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AZUMAYA ALGEBRAS OVER HENSEL RINGS

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In this paper we prove the following theorem.

Let $(R, \mathfrak{a})$ be an henselian couple and let $\mathscr{P}(R)$ be the set of isomorphism classes of Azumaya R-algebras; then the canonical map

$$\mathscr{P}(R) \longrightarrow \mathscr{P}(R/\mathfrak{a})$$

is bijective.

As a corollary we obtain that, if $(R, \mathfrak{a})$ is an henselian couple, then the canonical homomorphism

$$\mathscr{B}r(R) \longrightarrow \mathscr{B}r(R/\mathfrak{a})$$

between the Brauer groups, is an isomorphism.

Introduction. The corollary mentioned in the abstract generalizes a theorem of Azumaya ([2], Th. 31). The proof is similar to the one used by Grothendieck in proving the above theorem in case that R is a local ring and $\mathfrak{a}$ is its maximal ideal ([6], Th. 6.1).

Concerning the definition of henselian couple and Azumaya algebra we refer to [10] and [9] respectively.

All the rings and algebras are supposed to have unity.

In § 1 we recall some properties of representable functors and smooth morphisms we shall need later.

In §§ 2, 3 we study two particular functors F_1, F_2 from the category of commutative R-algebras to the category of sets and we prove that F_1 and F_2 are represented by smooth commutative R-algebras. These functors will be used to prove the theorem.

In § 4, applying a known property of henselian couples, we obtain the theorem stated before and deduce some corollaries.

1. In this section we give some properties of representable functors and smooth morphisms.

Let R be a commutative ring; if F: (comm. R-alg.) $\rightarrow$ (sets) is a functor we will say shortly that F is a sheaf if F is a sheaf of sets on the category of affine schemes over Spec R in the Zariski topology ([1] Def. 0.1 and 0.2).

PROPOSITION 1. *Let F: (comm. R-alg.) $\rightarrow$ (sets) be a functor and suppose that F is a sheaf. Suppose that there exists a family $[f_i]_{i\in I}$ of elements of R generating the unity ideal in R, such that the functor F_i: (comm. R_{f_i}-alg.) $\rightarrow$ (sets) induced by F is representable for all $i \in I$; then F is representable.*

Proof. The proof is straightforward and we omit it.

Now we recall the definition of smooth R-algebra.

DEFINITION 1. Let U be a commutative R-algebra. We say that U is *smooth* if

(a) U is of finite presentation.

(b) U is *formally smooth*, i.e. for every commutative R-algebra S, for every nilpotent ideal I of S, and for every R-homomorphism $U \to S/I$, there exists an R-homomorphism $U \to S$ such that the diagram

$$\begin{tikzcd} & S \arrow[d] \\ U \arrow[ur] \arrow[r] & S/I \end{tikzcd}$$

commutes.

PROPOSITION 2. *Let U be a commutative R-algebra of finite presentation and S a faithfully flat commutative R-algebra; then U is a smooth R-algebra if and onlf if $U \otimes S$ is a smooth S-algebra.*

Proof. See [5] Corollary 17.7.2.

PROPOSITION 3. *Let U be a commutative R-algebra of finite presentation; if for every prime ideal $\mathfrak{p}$ of R, $U_{\mathfrak{p}}$ is a smooth $R_{\mathfrak{p}}$-algebra, then U is a smooth R-algebra.*

Proof. Let $\mathfrak{P}$ be a prime ideal of U and let $\mathfrak{p} = \mathfrak{P} \cap R$. $U_{\mathfrak{p}}$ is a smooth $R_{\mathfrak{p}}$-algebra by hypothesis and it is easy to prove that $U_{\mathfrak{P}}$ is a formally smooth $U_{\mathfrak{p}}$-algebra. Hence $U_{\mathfrak{p}}$ is a formally smooth $R_{\mathfrak{p}}$-algebra and, by [5] Th. 17.5.1, U is a smooth R-algebra.

2. In this section we consider the functor F_1 defined as follows. Let A and A' be two Azumaya R-algebras; let $\mathfrak{a}$ be an ideal of R and suppose that

$$A/\mathfrak{a}A \approx A'/\mathfrak{a}A' .$$

For every commutative R-algebra S, define

$$F_1(S) = \mathrm{Isom}_{S\text{-alg}}(A \otimes S, A' \otimes S)$$

i.e. $F_1(S)$ is the set of isomorphisms of the S-algebra $A \otimes S$ onto $A' \otimes S$. It is easy to see that F_1 is a sheaf. The functor F_1 satisfies the following properties.

(1) *F_1 is representable.*

By Proposition 1 we can suppose that A and A' are free as R-modules and with the same rank n, because of the hypothesis $A/\mathfrak{a}A \approx A'/\mathfrak{a}A'$. Let $\{e_i\}$ and $\{e'_i\}$, $i = 1, \cdots, n$, be bases for A and A' respectively and let

$$e_i e_j = \sum_k m_{ijk} e_k \,, \qquad e'_i e'_j = \sum_k m'_{ijk} e'_k$$

be the multiplication laws in A and A' respectively. Let $\varphi: A \otimes S \to A' \otimes S$ be an isomorphism; we can write

$$\varphi(e_i) = \sum_j x_{ij} e'_j \,, \qquad x_{ij} \in S$$

where the x_{ij}'s must satisfy the following conditions:

(a) since φ must satisfy $\varphi(e_i e_j) = \varphi(e_i)\varphi(e_j)$ we have

$$\sum_k m_{ijk} x_{kt} = \sum_{kl} m'_{klt} x_{it} x_{jl}$$

for all $i, j, t = 1, \cdots, n$.

(b) $\det(x_{ij})$ is invertible in S.

Then consider the ring $R[\cdots, X_{ij}, \cdots]$ where the X_{ij}'s $(i, j = 1, \cdots, n)$ are indeterminate and let

$$f_{ijt} = \sum_k m_{ijk} X_{kt} - \sum_{kl} m'_{klt} X_{ik} X_{jl}$$

and

$$d = \det(X_{ij}) \,.$$

We set

$$U_1 = \left(\frac{R[\cdots, X_{ij}, \cdots]}{(\cdots, f_{ijt}, \cdots)} \right)_d$$

and define the isomorphism

$$\varphi: A \otimes U_1 \longrightarrow A' \otimes U_1$$

by

$$\varphi(e_i) = \sum_j X_{ij} e'_j \,.$$

It is immediate to see that the couple (U_1, φ) represents the functor F_1.

(2) *The R-algebra U_1 which represents F_1 is smooth.*

(a) By the definition of U_1 we have that U_1 is locally of finite presentation, hence U_1 is of finite presentation ([4] Prop. 1.4.6).

(b) To prove that U_1 is formally smooth, by Prop. 3 we can suppose R local ring. Consider the strict henselization $\tilde{R}$ of R; it is known that, if $\mathfrak{m}$ is the maximal ideal of R, then $\mathfrak{m}\tilde{R}$ is the maximal ideal of $\tilde{R}$ and the residue field Ω of $\tilde{R}$ is a separable closure of the residue field k of R ([11], Chap. VIII § 2). We have $A \otimes \Omega \simeq M_r(\Omega)$, i.e. the full matrix algebra of rank r over Ω ([9], Chap. III, Cor. 6.3); by this we have

$$A \otimes \tilde{R} \simeq M_r(\tilde{R})$$

([3] Cor. 5.6).

By Proposition 2 we can suppose that

$$A \simeq M_r(R) \simeq A'$$

then U_1 represents the functor

$$\underline{\mathrm{Aut}}\,(M_r)\colon (\text{comm. } R\text{-alg.}) \longrightarrow (\text{sets})$$

defined by

$$\underline{\mathrm{Aut}}\,(M_r)(S) = \mathrm{Aut}_{S\text{-alg}}(M_r(S)) \ .$$

We must prove that, if I is a nilpotent ideal of S, the map

$$\mathrm{Aut}_{S\text{-alg}}\,((M_r(S)) \longrightarrow \mathrm{Aut}_{S/I\text{-alg}}\,(M_r(S/I))$$

is surjective.

This is an immediate consequence of the following proposition, because there is a bijection between

$$\mathrm{Aut}_{S\text{-alg}}\,(M_r(S))$$

and the set of all systems $\{e_{ij}\}$ $(i, j = 1, \cdots, r)$ of matrix units in $M_r(S)$.

PROPOSITION 4. *Let (S, I) be an henselian couple and C a finite S-algebra. If $\{\bar{e}_{ij}\}$ $(i, j = 1, \cdots, r)$ is a system of matrix units in C/IC, then $\{\bar{e}_{ij}\}$ can be lifted to a system $\{e_{ij}\}$ of matrix units in C.*

Proof. The proof is the same as in [3] Th. 3.3.

3. In this section we consider the functor F_2 defined as follows. Let P be a finite projective R-module and, for every commutative R-algebra S, define $F_2(S) =$ set of multiplication laws m which can be defined on $S \otimes P$ such that $(S \otimes P, m)$ is an Azumaya S-algebra. Note that F_2 is a sheaf: this is an easy consequence of the fact that the property of being an Azumaya R-algebra is a local property on Spec R([9], Chap. III, Th. 6.6). The functor F_2 satisfies the following properties.

(1) *F_2 is representable.*

By Proposition 1 we can suppose that P is a free R-module of rank n. Let $\{e_i\}$ $(i = 1, \cdots, n)$ be a basis for P. A multiplication law on $P \otimes S$ is defined by

$$e_i e_j = \sum_k m_{ijk} e_k \ , \qquad m_{ijk} \in S$$

where the elements m_{ijk} must satisfy the following properties. By the associative law $(e_i e_j)e_k = e_i(e_j e_k)$ we have

$$\sum_l (m_{ijl} m_{lkt} - m_{jkl} m_{ilt})$$

for all $i, j, k, t = 1, \cdots, n$.

Let $1 = \sum_i x_i e_i$ be the identity element; we have

$$\sum_i x_i m_{ijk} = \sum_i x_i m_{jik} = \delta_{ik}$$

for all $i, k = 1, \cdots, n$.

In order to express the condition that $(P \otimes S, m)$ is an Azumaya S-algebra, we recall the following proposition.

PROPOSITION 5. *Let A be an R-algebra and suppose that, as R-module, A is free of rank n; let $\{e_i\}$ $(i = 1, \cdots, n)$ be a basis. Then A is an Azumaya R-algebra if and only if the matrix (a_{ij}), defined by $a_{ij} = e_j e_i$, is an invertible matrix in the ring $M_n(A)$.*

Proof. See [2] Theorem 12.

Then if we denote by $(b_{kl}) = (\sum_t m'_{klt} e_t)$ the inverse matrix of $(a_{ij}) = (\sum_s m_{jis} e_s)$, we have

$$\sum_{jkt} m_{jik} m_{kts} m'_{jlt} = \delta_{il} x_s$$

for all $i, l, s = 1, \cdots, n$.

Then consider the ring

$$R[\cdots, X_i, \cdots; \cdots, Y_{ijk}, \cdots; \cdots, Y'_{ijk}, \cdots]$$

where the X_i's, Y_{ijk}'s, Y'_{ijk}'s are indeterminate $(i, j, k = 1, \cdots, n)$. Set $f_{ijkt} = \sum_l (Y_{ijl} Y_{lkt} - Y_{jkl} Y_{ilt})$

$$g_{jk} = \sum_i X_i Y_{ijk} - \delta_{jk} \ , \quad g'_{jk} = \sum_i X_i Y_{jik} - \delta_{jk}$$

$$h_{ils} = \sum_{jkt} Y_{jik} Y_{kts} Y'_{jlt} - \delta_{il} X_s$$

and set

$$U_2 = \frac{R[\cdots, X_i, \cdots; \cdots, Y_{ijk}, \cdots; \cdots, Y'_{ijk}, \cdots]}{(\cdots, f_{ijkt}, \cdots; \cdots, g_{jk}, \cdots; \cdots, g'_{jk}, \cdots, h_{ils}, \cdots)} \ .$$

Define on $P \otimes U_2$ a multiplication law m by

$$e_i e_j = \sum_k X_{ijk} e_k \;;$$

then it is easy to see that (U_2, m) represents F_2

(2) *The R-algebra U_2 which represents F_2 is smooth.*

(a) As with the algebra U_1, U_2 is of finite presentation.

(b) To see that U_2 is formally smooth, consider the following proposition.

PROPOSITION 6. *Let S be a commutative R-algebra and I a nilpotent ideal of S; then if $\bar{A}$ is an Azumaya S/I-algebra, there exists an Azumaya S-algebra A such that $A/IA \simeq \bar{A}$.*

First we prove that the proposition implies U_2 formally smooth, i.e. the map $F_2(S) \to F_2(S/I)$ surjective. Let $\bar{m} \in F_2(S/I)$; call $\bar{A}$ the algebra $(P \otimes S/I, \bar{m})$. By Prop. 6 there exists an Azumaya S-algebra A such that $A/IA \simeq \bar{A}$. Call Q the S-module underlying to A; Q is finite and projective and $Q/IQ \simeq P \otimes S/I$. Since Q is projective the above isomorphism lifts to an S-module homomorphism $\varphi\colon Q \to P \otimes S$ and it is easy to prove that φ is an isomorphism. Hence the multiplicative structure on A is carried by φ to a multiplication m on $P \otimes S$ whose image in $F_2(S/I)$ is $\bar{m}$.

Proof of Proposition 6. We can suppose that $\bar{A}$, as a projective S/I-module has constant rank n (by [9] Chap. I. Lemma 6.3 and [3] Cor. 3.2). It is known that there exists a faithfully flat étale extension $\bar{S}'$ of $\bar{S} = S/I$ such that

$$\bar{A} \otimes \bar{S}' \simeq M_r(\bar{S}')$$

with $r^2 = n$ ([9] Chap. III Cor. 6.3).

By a known theorem ([11] Chap. V, Th. 4) there exists an étale S-algebra S' such that $S'/IS' \simeq \bar{S}'$ and it is easy to see that S' is faithfully flat S-algebra. Now recall that, if S' is a faithfully flat extension of S, the isomorphism classes of Azumaya S-algebras A such that

$$A \otimes S' \simeq M_r(S')$$

are classified by

$$H^1(S'/S, \underline{\mathrm{Aut}}\,(M_r))$$

where $\underline{\mathrm{Aut}}\,(M_r)$: (comm. S-alg.) $\to$ (groups) is the functor defined before ([9] Chap. II, Rem. 8.2). Then the Proposition 6 follows from the lemma.

LEMMA. *Let S' be a faithfully flat extension of S, I a nilpotent ideal of S, F*: (comm. S-alg.) $\to$ (groups) *a functor represented by a smooth S-algebra. Let $\bar{S} = S/I$, $\bar{S}' = S'/IS'$ and $\bar{F}$*: (comm. S/I-alg.) $\to$ (groups) *be the functor induced by F. Then the canonical map*

$$H^1(S'/S, F) \longrightarrow H^1(\bar{S}'/\bar{S}, \bar{F})$$

is bijective.

Proof. [7] Lemma 8.1.8, page 404.

4. In this section we prove the theorem enunciated in the introduction and deduce some corollaries.

First we recall a result on henselian couples.

THEOREM 1. *Let $(R, \mathfrak{a})$ be an henselian couple and U a smooth R-algebra; then the canonical map*

$$\mathrm{Hom}_{R\text{-alg}}(U, R) \longrightarrow \mathrm{Hom}_{R\text{-alg}}(U, R/\mathfrak{a})$$

is surjective.

Proof. See [8] Theorem 1.8.

Now we are able to prove the following propositions.

PROPOSITION 7. *Let $(R, \mathfrak{a})$ be an henselian couple and A, A' two Azumaya R-algebras such that $A/\mathfrak{a}A \simeq A'/\mathfrak{a}A$; then $A \simeq A'$.*

Proof. By Theorem 1 and § 2.

PROPOSITION 8. *Let $(R, \mathfrak{a})$ be an henselian couple and $\bar{A}$ an Azumaya $R/\mathfrak{a}$-algebra; then there exists an Azumaya R-algebra A such that $A/\mathfrak{a}A \simeq \bar{A}$.*

Proof. Let $\bar{P}$ be the finite projective $R/\mathfrak{a}$-module underlying to $\bar{A}$; then by [3] Theorem 4.1 there exists a finite projective R-module P such that $P/\mathfrak{a}P \simeq \bar{P}$. Then the proposition follows from Theorem 1 and § 3.

THEOREM 2. *Let $(R, \mathfrak{a})$ be an henselian couple and let $\mathscr{P}(R)$ be the set of isomorphism classes of Azumaya R-algebras. Then the canonical map*

$$\mathscr{P}(R) \longrightarrow \mathscr{P}(R/\mathfrak{a})$$

is bijective.

Proof. By Propositions 7 and 8.

COROLLARY 1. *Let* $(R, \mathfrak{a})$ *be an henselian couple; then the canonical homomorphism*

$$\mathscr{B}_r(R) \longrightarrow \mathscr{B}_r(R/\mathfrak{a})$$

between the Brauer groups is an isomorphism.

Proof. The injectivity is in [3] Proposition 5.7; the surjectivity follows from Theorem 2.

COROLLARY 2. *Let* $(R, \mathfrak{a})$ *be an henselian couple and let*

$$G\colon (\text{Azumaya } R\text{-alg.}) \longrightarrow (\text{Azumaya } R/\mathfrak{a}\text{-alg.})$$

be the functor defined by $G(A) = A/\mathfrak{a}A$ *for every Azumaya* R*-algebra* A. *Then* G *is essentially bijective and full, but, if* $\mathfrak{a} \neq (0)$, *is not faithful.*

Proof. G is essentially bijective means exactly what we proved in Theorem 2. In order to prove that G is full consider two Azumaya R-algebras A and A' and define the functor

$$F'\colon (\text{comm. } R\text{-alg.}) \longrightarrow (\text{sets})$$

by

$$F'(S) = \mathrm{Hom}_{S\text{-alg}}(A \otimes S, A' \otimes S) .$$

As with the functor F_1 we can prove that F' is represented by an R-algebra U' of finite presentation.

To prove that U' is a smooth R-algebra we can suppose, as with the algebra U_1, $A \simeq M_n(R)$ and $A' \simeq M_m(R)$. Now observe that, if $\varphi \in F'(S)$ and $\{e_{ij}\}$ $(i, j = 1, \cdots, n)$ is a system of matrix units in A, then $\{\varphi(e_{ij})\}$ is a system of matrix units in A', hence we have

$$F'(S) = \emptyset \quad \text{if } m \neq n .$$
$$F'(S) = \mathrm{Aut}_{S\text{-alg}}(M_n(S)) \quad \text{if } m = n .$$

Hence U' is a smooth R-algebra and by Theorem 1 we have that G is full.

Now let $a \in \mathfrak{a}$, $a \neq 0$. Consider the inner automorphism α of $M_2(R)$ given by the element

$$\begin{pmatrix} 1+a & 0 \\ 0 & 1 \end{pmatrix} \in M_2(R) ;$$

the induced automorphism $\bar{\alpha}$ of $M_2(R/\mathfrak{a})$ is the identity automorphism while α is not the identity automorphism of $M_2(R)$. This proves that G is not faithful.

Now suppose R connected and recall that two Azumaya R-algebras A and A' are said to be *stable isomorphic* if there exist integers m and n such that

$$M_n(A) \simeq M_m(A') .$$

Denote by $\mathscr{K}\mathscr{P}(R)$ the set of stable isomorphism classes of Azumaya R-algebras ([6] Remark 1.8).

COROLLARY 3. *Let $(R, \mathfrak{a})$ be an henselian couple and suppose that $R/\mathfrak{a}$ is connected. Then the canonical map*

$$\mathscr{K}\mathscr{P}(R) \longrightarrow \mathscr{K}\mathscr{P}(R/\mathfrak{a})$$

is bijective.

Proof. First we observe that if $R/\mathfrak{a}$ is connected then R is connected. Now we show that $M_n(A)$ is an Azumaya R-algebra, if A is an Azumaya R-algebra: in fact we know that there exists a faithfully flat extension S of R such that $A \otimes S \simeq M_r(S)$; then $M_n(A) \otimes S \simeq M_{n\times r}(S)$, i.e. $M_n(A)$ is an Azumaya R-algebra. Then the Corollary 3 follows from the Propositions 7 and 8.

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