.6.3), (2.10.2), and (3.1.2), since P_{k-1} is a local ring and $P_k = P_{k-1}[X_k]_{(M_{k-1},X_k)}$, where $M_{k-1} = (M, X_1, \cdots, X_{k-1})P_{k-1}$, q.e.d.

We will now prove the following generalization of (2.10.2).

THEOREM 4.2. *Let (R, M) be a local ring, and let i and k be positive integers. If $P_k = R[X_1, \cdots, X_k]_{(M,X_1,\cdots,X_k)}$ is H_{i+k-1}, then, for all proper ideals $B = (b_1, \cdots, b_k)R$ such that height $B \geqq 1$, $\mathscr{L}(R, B)$ is H_i.*

Proof. Assume that P_k is H_{i+k-1}, let $B = (b_1, \cdots, b_k)R$ be a proper ideal in R such that height $B \geqq 1$, and let $\mathscr{L} = \mathscr{L}(R, B)$. Then height $b_jR = 1$, for some $j = 1, \cdots, k$ (since $B \not\subseteq \cup\{z\,;\, z$ is a minimal prime ideal in $R\}$). Say height $b_kR = 1$, and let f be the natural homomorphism from P_{k+1} onto $\mathscr{L}' = \mathscr{L}(P_{k-1}, (b_k)) = P_{k-1}[tb_k, u]_{(M,X_1,\cdots,X_{k-1},tb_k,u)}$ (that is, $f(X_k) = tb_k$ and $f(X_{k+1}) = u$), and let g be the natural homomorphism from P_{k+1} onto $\mathscr{L}$ ($g(X_i) = tb_i$ $(i = 1, \cdots, k)$ and $g(X_{k+1}) = u$). Let $K_1 = \operatorname{Ker} f$ and $K = \operatorname{Ker} g$. Then, with $u = X_{k+1}$, $uX_k - b_k \in K_1$ and $(uX_1 - b_1, \cdots, uX_k - b_k)P_{k+1} \subseteq K$. Also, $K_1 \subseteq K$, since g induces the natural homomorphism from $\mathscr{L}'$ onto $\mathscr{L}$ (so $\mathscr{L} \cong \mathscr{L}'/(K/K_1)$).

Assume temporarily that R is a local domain. Then K is prime and $K \cap R = (0)$, so by the altitude formula for K relative to R [**19**, Proposition 2, p. 326], height $K + \operatorname{trd}$ $(P_{k+1}/K)/R =$ height $K \cap R + (k + 1)$, hence height $K = k$ (since $P_{k+1}/K = \mathscr{L}$ and $\operatorname{trd} \mathscr{L}/R = 1$). Also, K_1 is prime and $K_1 \cap P_{k-1} = (0)$, so by the altitude formula for K_1 relative to P_{k-1}, height $K_1 = 1$. Further, $K_1 \subseteq K$ and $(P_{k+1})_K$ is a regular local ring (since $K \cap R = (0)$), so height $K/K_1 = k - 1$ (since regular local rings satisfy the f.c.c.). Therefore, since $\mathscr{L}'$ is H_{i+k-1} (4.1), $\mathscr{L} \cong \mathscr{L}'/(K/K_1)$ is H_i (2.4.2).

Now assume that R has non-zero divisors of zero, and let w be a minimal prime ideal in $\mathscr{L}$, so $w = z^*\mathscr{L}$, for some minimal prime ideal z in R. Let P be the minimal prime divisor of K such that $P/K = z^*\mathscr{L}$, so $P \cap R = z$. Let $z' = zP_{k+1}$. Then $K_1' = (uX_k - b_k, z')P_{k+1}/z' \subseteq (K, z')/z' \subseteq P/z'$ and K_1' is prime [**2**, Ex. 3, p. 102] (since $b_k \notin z$ implies $u, b_k + zA$ is a prime sequence in A/zA and $zA = z' \cap A$, where $A = P_{k-1}[u]_{(M,X_1,\cdots X_{k-1},u)}$). Also,

$$K_1' = \operatorname{Ker}(P_{k+1}/z' \to \mathscr{L}'' = \mathscr{L}(P_{k-1}/zP_{k-1}, (b_k + zP_{k-1})))$$

and $\mathscr{L}'' \cong (2.2.3)$ $\mathscr{L}'/(zP_{k-1})^*\mathscr{L}'$ (where $(zP_{k-1})^* = (zP_{k-1})P_{k-1}[t, u] \cap \mathscr{R}(P_{k-1}, (b_k)))$. Therefore $p = (uX_k - b_k, z')P_{k+1}$ is prime and $K_1 \subseteq p \subseteq P$ (since $K_1' = p/z' \subseteq P/z'$ and $p = \operatorname{Ker}(P_{k+1} \to \mathscr{L}'/(zP_{k-1})^*\mathscr{L}')$). Also, $P/z' = \operatorname{Ker}(P_{k+1}/z' \to \mathscr{S} = \mathscr{L}(R/z, (B+z)/z))$ and $\mathscr{S} \cong \mathscr{L}/z^*\mathscr{L}$ (2.2.3). Hence, by the domain case, height $P/z' = k$, and height $P =$ height P/z' (since z' is the only minimal prime ideal contained in P (since $P \cap R = z$ and $z' = zP_{k+1}$)). Likewise, height $p =$ height $p/z' = 1$. Further, every maximal chain of prime ideals in $(P_{k+1})_P =$ (say) D has length equal to height P (since D/zD is a regular local ring and z is the only minimal prime ideal in D). Therefore height $P/p =$ height $P -$ height $p = k-1$. Therefore, $\mathscr{L}/z^*\mathscr{L} = P_{k+1}/P = (P_{k+1}/K_1)/(P/K_1) = \mathscr{L}'/(P/K_1)$ is H_i (2.4.2), (since P_{k+1}/K_1 is H_{i+k-1} (4.1) and $k-1 =$ height $P/p \leqq$ height $P/K_1 \leqq k-1$ (since height $P =$ height $P/z' = k$ and height $K_1 \geqq 1$)).

Hence, for each minimal prime ideal w in $\mathscr{L}$, $\mathscr{L}/w$ is H_i. Further, if w is a minimal prime ideal in $\mathscr{L}$, then with altitude $R = a$, depth $w = a+1$ or depth $w \leqq i$; for, $w = z^*\mathscr{L}$, for some minimal prime ideal z in R, so depth $w =$ depth $z + 1$ (2.2.3), and depth $z =$ depth $zP_k - k$ and either depth $zP_k = a + k$ or depth $zP_k \leqq i + k - 1$ (by (2.4.2) for the case $j = 0$ (since P_k is H_{i+k-1})). Therefore, $\mathscr{L}$ is H_i (2.4.2), q.e.d.

The following remark (which is obvious from (4.2)) will be useful in the proof of (4.4).

REMARK 4.3. If, in (4.2), P_k is $H_{i+k-1}, \cdots, H_{i+k+h}$ $(h \geqq 0)$, then $\mathscr{L}(R, B)$ is $H_i, \cdots, H_{i+h+1}$.

COROLLARY 4.4. *With the notation of* (4.2), *assume that* P_k is H_{i+k-1} *and let* $B' = (b_1, \cdots, b_j)R$ $(1 \leqq j \leqq k)$ *be a proper ideal in* R *such that height* $B' > 0$. *Then* $\mathscr{L} = \mathscr{L}(R, B')$ *is* $H_i, \cdots, H_{i+k-j}$.

Proof. Let, say, b_j such that height $b_jR = 1$, let f be the natural homomorphism from P_{j+1} onto $\mathscr{L}' = \mathscr{L}(P_{j-1}, (b_j))$ $(f(X_j) = tb_j, f(X_{j+1}) = u)$, let g be the natural homomorphism from P_{j+1} onto $\mathscr{L}$ $(g(X_h) = tb_h$ $(h = 1, \cdots, j)$, $g(X_{j+1}) = u)$, let $K_1 = \operatorname{Ker} f$, and let $K = \operatorname{Ker} g$. Then, as at the end of the first paragraph of the proof of (4.2), $\mathscr{L} \cong$

$\mathscr{L}'/(K/K_1)$. Also, as in the third paragraph of the proof of (4.2), every minimal prime divisor P/K_1 of K/K_1 has height $j-1$. Further, since P_k is H_{i+k-1}, P_j is $H_{i+j-1}, \cdots, H_{i+k-1}$ (by (2.6.3)). Moreover, as in the last paragraph of the proof of (4.2), if P is a minimal prime divisor of K, then either depth $P =$ depth $P/K =$ depth $w = a+1$ or $\leqq i$. Therefore, by (4.3), $\mathscr{L}$ is $H_i, \cdots, H_{i+k-j}$, q.e.d.

REMARK 4.5. With the notation of (4.2), assume that P_k is H_{i+k-1} and let B' be as in (4.4) with $j<k$. Then $\mathscr{L}(R, B')$ is $C_i, C_{i+1}, \cdots, C_{i+k-j-2}$.

Proof. By (2.6.3), P_{j+1} is $C_{i+j}, \cdots, C_{i+k-2}$. Also, as in the third and fourth paragraphs of the proof of (4.2), every minimal prime divisor P of K (the kernel of the natural homomorphism from P_{j+1} onto $\mathscr{L}(R, B')$) is such that height $P = j$ and either depth $P = a+1$ or $\leqq i$. Therefore it follows from (2.6.2) that $\mathscr{L}(R,B')$ is $C_i, \cdots, C_{i+k-j-2}$, q.e.d.

The following known result is an easy corollary to (4.4) (the case $k = a-1$ and $j = i = 1$).

COROLLARY 4.6. (cf. [**15**, (3.10)].) *With the notation of* (4.2), *let $a =$ altitude R and assume that P_{a-1} is H_{a-1}. Then R satisfies the s.c.c.*

Proof. By (4.4), for all $b \in E'$ (see (3.1)), $\mathscr{L} = \mathscr{L}(R, bR)$ is $H_1, \cdots, H_{a-1}$, so R is $C_0, \cdots, C_{a-2}$ (3.1.2), hence $R[u]_{(M,u)}$ is $H_1, \cdots, H_{a-1}$ (2.10.2). Thus, for all $b \in E$, $\mathscr{L}(R, bR)$ is taut (2.4.1), so R satisfies the s.c.c. (3.18.6), q.e.d.

Also, the following known result follows from (4.4) (the case $k = a-2$, $i = 2$, and $j = 1$).

COROLLARY 4.7. (cf. [**15**, (3.12)].) *With the notation of* (4.2), *let $a =$ altitude R and assume that P_{a-2} is H_{a-1}. Then R is taut and, for each minimal prime ideal z in R and for each maximal ideal N in the integral closure $(R/z)'$ of R/z, $(R/z)'_N$ satisfies the s.c.c. and height $N \in \{1, a\}$.*

Proof. By (4.4), for all $b \in E'$, $\mathscr{L}(R, bR)$ is $H_2, \cdots, H_{a-1}$. Therefore R is $C_1, \cdots, C_{a-2}$ (3.1.2), hence the conclusion follows from [**15**, (3.13)] (see the paragraph preceding (3.21)), q.e.d.

The converses of (4.6) and (4.7) are true (and are given in the cited references). Thus, it seems natural to ask if the converse of (4.2) is true. The author doesn't know the answer.

The following corollary to (4.2) is a generalization of (2.14).

COROLLARY 4.8. *With the hypotheses of* (4.4), *the following statements hold (where $a =$ altitude R):*

(4.8.1) *For all maximal ideals* N *in* $\mathcal{R} = \mathcal{R}(R, B')$ *such that* $N \cap R = M$, $\mathcal{R}_N$ *is* $C_i, \cdots, C_{i+k-j-2}$ *and either altitude* $\mathcal{R}_N = a+1$ *or altitude* $\mathcal{R}_N \leqq i$.

(4.8.2) *For all maximal ideals* N *in* $\mathcal{F} = \mathcal{F}(R, B')$ *such that* $N \cap (R/B') = M/B'$, $\mathcal{F}_N$ *is* $C_{i-1}, \cdots, C_{i+k-j-3}$ *and either altitude* $\mathcal{F}_N = a$ *or altitude* $\mathcal{F}_N \leqq i-1$.

(4.8.3) *For all non-zero-divisors* c *in* R *and for all maximal ideals* N *in* $A = R[b_1/c, \cdots, b_j/c]$ *such that* $N \cap R = M$, A_N *is* $C_{i-1}, \cdots, C_{i+k-j-3}$ *and either altitude* $A_N = a$ *or altitude* $A_N \leqq i-1$.

Proof. (4.8.1) Let K be the kernel of the natural homomorphism from $R_{j+1} = R[X_1, \cdots, X_{j+1}]$ onto $\mathcal{R} = \mathcal{R}(R, B')$, let N be a maximal ideal in $\mathcal{R}$ such that $N \cap R = M$, and let Q be the pre-image of N in R_{j+1}. Then $\mathcal{R}_N \cong L/KL$, where $L = (R_{j+1})_Q$. Now there exist polynomials $f_1, \cdots, f_{j+1} \in R_{j+1}$ such that $Q = (M, f_i, \cdots, f_{j+1})$ and R_{j+1} is integral over $T = R[f_1, \cdots, f_{j+1}]$ [**5**, (14.7)]. (Let $f_i' = f_i'(T_i)$ be the minimum polynomial for $x_i = X_i + Q$ over $(R/M)[x_1, \cdots, x_{i-1}]$ and let f_i be obtained from f_i' by replacing $x_1, \cdots, x_{i-1}, T_i$ by $X_1, \cdots, X_i$.) Then $(R_{j+1})_Q$ is integral over $T_{Q\cap T} \cong P_{j+1}$, so $T_{Q\cap T}$ is $C_{i+j}, \cdots, C_{i+k-2}$, since P_{j+1} is (as in the proof of (4.5)). Therefore L is $C_{i+j}, \cdots, C_{i+k-2}$ [**15**, (3.18)]. Also, if P is a minimal prime divisor of K, then height $P = j$ and either depth $P = a+1$ or $\leqq i$ (as in the proof of (4.5), since all prime divisors of K are contained in $(M, X_1, \cdots, X_{j+1})R_{j+1}$ (since all prime divisors of (0) in $\mathcal{R}$ are contained in $\mathcal{M}$)). Therefore, since $\mathcal{R}_N \cong L/KL$, the conclusion follows from (2.6.2).

(4.8.2) follows as in the proof of (2.14.3), and (4.8.3) follows as in the proof of (2.14.4), q.e.d.

In (2.10.2) it was seen that if $R[X]_{(M,X)}$ is H_i, then all $\mathcal{L} = \mathcal{L}(R, bR)$ (with $b \in E$) are H_i. It seems natural to ask are all such $\mathcal{L}$ C_i when $R[X]_{(M,X)}$ is C_i? The author doesn't know the answer. However, if the answer is yes, and if P_k is C_{i+k-1}, then all $\mathcal{L}(R, B)$ of (4.2) are C_i (much as in the proof of (4.2) and using (2.6.2)).

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