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A CONSTRUCTION OF THE IDEMPOTENT-SEPARATING CONGRUENCES ON A BISIMPLE ORTHODOX SEMIGROUP

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For any bisimple orthodox semigroup S we show how to construct all idempotent-separating congruences on S, and give an explicit construction for the quotient semigroup of S modulo such a congruence.

Introduction. For an arbitrary bisimple inverse semigroup S, Reilly and Clifford **[8]** have shown (1) how to construct all idempotent-separating congruences on S, and (2) have obtained an explicit construction for the quotient semigroup of S modulo such a congruence. Their work is based on the construction of all bisimple inverse semigroups given by Reilly in **[7]**.

The purpose of this article is to extend the above results to the case where the semigroup S is a bisimple orthodox semigroup, by making use of the elegant construction theorem for all such semigroups due to Clifford **[1; 2]**. The construction of the idempotent-separating congruences on S yields an immediate one-to-one correspondence between these congruences and certain pairs (V, V') of normal subgroups of some of the components used in Clifford's construction of S. When this correspondence is applied to an abstract bisimple orthodox semigroup, it reduces to the one given by Munn **[5]** for bisimple regular semigroups.

1. Preliminaries. We shall adopt the notation and terminology of **[3]**. Clifford's construction of any bisimple orthodox semigroup is given in Theorem A of **[1; 2]**. As this construction is basic for our study, we begin by reviewing it and certain associated concepts.

By a *right Reilly groupoid* we mean a partial groupoid R satisfying the following four axioms.

(R1) If a, b, c are elements of R such that bc and $a(bc)$ are defined, then ab and $(ab)c$ are defined, and $(ab)c = a(bc)$.

(R2) If a is an element of R such that ab is defined for some b in R, then ax is defined for all x in R.

(R3) If a, b, c are elements of R such that $ac = bc$, then $ax = bx$ for all x in R.

(R4) R contains at least one left identity element.

By the *core* $\hat{R}$ of a right Reilly groupoid R we mean the set of all a in R such that ab is defined for some, hence for all, b in R. Now $\hat{R}$ is a subsemigroup of R containing all left identities of R. For any left identity e of R let H_e denote the group of units of the semigroup $\hat{R}e$. Define $a\mathscr{L}B$ $(a, b$ in $R)$ to mean that $a = xb$ and $b = ya$ for some x, y in $\hat{R}$. By Proposition 1.2 of **[2]**, $a\mathscr{L}b$ if and only if $a = ub$ for some $u \in H_e$.

A *left Reilly groupoid* L and its core $\hat{L}$ are defined dually. If e' is a right identity of a left Reilly groupoid L, let $H_{e'}$ denote the group of units of the semigroup $e'\hat{L}$. The relation $\mathscr{R}$ on L is defined dually.

Let $R[L]$ be a right [left] Reilly groupoid. Elements of L will be denoted by primed letters. By an *anti-correlation* between L and R we mean a subset K of $L \times R$ satisfying the following conditions.

(AC1) The projection of K into $L[R]$ is onto $L[R]$.

(AC2) $(a', a) \in K$, $(b', b) \in K$, $(b', a) \in K$ imply $(a', b) \in K$.

(AC3) If $(a', a) \in K$ then $a' \in \hat{L}$ if and only if $a \in \hat{R}$.

(AC4) Let $\hat{K} = K \cap (\hat{L} \times \hat{R})$. Then $(a', a) \in \hat{K}$, $(b', b) \in K$ imply $(b'a', ab) \in K$.

The following axioms are needed.

(AI) $R[L]$ is a right [left] Reilly groupoid, and K is an anti-correlation between them; $e[e']$ is an arbitrary but fixed left [right] identity of $R[L]$.

We write $\kappa = K^{-1} \circ K$ and $\lambda = K \circ K^{-1}$, and let $H_e[H_{e'}]$ be the group of units of $\hat{\mathrm{R}}\mathrm{e}[e'\hat{L}]$.

(AII) $ac\kappa bc$ $(a, b \in \hat{R}, c \in R)$ imply $ae\kappa be$; and $c'a'\lambda c'b' (a', b' \in \hat{L}, c' \in L)$ imply $e'a'\lambda e'b'$.

(AIII) For $a' \in L$, $(a', e) \in K$ if and only if a' is a right identity of L; for $a \in R$, $(e', a) \in K$ if and only if a is a left identity of R.

Using (AI–III), one can show that K induces an anti-isomorphism $u \to u'$ from H_e onto $H_{e'}$. Here, for $u \in H_e$, u' denotes the unique element of $H_{e'}$ such that $(u', u) \in K$.

Define an equivalence relation τ on $L \times R$ by

(1.1) $(a', b)\tau(c', d)$ if and only if $c' = a'u'$ and $d = ub$ for some $u \in H_e$.

Let $(a', b)_\tau$ be the τ-class containing (a', b). Proposition 3.2 of **[2]** shows that the subsets $\hat{L} \times \hat{R}$, $e'\hat{L} \times \hat{R}e$, and K of $L \times R$ are unions of τ-classes, and so we write

$$T = (L \times R)/\tau, \qquad \hat{T} = (\hat{L} \times \hat{R})/\tau, \qquad T^0 = (e'\hat{L} \times \hat{R}e)/\tau,$$
$$K_\tau = K/\tau, \qquad \hat{K}_\tau = \hat{T} \cap K_\tau, \qquad K_\tau^0 = T^0 \cap K_\tau,$$

and $\hat{K} = K \cap (\hat{L} \times \hat{R})$. For arbitrary $(x', y)_\tau$ in $\hat{T}$ and (a', b) in $L \times R$, Proposition 3.3 of **[2]** allows us to define

$$(x', y)_\tau (a', b) = (a'x', yb)_\tau. \tag{1.2}$$

We now postulate a binary operation $(\circ)$ on K_τ such that the following axioms hold.

(AIV) $K_\tau(\circ)$ is a band; and, for all $(a', a) \in \hat{K}$,

$$(e', e)_\tau \circ (a', a)_\tau = (e'a', a)_\tau,$$
$$(a', a)_\tau \circ (e', e)_\tau = (a', ae)_\tau.$$

(AVI) If $(a', a)_\tau$ and $(b', b)_\tau$ belong to K_τ^0 and $(c', c) \in K$, then $[(a', a)_\tau \circ (b', b)_\tau](c', c) = (a', a)_\tau(c', c) \circ (b', b)_\tau(c', c)$.

(AVII) For each element (c', b) of $L \times R$, there exists an element $(x', y)_\tau$ of $\hat{T}$ such that $(b', b)_\tau \circ (c', c)_\tau = (x', y)_\tau(b', c) = (b'x', yc)_\tau$, for any $b' \in L$ and $c \in R$ such that (b', b) and (c', c) belong to K.

REMARK. In **[1; 2]** an axiom (AV) is also postulated. However, we have shown in **[4]** that this axiom is a consequence of axioms (AI–IV, VII).

Using these axioms, Proposition 3.5 of **[2]** shows that an element $(x', y)_\tau$ satisfying (AVII) exists in T^0, and this element is uniquely determined by b and c'; denoting it by $b \star c'$ the equation in (AVII) becomes

$$(b', b)_\tau \circ (c', c)_\tau = (b \star c')(b', c). \tag{1.3}$$

By a *box frame* we mean a system $(L, e'; R, e; K)$ satisfying axioms (AI–III). By a *banded box frame* $(L, e'; R, e; K_\tau(\circ))$ we mean a box frame $(L, e'; R, e; K)$ together with a binary operation $(\circ)$ on K_τ satisfying (AIV–VII).

THEOREM A **[1]**. *Let $(L, e'; R, e; K_\tau(\circ))$ be a banded box frame, and let $T = (L \times R)/\tau$. Define a binary operation $(\circ)$ on T by*

$$(1.4) \qquad (a', b)_\tau \circ (c', d)_\tau = (b \star c')(a', d),$$

where $b \star c'$ *is the element of* T^0 *defined by* (1.3). *Then the following hold*:

(i) $T(\circ)$ *is a bisimp e orthodox semigroup.*

(ii) *The band of idempotents of* $T(\circ)$ *is precisely* $K_\tau(\circ)$.

(iii) *The mapping* $b \to (e', b)_\tau$ *is an isomorphism of* R *onto the* $\mathcal{R}$*-class* R_ϵ *of* $T(\circ)$ *containing the idempotent* $\epsilon = (e', e)_\tau$, *and* $a' \to (a', e)_\tau$ *is an isomorphism of* L *onto* L_ϵ.

(iv) *For arbitrary* $a' \in L$ *and* $b \in R$, $(a', b) \in K$ *if and only if* $(a', e)_\tau$ *and* $(e', b)_\tau$ *are inverse to each other in* $T(\circ)$.

Conversely, if S *is a bisimple orthodox semigroup, if* e *is an idempotent of* S, *and* K_e *is the set of all mutually inverse pairs* (a', a) *in* $L_e \times R_e$, *then* K_e *is an anti-correlation between* L_e *and* R_e *and* $(L_e, e; R_e, e; K_e)$ *is a box frame. Define* τ *on* $L_e \times R_e$ *by* $(a', b)\tau(c', d)$ *if and only if* $c' = a'u^{-1}$ *and* $d = ub$ *for some* u *in* H_e, *and let* $T_e = (L_e \times R_e)/\tau$, $K_\tau = K_e/\tau$. *The mapping* $\theta\colon T_e \to S$ *defined by* $(a', b)_\tau\theta = a'b$ *is a bijection, and maps* K_τ *onto* E_S. *The latter enables us to define a binary operation* $(\circ)$ *on* K_τ *by*

$$(a', a)_\tau \circ (b', b)_\tau = (c', c)_\tau \text{ if and only if } (a'a)(b'b) = c'c.$$

Then $(L_e, e; R_e, e; K_\tau(\circ))$ *is a banded box frame. Defining* $\star\colon R \times L \to T^0$ *by* (1.3), *and then* $(\circ)$ *on* T_e *by* (1.4), *the above mapping* θ *is an isomorphism of* $(T_e, \circ)$ *onto* S.

REMARKS. (1) From the proof of Theorem A in **[2]**, it is clear that the operation $(\circ)$ on T extends the band operation $(\circ)$ on $K_\tau(\circ)$.

(2) For purposes in the next section, we reformulate the operation $(\circ)$ on T as follows. According to equations (1.4), (1.3), (1.2), and axiom (AVII), for any $(a', b)_\tau$ and $(c', d)_\tau$ in T, we have $(a', b)_\tau \circ (c', d)_\tau = (a'x', yd)_\tau$, where $(x', y)_\tau$ is the unique element in T^0 such that

$$(b', b)_\tau \circ (c', c)_\tau = (b'x', yc)_\tau$$

for any $b' \in L$ and $c \in R$ such that (b', b), (c', c) are in K.

The following lemma is immediate from (R3); we shall use it and its left-right dual frequently without explicit mention.

LEMMA 1.1. *Let* e *be a left identity for a right Reilly groupoid* R. *If* $ac = bc$ *with* a, b *in* $\hat{R}e$ *and* c *in* R, *then* $a = b$. *In particular,* $\hat{R}e$ *is right cancellative.*

LEMMA 1.2. *Let* $T(\circ)$ *be a bisimple orthodox semigroup constructed from a banded box frame* $(L, e'; R, e; K_\tau(\circ))$ *as in Theorem A. Then in* $T(\circ)$

(i) $(a', b)_\tau \mathcal{R} (c', d)_\tau$ *if and only if* $a' = c'u'$ *for some* $u \in H_e$;
(ii) $(a', b)_\tau \mathcal{L} (c', d)_\tau$ *if and only if* $b = vd$ *for some* $v \in H_e$;
(iii) $(a', b)_\tau \mathcal{H} (c', d)_\tau$ *if and only if* $a' = c'u'$ *and* $b = vd$ *for some* u, v *in* H_e.

Proof. Suppose $(a', b)_\tau \mathcal{R} (c', d)_\tau$ in $T(\circ)$; then there are elements $(x', y)_\tau$ and $(w', z)_\tau$ in T such that $(a', b)_\tau = (c', d)_\tau \circ (x', y)_\tau$ and $(c', d)_\tau = (a', b)_\tau \circ (w', z)_\tau$. Thus $(a', b)_\tau = (c'p', qy)_\tau$ for some $(p', q)_\tau \in T^0 = (e'\hat{L} \times \hat{R}e)/\tau$ and $(c', d)_\tau = (a'r', sz)_\tau$ for some $(r', s)_\tau \in T^0$. Then $a' = (c'p')u'$ and $b = u(qy)$, some $u \in H_e$, and $c' = (a'r')v'$ and $d = v(sz)$, some $v \in H_e$, so

$$a'e' = a' = c'(p'u') = (a'r'v')(p'u') = a'(r'v'p'u').$$

Since e' and $r'v'p'u'$ belong to $e'\hat{L}$, the dual of Lemma 1.1 gives $e' = r'v'p'u'$, whence $p'u'$ is a unit in $H_{e'}$. Therefore $a' = c'(p'u')$ with $p'u' \in H_{e'}$, as desired.

Conversely, suppose $(a', b)_\tau$, $(c', d)_\tau$ belong to $T(\circ)$ with $a' = c'u'$ for some $u \in H_e$ (hence $u' \in H_{e'}$). By (AC1), let $b' \in L$ such that $(b', b) \in K$. Then

$$\begin{aligned}(a', b)_\tau \circ (b', ud)_\tau &= (b \star b')(a', ud) \\ &= (e', e)_\tau (a', ud) \text{ by Proposition 4.5 of [2]} \\ &= (a'e', eud)_\tau \\ &= (a', ud)_\tau \\ &= (a'(u')^{-1}, d)_\tau \\ &= (c', d)_\tau.\end{aligned}$$

By similarity we conclude that $(a', b)_\tau \mathcal{R} (c', d)_\tau$ in $T(\circ)$. Assertion (ii) is dual to (i), and (iii) is immediate from (i) and (ii).

2. Idempotent-separating congruences. Let e be a left identity of a right Reilly groupoid R, and let us call a subgroup V of H_e a *left normal divisor of* $\hat{R}e$ provided that $aV \subseteq Va$ for every a in $\hat{R}e$. Clearly, such a subgroup is a normal subgroup of H_e. Dually, if e' is a right identity for a left Reilly groupoid L, we call a subgroup V' of $H_{e'}$ a *right normal divisor* of $e'\hat{L}$ if $V'b' \subseteq b'V'$ for every $b' \in e'\hat{L}$. These concepts are due to Rees [6].

Let $T(\circ)$ be a bisimple orthodox semigroup constructed from a banded box frame $(L, e'; R, e; K_\tau(\circ))$ as in Theorem A. If $V[V']$ is a left [right] normal divisor of $\hat{R}e[e'\hat{L}]$ such that the anti-isomorphism from H_e to $H_{e'}$ maps V onto V', we call the pair (V, V') a *linked pair* of left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$. The following theorem is the analogue of Theorem 2.4 of **[8]**.

THEOREM 2.1. *Let $T(\circ)$ be a bisimple orthodox semigroup constructed from a banded box frame $(L, e'; R, e; K_\tau(\circ))$ as in Theorem A. Let (V, V') be a linked pair of left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$. Define a relation $\rho_{(V,V')}$ on $T(\circ)$ as follows:*

$(a', b)_\tau \rho_{(V,V')} (c', d)_\tau$ if and only if there are elements u, v of H_e such that $a' = c'u'$, $b = vd$, and $u^{-1}v \in V$. Then $\rho_{(V,V')}$ is an idempotent-separating congruence on $T(\circ)$. Moreover, if (V_1, V_1') and (V_2, V_2') are two pairs of linked left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$, then $\rho_{(V_1,V_1')} \subseteq \rho_{(V_2,V_2')}$ if and only if $V_1 \subseteq V_2$.

Conversely, if ρ is any idempotent-separating congruence on $T(\circ)$ then there is a linked pair (V, V') of left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$ such that $\rho = \rho_{(V,V')}$, namely, $V = \{v \in H_e : (e', v)_\tau \rho (e', e)_\tau\}$ and $V' = \{v' \in H_{e'}: (v', e)_\tau \rho (e', e)_\tau\}$.

Proof. It is clear from Lemma 1.2 (iii) that the relation $\rho_{(V,V')}$ is contained in $\mathcal{H}$, so that $\rho_{(V,V')}$ separates idempotents. It is also clear that $\rho_{(V,V')}$ is reflexive.

For symmetry, suppose $(a', b)_\tau \rho_{(V,V')} (c', d)_\tau$, say $a' = c'u'$ and $b = vd$ for some $u, v \in H_e$ with $u^{-1}v \in V$. Then $c' = a'(u')^{-1} = a'(u^{-1})'$, $d = v^{-1}b$, and since V is a normal subgroup of H_e, $u^{-1}v \in V$ implies $uv^{-1} \in V$.

For transitivity, let $(a', b)_\tau \rho_{(V,V')} (c', d)_\tau$ and $(c', d)_\tau \rho_{(V,V')} (g', f)_\tau$, say $a' = c'u'$, $b = vd$, $c' = g'w'$ and $d = xf$ with $u, v, w, x \in H_e$ and $u^{-1}v$, $w^{-1}x \in V$. Then

$$a' = c'u' = (g'w')u' = g'(w'u') = g'(uw)' \quad \text{and}$$
$$b = vd = v(xf) = (vx)f \quad \text{with}$$

$(uw)^{-1}(vx) = w^{-1}(u^{-1}v)w\, w^{-1}x \in V$ since $u^{-1}v$, $w^{-1}x \in V$. Thus $\rho_{(V,V')}$ is an equivalence on $T(\circ)$.

To see that $\rho_{(V,V')}$ is a congruence on $T(\circ)$, let $(a', b)_\tau \rho_{(V,V')} (c', d)_\tau$, say $a' = c'u'$ and $b = vd$ with $u, v \in H_e$, $u^{-1}v \in V$. Let $(x', y)_\tau$ be arbitrary in $T(\circ)$; for right compatibility we must show $(a', b)_\tau \circ (x', y)_\tau \rho_{(V,V')} (c', d)_\tau \circ (x', y)_\tau$. Now $(a, b)_\tau \circ (x', y)_\tau = (a'p', qy)_\tau$ where $(p', q)_\tau$ is the *unique* element of $T^0 = (e'\hat{L} \times \hat{R}e)/\tau$ such that

$$(b', b)_\tau \circ (x', x)_\tau = (b'p', qx)_\tau \tag{2.1}$$

for any $b' \in L$ and $x \in R$ such that (b', b) and (x', x) are in K.

Likewise, $(c', d)_\tau \circ (x', y)_\tau = (c'r', sy)_\tau$ where $(r', s)_\tau$ is the *unique* element of T^0 such that

$$(d', d)_\tau \circ (x', x)_\tau = (d'r', sx)_\tau \tag{2.2}$$

for any $d' \in L$ and $x \in R$ such that (d', d) and (x', x) are in K.

By (AC1), choose $d' \in L$ and $x \in R$ such that (d', d), (x', x) are in K. Since $(d', d) \in K$ and $(v', v) \in \hat{K}$, (AC4) implies $(d'v', vd) \in K$, i.e., $(d'v', b) \in K$. Thus from (2.1) we have

$$(d'\,v';\, b)_\tau \circ (x', x)_\tau = (d'\,v'\,p', qx)_\tau, \quad \text{i.e.,}$$

$$(d'v', vd)_\tau \circ (x', x)_\tau = (d'v'p', qx)_\tau, \quad \text{so that}$$

$$(d', d)_\tau \circ (x', x)_\tau = (d'\,v'\,p', qx)_\tau. \quad \text{Comparing} \tag{2.3}$$

(2.3) with (2.2), the uniqueness of $(r', s)_\tau$ in T^0 forces $(r', s)_\tau = (v'p', q)_\tau$ [note that $v' \in V' \subseteq H_{e'} \subseteq e'\hat{L}$, and $p' \in e'\hat{L}$, implies $v'p' \in e'\hat{L}$; and $q \in \hat{R}e$, so that $(v'p', q)_\tau \in (e'\hat{L} \times \hat{R}e)/\tau = T^0$].

From $(r', s)_\tau = (v'p', q)_\tau$, there exists $\alpha \in H_e$ such that $q = \alpha s$ and $v'p' = r'\alpha'$. Then $p' = (v')^{-1}r'\alpha' = (v^{-1})'r'\alpha'$. Now

$$\begin{aligned} a'p'(\alpha')^{-1} &= a'[(v^{-1})'r'\alpha'](\alpha')^{-1} \\ &= a'(v^{-1})'r' \\ &= c'u'(v^{-1})'r' \\ &= c'(v^{-1}u)'r', \end{aligned}$$

and $u^{-1}v \in V$ implies $v^{-1}u \in V$, whence $(v^{-1}u)' \in V'$. Since $r' \in e'\hat{L}$ and V' is a right normal divisor of $e'\hat{L}$, $V'r' \subseteq r'V'$. Thus $(v^{-1}u)'r' = r'z'$, some $z' \in V'$ (so $z \in V$). Consequently, $a'p'(\alpha')^{-1} = c'(v^{-1}u)'r' = c'r'z'$ and so $a'p' = (c'r'z')\alpha' = (c'r')(\alpha z)'$ with $\alpha z \in H_e$ (since $\alpha \in H_e$ and $z \in V \subseteq H_e$).

Then $a'p' = (c'r')(\alpha z)'$ with $\alpha z \in H_e$, and $qy = (\alpha s)y = \alpha(sy)$ with $\alpha \in H_e$, and $(\alpha z)^{-1}\alpha = z^{-1}\alpha^{-1}\alpha = z^{-1} \in V$. Therefore $(a'p', qy)_\tau$ $\rho_{(V,V')}(c'r', sy)_\tau$ and $\rho_{(V,V')}$ is right compatible; left compatibility is dual.

If (V_1, V_1') and (V_2, V_2') are two pairs of linked left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$, the definitions of $\rho_{(V_1, V_1')}$ make it clear that if $V_1 \subseteq V_2$ (and hence $V_1' \subseteq V_2'$) then $\rho_{(V_1, V_1')} \subseteq \rho_{(V_2, V_2')}$. The converse is evident since, for any linked pair (V, V'), we have

$$V = \{v \in H_e\colon (e', v)_\tau \rho_{(V,V')} (e', e)_\tau\}, \qquad \text{and}$$

$$V' = \{v' \in H_{e'}\colon (v', e)_\tau \rho_{(V,V')} (e', e)_\tau\}.$$

Turning to the converse assertion of the theorem, let ρ be any idempotent-separating congruence on $T(\circ)$, and let N be the ρ-class containing $\epsilon = (e', e)_\tau$. Since $\rho \subseteq \mathcal{H}$ in $T(\circ)$, $N \subseteq H_\epsilon = R_\epsilon \cap L_\epsilon$, and by Theorem A(iii),

$$R_\epsilon = \{(e', b)_\tau : b \in R\} \qquad \text{and}$$
$$L_\epsilon = \{(a', e)_\tau : a' \in L\}.$$

For any $x \in N$, $x \in R_\epsilon$ implies $x = (e', b)_\tau$ for some $b \in R$; but $x \in L_\epsilon$ implies $(e', b)_\tau \mathscr{L} (e', e)_\tau$, whence Lemma 1.2(ii) gives $b = ue = u$, some $u \in H_e$. Thus any $x \in N$ can be written as

$$x = (e', u)_\tau = ((u^{-1})', e)_\tau, \quad \text{some} \quad u \in H_e.$$

Let
$$V = \{v \in H_e : (e', v)_\tau \text{ is in } N\} \qquad \text{and}$$
$$V' = \{v' \in H_e : (v', e)_\tau \text{ is in } N\}.$$

Then $V[V']$ is a subgroup of $H_e[H_{e'}]$, and we shall show that (V, V') is a linked pair of left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$ such that $\rho = \rho_{(V, V')}$.

To see that the anti-isomorphism $u \to u'$ from H_e to $H_{e'}$ maps V onto V', let $v \in V$. Then $(e', v)_\tau \in N$, and

$$(e', v)_\tau = [e'(v^{-1})', v^{-1}v]_\tau = [(v^{-1})', e]_\tau$$

implies $(v^{-1})' \in V'$. Since $(v^{-1})' = (v')^{-1}$ and V' is a group, we have $v' \in V'$. Thus $u \to u'$ maps V into V'. On the other hand, if $v' \in V'$ then $(v', e)_\tau \in N$, so $(v', e)_\tau = [v'(v^{-1})', v^{-1}e]_\tau = (e', v^{-1})_\tau$ shows v^{-1}, and hence v, is in V. Thus V maps onto V'.

To see that V is a left normal divisor of $\hat{R}e$, first note that by Lemma 3 of **[5]**, N is a left normal divisor of

$$P_\epsilon = \{x \text{ in } R_\epsilon : x \circ \epsilon = x\}$$
$$= \{(e', b)_\tau : b \in R \text{ and } (e', b)_\tau \circ (e', e)_\tau = (e', b)_\tau\}$$

and a right normal divisor of

$$Q_\epsilon = \{x \text{ in } L_\epsilon : \epsilon \circ x = x\}$$
$$= \{(a', e)_\tau : a' \in L \text{ and } (e', e)_\tau o (a', e)_\tau = (a', e)_\tau\}.$$

Since the isomorphism $a \to (e', a)_\tau$ of R onto R_ϵ carries $\hat{R}e$ onto P_ϵ and V onto N, it follows that V is a left normal divisor of $\hat{R}e$. Similarly, the isomorphism $a' \to (a', e)_\tau$ of L onto L_ϵ carries $e'\hat{L}$ onto Q_ϵ and V' onto N, so V' is a right normal divisor of $e'\hat{L}$.

Before proceeding to show that $\rho = \rho_{(V, V')}$, we note a lemma and corollary which will shorten our work.

LEMMA 2.2. *If* $(b', b) \in K$ *then* $(a', b)_\tau \circ (b', c)_\tau = (a', c)_\tau$.

Proof.

$$(a', b)_\tau \circ (b', c)_\tau = (b \star b')(a', c)_\tau = (e', e)_\tau (a', c)$$
$$= (a'e', ec)_\tau = (a', c)_\tau.$$

COROLLARY 2.3. *If* $u, v \in H_e$, *and* (c', c) *and* (d', d) *are in* K *then*

$$(a', c)_\tau \circ (c'u', vd)_\tau \circ (d', b)_\tau = (a', u^{-1}vb)_\tau.$$

Proof.

$$\begin{aligned}(a', c)_\tau \circ (c'u', vd)_\tau \circ (d', b)_\tau &= (a', c)_\tau \circ (c', u^{-1}vd)_\tau \circ (d', b)_\tau \\ &= (a', u^{-1}vd)_\tau \circ (d', b)_\tau \\ &= (a'(v^{-1}u)', d)_\tau \circ (d', b)_\tau \\ &= (a'(v^{-1}u)', b)_\tau \\ &= (a', u^{-1}vb)_\tau.\end{aligned}$$

To see that $\rho = \rho_{(V, V')}$, let $(a', b)_\tau \rho_{(V, V')} (c', d)_\tau$, say $a' = c'u'$, $b = vd$, $u, v \in H_e$, and $u^{-1}v \in V$. Then $(e', u^{-1}v)_\tau \in N$, so $(e', u^{-1}v)_\tau \rho (e', e)_\tau$. Therefore

$$(e', u^{-1}v)_\tau \circ (e', d)_\tau \rho (e', e)_\tau \circ (e', d)_\tau,$$

that is, $(e', u^{-1}vd)_\tau \rho (e', d)_\tau$. But then

$$(c', e)_\tau \circ (e', u^{-1}vd)_\tau \rho (c', e)_\tau \circ (e', d)_\tau,$$

that is, $(c', u^{-1}vd)_\tau \rho (c', d)_\tau$. But $(c', u^{-1}vd)_\tau = (c'u', vd)_\tau = (a', b)_\tau$, whence $(a', b)_\tau \rho (c', d)_\tau$ and $\rho_{(V, V')} \subseteq \rho$.

Conversely, let $(a', b)_\tau \rho (c', d)_\tau$. Since $\rho \subseteq \mathcal{H}$, Lemma 1.2(iii) says $a' = c'u'$ and $b = vd$ for some $u, v \in H_e$. Then $(a', b)_\tau = (c'u', vd)_\tau$ so that $(c'u', vd)_\tau \rho (c', d)_\tau$. By (AC1) let $c \in R$ and $d' \in L$ such that (c', c) and (d', d) are in K. Then

$$(e'c)_\tau \circ (c'u', vd)_\tau \circ (d', e)_\tau \rho (e', c)_\tau \circ (c', d)_\tau \circ (d', e)_\tau.$$

Applying Corollary 2.3 to both sides of this last relation we obtain

$$(e', u^{-1}v)_\tau \rho (e', e)_\tau.$$

This puts $u^{-1}v \in V$, and so $\rho \subseteq \rho_{(V,V')}$. The proof is complete.

As an immediate corollary we obtain

COROLLARY 2.4. *Let $T(\circ)$ be a bisimple orthodox semigroup constructed from a banded box frame $(L, e'; R, e; K_\tau(\circ))$ as in Theorem A. There is a one-to-one, inclusion preserving correspondence between the idempotent-separating congruences on $T(\circ)$ and the linked pairs (V, V') of left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$.*

COROLLARY 2.5. *Let $T(\circ)$ be as in Corollary 2.4. Then $\mathcal{H}$ is a congruence on $T(\circ)$ if and only if $(H_e, H_{e'})$ is a linked pair of left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$.*

REMARK. Let S be any bisimple orthodox semigroup, and e any idempotent in S. By Theorem A, $S \approx T_e$ where T_e is constructed from the box frame $(L_e, e; R_e, e; K_e)$ as in Theorem A. Now by definition, $\hat{R}_e = \{a \in R_e \colon ae \in R_e\}$ and clearly $\hat{R}_e e = \{x \in R_e \colon xe = x\} = R_e \cap eSe = P_e$. Similarly, $e\hat{L}_e = \{x \in L_e \colon ex = x\} = L_e \cap eSe = Q_e$. Since H_e is the group of units of eSe, H_e is then the group of units of $\hat{R}_e e = P_e$ and of $e\hat{L}_e = Q_e$. Thus the anti-isomorphism of Proposition 3.1 in **[2]** is the mapping $u \to u^{-1}$ on H_e.

Suppose we identify S with T_e. Then if (V, V') is a linked pair of left and right normal divisors of $\hat{R}_e e$ and $e\hat{L}_e$, we have $V' = V^{-1} = V$, so that V is a subgroup of H_e satisfyiing (1) $aV \subseteq Va$ for every a in $\hat{R}_e e = P_e$, and (2) $Vb \subseteq bV$ for every b in $e\hat{L}_e = Q_e$. Thus V is a subgroup of H_e which is a left normal divisor of P_e and a right normal divisor of Q_e.

So the one-to-one correspondence in Corollary 2.4 is just the one stated in Munn's theorem **[5]**.

3. A construction of the quotient semigroup. For any bisimple orthodox semigroup T and any idempotent-separating congruence ρ on T, T/ρ is also a bisimple orthodox semigoup, and so the converse half of Theorem A gives a construction for T/ρ in terms of a banded box-frame whose components are internal to T/ρ. In this section we show how Theorem A may be used to describe a construction of T/ρ in terms involving the original box-frame used to construct T.

Recall from **[1]** that by a congruence on a right Reilly groupoid R we mean an equivalence relation ρ on R satisfying the following conditions.

(CR1) $\hat{R}$ is a union of ρ-classes.

(CR2) If $(a, b) \in \rho \cap (\hat{R} \times \hat{R})$ and $c \in R$, then $(ac, bc) \in \rho$.

(CR3) If $(a, b) \in \rho$ and $c \in \hat{R}$, then $(ca, cb) \in \rho$.

A congruence on a left Reilly groupoid is defined dually. The following lemma generalizes Lemma 2.2 of **[8]** to right Reilly groupoids.

LEMMA 3.1. *Let R be a right Reilly groupoid, e a fixed left identity for R, and V a left normal divisor of $\hat{R}e$. Then*

$$\sigma_V = \{(a, b) \in R \times R\colon a = ub, \text{ some } u \in V\} \tag{3.1}$$

is a congruence on R such that $\sigma_V \subseteq \mathscr{L}$ and (R3) *holds for R/σ_V.*

Conversely, if σ is a congruence on R such that $\sigma \subseteq \mathscr{L}$ and (R3) *holds for R/σ, then $\sigma = \sigma_V$ where $V = e\sigma$, and V is a left normal divisor of $\hat{R}e$.*

Proof. Clearly σ_V is an equivalence relation on R. For (CR1), suppose $a\sigma_V b$ with $a \in \hat{R}$. Then $b = ua$, some $u \in V$. Since $a, u \in \hat{R}$, so is $b = ua$, whence it follows that $\hat{R}$ is a union of σ_V-classes.

To show (CR2), let $a\sigma_V b$ with $a, b \in \hat{R}$ and let $c \in R$. Then $a = ub$, some $u \in V$, so that $ac = (ub)c = u(bc)$ and $ac\sigma_V bc$.

Turning to (CR3), let $a\sigma_V b$ and $c \in \hat{R}$, say $a = ub$, some $u \in V$. Now ce lies in $\hat{R}e$, so by left normality, $(ce)u = v(ce)$ for some $v \in V$. Then $ca = c(ea) = (ce)a = (ce)(ub) = vceb = v(cb)$, and so $ca\sigma_V cb$. Thus σ_V is a congruence on R, and $\sigma_V \subseteq \mathscr{L}$ is clear.

As noted on p. 15 of **[2]**, R/σ_V becomes a partial groupoid satisfying (R1), (R2), and (R4) if we define $(a\sigma_V)(b\sigma_V) = (ab)\sigma_V$ if $a \in \hat{R}$, and $(a\sigma_V)(b\sigma_V)$ is undefined otherwise. To see that R/σ_V satisfies (R3), let $a\sigma_V$, $b\sigma_V$, and $c\sigma_V$ belong to R/σ_V such that $(a\sigma_V)(c\sigma_V) = (b\sigma_V)(c\sigma_V)$. Then $a, b \in \hat{R}$ and $(ac)\sigma_V(bc)$. Thus $ac = u(bc) = (ub)c$, some $u \in V$. Condition (R3) in R then implies $ax = (ub)x$ for any $x \in R$, so $ax = u(bx)$. Thus $ax\sigma_V bx$, or $(a\sigma_V)(x\sigma_V) = (b\sigma_V)x\sigma_V)$ as desired.

Conversely, let σ be a congruence on R such that $\sigma \subseteq \mathscr{L}$ and (R3) holds in R/σ. Let $V = e\sigma$. If $a\sigma b$ then $a\mathscr{L}b$, whence $a = ub$ for some $u \in H_e$ by Proposition 1.2 of **[2]**. Then

$$(e\sigma)(b\sigma) = (eb)\sigma = b\sigma = a\sigma = (ub)\sigma = (u\sigma)(b\sigma). \tag{3.2}$$

Since (R3) holds for R/σ, $e\sigma = (e\sigma)(e\sigma) = (u\sigma)(e\sigma) = (ue)\sigma = u\sigma$ so $u \in V$. Thus $a\sigma_V b$ and $\sigma \subseteq \sigma_V$. On the other hand, if $a\sigma_V b$, say $a = ub$ with $u \in V = e\sigma$, then $a\sigma = (ub)\sigma = (u\sigma)(b\sigma) = (e\sigma)(b\sigma) = b\sigma$. Thus $\sigma = \sigma_V$.

Now $\sigma \subseteq \mathscr{L}$, so $x \in V$ implies $x = ue = u$ for some $u \in H_e$. Also, conditions (CR1)–(CR3) may be applied to show that V is a subgroup of H_e. If $a \in \hat{R}e$ and $v \in V$, then $(av)\sigma = (a\sigma)(v\sigma) = (a\sigma)(e\sigma) = (ae)\sigma = a\sigma$ so that $av\sigma_V a$. Thus $av = ua$ for some $u \in V$, and V is a left normal divisor of $\hat{R}e$. The proof is complete.

Now let $T(\circ)$ be a bisimple orthodox semigroup constructed from a banded box-frame $(L, e'; R, e; K_\tau(\circ))$ as in Theorem A. Let $\rho = \rho_{(V, V')}$ be an idempotent-separating congruence on $T(\circ)$, where (V, V') is a linked pair of left and right normal divisors of $\hat{R}e$ and $e'\hat{L}$. Define $\sigma = \sigma_V$ as in Lemma 3.1 and $\sigma' = \sigma_{V'}$ dually; i.e.,

$$\sigma' = \sigma_{V'} = \{(a', b') \in L \times L\colon a' = b'u', \text{ some } u' \in V'\}.$$

Since (V, V') is a linked pair, we may write

$$\sigma' = \sigma_{V'} = \{(a', b') \in L \times L\colon a' = b'u', \text{ some } u \in V\}$$

where, as usual, u' is the unique element of $H_{e'}$ such that $(u', u) \in K$. By Lemma 3.1 $R_1 = R/\sigma$ is a right Reilly groupoid with left identity $e\sigma$; $\hat{R}_1 = \hat{R}/\sigma$, and $\hat{R}_1(e\sigma) = (\hat{R}e)/\sigma$ since $\hat{R}e$, like $\hat{R}$, is a union of σ-classes. Moreover, the group of units $H_{e\sigma}$ of $\hat{R}_1(e\sigma)$ is precisely H_e/σ. To verify this last assertion, assume $a\sigma \in H_{e\sigma}$ say $(b\sigma)(a\sigma) = e\sigma$ for some $b\sigma$ in $\hat{R}_1(e\sigma) = (\hat{R}e)/\sigma$. Since $a\sigma$, $b\sigma$ are in $(\hat{R}e)/\sigma$ we have $a, b \in \hat{R}e$. From $(ba)\sigma e$ there exists $u \in V$ such that $e = u(ba) = (ub)a$. Now $u \in V \subseteq \hat{R}e \subseteq \hat{R}$ and $b \in \hat{R}e$ imply $ub \in \hat{R}\hat{R}e \subseteq \hat{R}e$. Then $e = (ub)a$ with ub, a in $\hat{R}e$ implies, by Proposition 1.3 of **[2]** that $a \in H_e$, whence $a\sigma \in H_e/\sigma$. The converse is immediate.

Dually, $L_1 = L/\sigma'$ is a left Reilly groupoid with right identity $e'\sigma'$ and $\hat{L}_1 = \hat{L}/\sigma'$, $(e'\sigma')\hat{L}_1 = (e'\hat{L})/\sigma'$, and the group of units $H_{e'\sigma'}$ of $(e'\sigma')\hat{L}_1$ is just $H_{e'}/\sigma'$.

We now define an anti-correlation K_1 between L_1 and R_1 as follows:

(3.3) $K_1 = \{(A', B) \in L_1 \times R_1$: there exists $a' \in A'$, $b \in B$ with $(a', b) \in K\}$.

Instead of checking directly to see that K_1 is an anti-correlation by showing (AC1)–(AC4), we shall deduce the result from the converse half of Theorem A. This approach will likewise be taken throughout the remainder of this section.

Let $\epsilon = (e', e)_\tau$ in T. From the definition of $\rho = \rho_{(V, V')}$ we have

$$(e', b)_\tau \rho (e', d)_\tau \Leftrightarrow b = vd \text{ with } v \in V \Leftrightarrow b\sigma d,$$
$$(a', e)_\tau \rho (c', e)_\tau \Leftrightarrow a' = c'u' \text{ with } u \in V \Leftrightarrow a'\sigma' c'.$$

Thus it follows that the map $b\sigma \to (e', b)_\tau \rho$ is a (partial) isomorphism of $R_1 = R/\sigma$ onto R_ϵ/ρ, and $a'\sigma' \to (a', e)_\tau \rho$ is an isomorphism of $L_1 = L/_{\sigma'}$ onto L_ϵ/ρ. Moreover, since ρ is idempotent separating, $R_\epsilon/\rho = R_{\epsilon\rho}$ and $L_\epsilon/\rho = L_{\epsilon\rho}$ in $T(\circ)/\rho$.

By the converse part of Theorem A, there is an anti-correlation, say K_1, between $L_{\epsilon\rho}$ and $R_{\epsilon\rho}$ in $T(\circ)/\rho$, namely

$$K_1 = \{(\alpha', \alpha) \in L_{\epsilon\rho} \times R_{\epsilon\rho}\colon \alpha' \text{ and } \alpha \text{ are mutually inverse}\}.$$

Since $\alpha \in R_{\epsilon\rho}$ implies $\alpha = (e', b)_\tau \rho$ for some $(e', b)_\tau$ in R_ϵ, and $\alpha' \in L_{\epsilon\rho}$ implies $\alpha' = (a', e)_\tau \rho$ for some $(a', e)_\tau$ in L_ϵ, K_1 consists of all mutually inverse pairs $((a',e)_\tau\rho, (e',b)_\tau\rho)$ in $T(\circ)/\rho$. So identifying $a'\sigma'$ with $(a',e)_\tau\rho$, and $b\sigma$ with $(e',b)_\tau\rho$ by the above isomorphisms, we may regard K_1 as consisting of all mutually inverse pairs $(a'\sigma', b\sigma)$ in $L_1 \times R_1$. Therefore,

$$\begin{aligned}(a'\sigma', b\sigma) \in K_1 &\Leftrightarrow (a',e)_\tau\rho \text{ and } (e',b)_\tau\rho \text{ are inverse pairs in } T(\circ)/\rho\\ &\Leftrightarrow (e',b)_\tau\rho \cdot (a',e)_\tau\rho = (e',e)_\tau\rho \text{ by Lemma 2.12 of [3]}\\ &\Leftrightarrow (b \star a')\rho(e',e)_\tau.\end{aligned}$$

Let $(b \star a') = (x',y)_\tau$. Now $(x',y)_\tau\rho(e',e)_\tau$ if and only if $x, y \in H_e$ and $x^{-1}y \in V$. Since $(x,y)_\tau = (e', x^{-1}y)_\tau$ it follows that

$$(a'\sigma', b\sigma) \in K_1 \Leftrightarrow b \star a' = (e',v)_\tau, \text{ some } v \in V.$$

Using Proposition 3.6 of [2] we have

$$(b \star a'v')(e',v) = b \star a' = (e',v)_\tau = (e',e)_\tau(e',v)$$

so Proposition 3.4 of [2] gives $b \star a'v' = (e',e)_\tau$. Conversely, $b \star a'v' = (e',e)_\tau$ implies $b \star a' = (e',v)_\tau$. By Proposition 4.5 of [2], $b \star a'v' = (e',e)_\tau$ is equivalent to $(a'v', b) \in K$. Since $(a'v')\sigma' a'$, we can choose a' (namely, replace it by $a'v'$) so that $(a',b) \in K$. Thus K_1 is the same as defined by (3.3), and is an anti-correlation between L_1 and R_1. Moreover, under the isomorphisms we have $e'\sigma' \to (e',e)_\tau\rho = \epsilon\rho$ and $e\sigma \to (e',e)_\tau\rho = \epsilon\rho$, so Theorem A assures us that $(L_1, e'\sigma'; R_1, e\sigma; K_1)$ is a box-frame.

Now $H_\epsilon = R_\epsilon \cap L_\epsilon = \{(e',u)_\tau : u \in H_e\}$. Therefore the ismorphism $b\sigma \to (e',b)_\tau\rho$ of R/σ onto R_ϵ/ρ, when restricted to H_e/σ, is an isomorphism of H_e/σ onto H_ϵ/ρ. Also, since ρ is idempotent-separating, we have $(L_\epsilon \cap R_\epsilon)/\rho = L_\epsilon/\rho \cap R_\epsilon/\rho$, whence

$$H_\epsilon/\rho = (L_\epsilon \cap R_\epsilon)/\rho = L_\epsilon/\rho \cap R_\epsilon/\rho = L_{\epsilon\rho} \cap R_{\epsilon\rho} = H_{\epsilon\rho}.$$

Similarly, the map $u'\sigma' \to (u',e)_\tau\rho$ is an isomorphism of $H_{e'}/\sigma'$ onto $H_\epsilon/\rho = H_{\epsilon\rho}$.

By the converse part of Theorem A, consider the relation τ_1 on $L_{\epsilon\rho} \times R_{\epsilon\rho}$ defined by

$$((a',e)_\tau\rho, (e',b)_\tau\rho)\tau_1((c',e)_\tau\rho, (e',d)_\tau\rho) \Leftrightarrow (c',e)_\tau\rho = (a',e)_\tau\rho \cdot [(e',u)_\tau\rho]^{-1}$$

and

$$(e',d)_\tau\rho = (e',u)_\tau\rho \cdot (e',b)_\tau\rho$$

for some $(e', u)_\tau\rho$ in $H_{e\rho}$. Note that $[(e', u)_\tau\rho]^{-1} = (e', u^{-1})_\tau\rho = (u', e)_\tau\rho$ in $H_{e\rho}$. Identifying under our isomorphisms, we may regard τ_1 as being defined on $L_1 \times R_1$ by

$$(3.4)\qquad (a'\sigma', b\sigma)\tau_1(c'\sigma', d\sigma) \Leftrightarrow c'\sigma' = (a'\sigma')(u'\sigma') \text{ and } d\sigma = (u\sigma)(b\sigma)$$

for some $u\sigma \in H_{e\sigma} = H_e/\sigma$.

Let $T_1 = (L_1 \times R_1)/\tau_1$ and $K_{1\tau_1} = K_1/\tau_1$. By Theorem A, the mapping $\theta: T_1 \to T(\circ)/\rho$ is a bijection, where θ is defined by $((a', e)_\tau\rho, (e', b)_\tau\rho)_{\tau_1}\theta = (a', e)_\tau\rho \cdot (e', b)_\tau\rho$, and maps $K_{1\tau_1}$ upon the band $E_{T/\rho}$ of idempotents of T/ρ. Thus θ is essentially given by

$$(3.5)\qquad (a'\sigma', b\sigma)_{\tau_1}\theta = (a', b)_\tau\rho.$$

Again, Theorem A shows that $(L_1, e'\sigma'; R_1, e\sigma; K_{1\tau_1}(\circ_1))$ becomes a banded box-frame, where we use θ to transfer the operation on $E_{T/\rho}$ to an operation $(\circ_1)$ on $K_{1\tau_1}$ in the obvious way. Under our identifications the definition of $(\circ_1)$ reads as follows:

$$\begin{aligned}&(a'\sigma', a\sigma)_{\tau_1} \circ_1 (b'\sigma', b\sigma)_{\tau_1} = (c'\sigma', c\sigma)_{\tau_1}\\ &\qquad \Leftrightarrow (a', a)_\tau\rho \cdot (b', b)_\tau\rho = (c', c)_\tau\rho \text{ in } T(\circ)/\rho, \text{ where } (a', a),\\ &\qquad (b, b), \text{ and } (c', c) \text{ are in } K.\end{aligned}$$

Since $K_\tau(\circ)$ is the band of idempotents of $T(\circ)$, and ρ is idempotent-separating on $T(\circ)$, we have

$$(3.6)\qquad \begin{aligned}&(a'\sigma', a\sigma)_{\tau_1} \circ_1 (b'\sigma', b\sigma)_{\tau_1} = (c'\sigma', c\sigma)_{\tau_1}\\ &\qquad \Leftrightarrow (a', a)_\tau \circ (b', b)_\tau = (c', c)_\tau \text{ in } K_\tau(\circ).\end{aligned}$$

Because $(L_1, e'\sigma'; R_1, e\sigma; K_{1\tau_1}(\circ_1))$ is a banded box-frame, Proposition 3.5 of **[2]** holds. That is, for each element $(c'\sigma', b\sigma)$ of $L_1 \times R_1$ there exists a unique element $(X', Y)_{\tau_1}$ of $T_1^0 = ((e'\sigma')\hat{L}_1 \times \hat{R}_1(e\sigma))/\tau_1$ such that $(b'\sigma', b\sigma)_{\tau_1}\sigma_1(c'\sigma', c\sigma)_{\tau_1} = ((b'\sigma')X', Y(c\sigma))_{\tau_1}$ for any $b'\sigma'$ in L_1 and $c\sigma \in R_1$ such that $(b'\sigma', b\sigma)$ and $(c'\sigma', c\sigma)$ are in K_1. If we denote $(X', Y)_{\tau_1}$ by $(b\sigma) \bigstar_1 (c'\sigma')$, and then extend the band operation $(\circ_1)$ on $K_{1\tau_1}$ to an operation $(\circ_1)$ on T_1 by

$$(3.7)\qquad (a'\sigma', b\sigma)_{\tau_1} \circ_1 (c'\sigma', d\sigma)_{\tau_1} = ((b\sigma) \bigstar_1 (c'\sigma'))(a'\sigma', d\sigma)$$

then θ becomes an isomorphism of $T_1(\circ_1)$ onto $T(\circ)/\rho$.

The following theorem summarizes all of the results in this section.

THEOREM 3.2. *Let $T(\circ)$ be a bisimple orthodox semigroup constructed from a banded box-frame $(L, e'; R, e; K_\tau(\circ))$ as in Theorem*

A. Let $\rho = \rho_{(V, V')}$ *be an idempotent-separating congruence on* $T(\circ)$ *where* (V, V') *is a linked pair of left and right normal divisors of* $\hat{R}e$ *and* $e'\hat{L}$. *Define* $\sigma = \sigma_V$ *on* R *by* (3.1) *and* $\sigma' = \sigma_{V'}$ *on* L *dually. Then* $\sigma[\sigma']$ *is a congruence on* $R[L]$ *and* $R_1 = R/\sigma[L_1 \times L/\sigma]$ *is a right [left] Reilly groupoid with* $e\sigma[e'\sigma']$ *as left [right] identity. Define* K_1 *by* (3.3); *then* $(L_1, e'\sigma'; R_1, e\sigma; K_1)$ *is a box-frame. Define the relation* τ_1 *on* $L_1 \times R_1$ *by* (3.4) *and define* $(\circ_1)$ *on* $K_{1\tau_1} = K_1/\tau_1$ *by* (3.6). *Then* $(L_1, e'\sigma'; R_1, e\sigma; K_{1\tau_1}(\circ_1))$ *is a banded box-frame. On* $T_1 = (L_1 \times R_1)/\tau_1$ *define* $(\circ_1)$ *by* (3.7). *Then* $T_1(\circ_1)$ *is a bisimple orthodox semigroup having* $K_{1\tau_1}(\circ_1)$ *as its band of idempotents, and the mapping* θ: $T_1(\circ_1) \rightarrow T(\circ)/\rho$ *defined by* (3.5) *is an isomorphism.*

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