PACIFIC JOURNAL OF MATHEMATICS
Vol. 61, No. 1, 1975

LOCAL CONNECTEDNESS IN DEVELOPABLE SPACES

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A space is spherically connected if and only if it has an admissible semi-metric d such that d-spheres of radius less than one are connected. It is shown that a developable space is locally connected if and only if it is spherically connected. A semi-metric space is K-semi-metrizable if and only if it admits a semi-metric d such that $d(A, B) > 0$ whenever A and B are disjoint compact sets. It is shown that in the class of locally connected rim compact spaces, the K-semi-metrizable spaces are precisely the developable γ-spaces. An example is given of a locally connected, locally compact K-semi-metrizable Moore space which is not metrizable.

1. Introduction. A topological space is said to be *rim compact* provided that each point has a local basis of open sets which have compact boundaries. A space is *locally connected* provided that each point of the space has a local basis of connected open sets. If R is the set of all rational points of the plane E^2, the $E^2 - R$ is an example of a locally connected, rim compact space which is nowhere locally compact.

If d is a semi-metric for a space X, then d is said to be a *K-semi-metric* provided that $d(A, B) > 0$ whenever A and B are disjoint compact subsets of X. It seems to be unknown whether every regular semi-metrizable space,[1] or even developable space,[2] has a compatible K-semi-metric. We define a topological space X to be *d-spherically connected* provided that X has a compatible semi-metric d such that every d-sphere $S_d(x, e) = \{y: d(x, y) < e\}$ of radius less than one is connected. A space is said to be *spherically connected* provided that it is d-spherically connected for some compatible semi-metric d.

Theorem 5.2 of [3] may be phrased as follows: let X be a rim compact space; if X is d-spherically connected by virtue of a K-semi-metric d, then X is metrizable. Also, P. Zenor has shown that a locally connected rim compact space is metrizable if and only if it

[1] A space X is semi-metrizable provided there exists a nonnegative, real-valued function d on $X \times X$, called a semi-metric, which satisfies the following three conditions: (i) $d(x, y) = d(y, x)$; (ii) $d(x, y) = 0$ iff $x = y$; (iii) for x in X and $A \subset X$, we have $x \in cl(A)$ iff $d(d(x, A)) = \inf \{x, a): a \in A\} = 0$.

[2] A sequence $G_1, G_2, \cdots$ of open covers of a space X is called a development provided that $\{St(x, G_n): n \in Z^+\}$ is a local base at x for each x in X. A space is developable provided it has a development. A regular developable space is called a Moore space.

has a regular G_δ-diagonal[3] [6]. A comparison of these two theorems suggests the question of whether local connectedness and spherical connectedness are equivalent concepts in the class of semi-metrizable spaces. The purpose of this note is to give a partial answer to this question by showing that local connectedness and spherical connectedness are equivalent in the class of developable spaces. Heath's theorem also suggests the question of whether a rim compact, spherically connected space is metrizable. We answer this question negatively by showing that there exists a locally compact, locally connected, completely regular Moore space X which is K-semi-metrizable but not metrizable. This same example shows that Zenor's theorem cannot be improved by replacing "regular G_δ-diagonal" by "G_δ^*-diagonal[4]". We shall also show that a locally connected rim compact space is K-semi-metrizable if and only if it is a developable γ-space.[5]

2. Theorems and example.

THEOREM 1. *Every spherically connected space is locally connected; every locally connected developable space is spherically connected.*

Proof. Let X be a d-spherically connected space. The space X is locally connected provided that each component of each open set is open. Let G be an arbitrary open subset of X and let C be an arbitrary component of the subspace G. Given $x \in C$, there exists $e > 0$ such that $e < 1$ and $S_d(x, e) \subset G$. Since $S_d(x, e)$ is connected, we have $S_d(x, e) \subset C$. It follows that $x \in \operatorname{int}(C)$, that is, that C is an open set, whence X is locally connected.

To prove the second part of Theorem 1, let X now denote a locally connected developable space. Let $V_1, V_2, \cdots$, be a development for X. Since X is a developable space, X has a compatible semi-metric p such that if $x \in X$ and $e > 0$ is given, then x has a neighborhood V such that $p(a, b) < e$ for every $a, b \in V$ [1, page 128]. For each $x \in X$ and natural number n, let $d_n(x)$ denote a connected open neighborhood of x such that $p(a, b) < 1/n$ for all $a, b \in d_n(x)$ and such that $d_n(x) \subset v$ for some $v \in V_n$; furthermore, choose the sets $d_n(x)$ so

[3] A space X has a regular G_δ-diagonal provided X has a sequence $G_1, G_2 \cdots$ of open covers such that if x and y are distinct points of X, then x has a neighborhood V for which $y \notin cl(St(V, G_n))$ for some natural number n.

[4] A space X has a G_δ^*-diagonal if X has a sequence $G_1, G_2, \cdots$ of open covers such that if x and y are distinct points of X, then $y \notin cl(St(x, G_n))$ for some natural number n.

[5] A space X is a γ-space iff there exists a function g from $Z^+ \times X$ into the open sets of X such that: (i) $\{g(n, x): n \in Z^+\}$ is a local base at x with $g(n+1, x) \subset g(n, x)$ and (ii) if $A \subset G$ where A is compact and G open, then there exists $n \in Z^+$ such that $g(n, a) \subset G$ for every $a \in A$.

that $d_{n+1}(x) \subset d_n(x)$ for all n. Let $G_n = \{d_n(x): x \in X\}$; then, $G_1, G_2, \cdots$, is a development for X. For $x \in X$ and $n = 1, 2, \cdots$, set $g_n(x) = St(x, G_n)$. If $x \in g_n(x_n)$ for $n \in Z^+$, then there exists a sequence $\{y_n\}$ in X such that $x, x_n \in d_n(y_n)$, whence $p(x, x_n) < 1/n$, that is, $\{x_n\}$ converges to x. Note also that $d_{n+1}(x) \subset d_n(x)$ implies that $g_{n+1}(x) \subset g_n(x)$. Consequently, we may define a semi-metric d for X, which is equivalent to p, in the following standard way [3, Theorem 3.2]: if $x \neq y$, let $d(x, y) = 1/n$ where n is the least natural number k such that $x \notin g_k(y)$ and $y \notin g_k(x)$.

Let $x \in X$ and $0 < e < 1$. The proof will be completed by showing that $S_d(x, e)$ is connected. Let n be the particular natural number which satisfies the relations $1/(n+1) < e \leqq 1/n$. Observe that $S_d(x, e) = S_d(x, 1/n)$. Let $y \in S_d(x, e)$; then $d(x, y) < 1/n$, so that $x \in g_n(y)$ or $y \in g_n(x)$. There must exist a set $S_y \in G_n$ with $x, y \in S_y$. If $z \in S_y$, then $x \in St(z, G_n) = g_n(z)$, so that $d(x, z) < 1/n$, that is, $z \in S_d(x, e)$; consequently, $S_y \subset S_d(x, e)$. It follows that $S_d(x, e) = \bigcup \{S_y: y \in S_d(x, e)\}$; but then $S_d(X, e)$ is connected since each set S_y is connected and contains the common point x, completing the proof.

In the proof of Theorem 1, we have $p(x, y) \leqq d(x, y)$ for all $x, y \in X$. Then, p is a K-semi-metric implies that d is a K-semi-metric. But, Zenor has shown that a developable space X has a regular G_δ-diagonal if and only if X has a K-semimetric p such that if $x \in X$ and $e > 0$ is given, then x has a neighborhood V such that $p(a, b) < e$ whenever $a, b \in V$, [7]. These remarks, together with the proof of Theorem 1, imply the following:

COROLLARY 2. *If X is a locally connected developable space which has a regular G_δ-diagonal, then X has a K-semi-metric d such that X is d-spherically connected.*

Corollary 2 suggests the question of whether a locally connected, K-semi-metrizable developable space has a K-semi-metric d such that the space is d-spherically connected. This question is important since an affirmative answer, in conjunction with Theorem 5.2 of [3], would imply that every K-semi-metrizable, locally connected, rim compact space is metrizable. This, however, is not the case, as shown by the following example, which is a variation of the space $\sum_B$ of [5, page 376], of a locally compact, locally connected, nonmetrizable, K-semi-metrizable, quasi-metric[6] Moore space.

Given a quasi-metric space (X, d) we let T_d denote the topology

[6] A space X is said to be quasi-metrizable provided that there exists a nonnegative real-valued function d on $X \times X$, called a quasi-metric, which satisfies the following conditions: (i) $d(x, y) = 0$ iff $x = y$; (ii) $d(x, y) \leqq d(x, z) + d(z, y)$; (iii) the collection $\{S_d(x, e): x \in X$ and $e > 0\}$ forms a base for the topology for X, where $S_d(x, e) = \{y \in X: d(x, y) < e\}$.

having sets of the form $\{y: d(x, y) < 1/n\}$ as an open basis, where x ranges over the set X and $n \in Z^+ = \{1, 2, \cdots\}$.

EXAMPLE 3. Let $P = \{(x, y): 0 < x < 1 \text{ and } 0 < y \leqq 1\}$. If x is a real number and $0 < a < \pi$, then let $R(x, a)$ denote the ray, emanating from the point $(x, 0)$ in the plane, which lies in the upper half plane and makes an angle of a radians with the positive direction along the x-axis. Let $R = \{R(x, a): 0 < x < 1 \text{ and } 0 < a < \pi\}$ and let $X = P \cup R$.

Let $I = \{(x, 0): 0 \leqq x \leqq 1\}$ and let p be a fixed point of I. Let x and y be elements of P. Define $d_p(x, y) = ||x - p| - |y - p|| + a$ where $|p - q|$ is the usual Euclidean distance between points p and q of the plane and a is the positive angle, in radians, between the line segment joining p to x and the line segment joining p to y. Now define $d(x, y) = \sup \{d_p(x, y): p \in I\}$.

If $y \in P$ and $R(x, a) \in R$, let $d(y, R(x, a)) = 2\pi$ and let $d(R(x, a), y) = |(x, 0) - y| + b$ where b is the positive angle, in radians, between the ray $R(x, a)$ and the line segment joining $(x, 0)$ to y.

If $R(x, a)$ and $R(y, b)$ are elements of R, define $d(R(x, a), R(y, b)) = 2\pi$ if $x \neq y$ and $d(R(x, a), R(y, b)) = |a - b|$ if $x = y$.

It is straightforward to show that d is a quasi-metric for the set X. The topological space (X, T_d) is a nonnormal, locally compact, locally connected, completely regular, quasi-metrizable Moore space.

If $R(x, a) \in R$, define $g_n(R(x, a))$ as follows: $R(y, b) \in g_n(R(x, a))$ if and only if $x = y$ and $|a - b| < 1/n$, and $(p, q) \in P$ belongs to $g_n(R(x, a))$ if and only if $\sqrt{(x - p)^2 + q^2} < 1/n$ and $|a - c| < 1/n$ where c is the angle in radians between the ray $R(x, a)$ and the line segment connecting the points (p, q) and $(x, 0)$. The topology T_d has the collections $\{g_n(R(x, a)): n \in Z^+\}$ as local open bases for points $R(x, a)$ in R and the points of P have their usual neighborhood systems. We shall now construct a compatible K-semi-metric for the space (X, T_d).

Let y and z denote distinct points of P and let I be defined as above. Let L denote the line $y = 1$ in the plane. If $x \in I$ and $p \in P$, let $\pi_x(p)$ be the point of intersection of the line L with the line connecting the points x and p. Now define $\delta(y, z) = \max \{|\pi_x(y) - \pi_x(z)|: x \in I\}$. If $p \in P$ and $R(x, a) \in R$, define $\delta(p, R(x, a)) = |p - (x, 0)| + b$ where b is the angle, in radians, between the ray $R(x, a)$ and the ray emanating from $(x, 0)$ and containing the point p. Finally, if $R(x, a)$ and $R(y, b)$ are points of R with $x \neq y$, let $\delta(R(x, a), R(y, b)) = 1$ and let $\delta(R(x, a), R(x, b)) = |a - b|$. It is easy to show that δ is a compatible semi-metric for the space (X, T_d). It remains only to show that δ is a K-semi-metric. Note that δ is a K-semi-metric for the space X if and only if whenever $\delta(w_n, z_n) \to 0$, $w_n \to$

w and $z_n \to z$, then $w = z$. With respect to this criterion for K-semi-metrizability, the only nonobvious case is that in which $w = R(x, a)$, $z = R(x, b)$ and the sequences $\{w_n\}$ and $\{z_n\}$ are in the set P. Therefore, suppose that $w_n \to R(x, a)$ and $z_n \to R(x, b)$ where $w_n, z_n \in P$ for all n and $a \neq b$. Let $p(a)$ be the point of intersection of the line L and the ray $R(x, a)$ and likewise let $p(b)$ be the point of intesection of L with the ray $R(x, b)$. Since $w_n \to R(x, a)$, we must have $\pi_{(x,0)}(w_n) \to p(a)$ along the line L; similarly, $\pi_{(x,0)}(z_n) \to p(b)$ along the line L. Since $\delta(w_n, z_n) = \max\{|\pi_y(w_n) - \pi_y(z_n)|: y \in I\}$, we have

$$|\pi_{(x,0)}(w_n) - \pi_{(x,0)}(z_n)| \leqq \delta(w_n, z_n) .$$

It follows readily that $\delta(w_n, z_n) > 1/2(|p(a) - p(b)|)$ for sufficiently large values of n, completing the proof that δ is a K-semi-metric.

The space of Example 3 is K-semi-metrizable and spherically connected, but not d-spherically connected whenever d is a K-semi-metric for X.

In the class of locally connected, rim compact spaces, we have the following coincidence theorem.

THEOREM 4. *A locally connected, rim compact space is K-semi-metrizable if and only if it is a developable γ-space.*

Proof. It is easy to show that a developable γ-space is K-semi-metrizable and we therefore omit the details. Let d be a compatible K-semi-metric for a locally connected, rim compact space X. We may choose a system $\{g(n, x): x \in X; n \in Z^+\}$ of open, connected subsets of X such that the following two conditions hold: (i) $\{g(n, x): n \in Z^+\}$ is a local base at x with $g(n+1, x) \subset g(n, x)$; (ii) if $y \in g(n, x)$, then $d(x, y) < 1/n$. Note that if C and D are disjoint compact subsets of X, then there exists a natural number n such that $g(n, C) \cap D = \emptyset$, where $g(n, C) = \bigcup \{g(n, c): c \in C)$.

Let A be an arbitrary compact subset of X and n be a natural number. Choose a finite number of points in A, say $x_1, x_2, \cdots, x_m$, such that $A \subset \bigcup \{g(n, x_i): i = 1, 2, \cdots, m\}$. Let $A_i = A \cap g(n, x_i)$; the sets $g(n, A_i)$ are connected for $1 \leqq i \leqq m$.

We are now in a position to show that X is a γ-space. Let K be a compact subset of X and let W be an open set containing K. Choose a finite number of open sets $G_1, G_2, \cdots, G_p$ such that for each $i = 1, 2, \cdots, p$, the set $Bd(G_i)$ is compact, $G_i \subset W$, and $K \subset \bigcup \{G_i: i = 1, 2, \cdots, p\} = G$. Since $Bd(G) \subset \bigcup \{Bd(G_i): i = 1, 2, \cdots, p\}$, we have that $Bd(G)$ is compact. Then, $K \cap Bd(G) = \emptyset$ and there exists a natural number n such that $g(n, K) \cap Bd(G) = \emptyset$. By the preceding paragraph, we have $K = K_1 \cup \cdots \cup K_m$ for some natural number m, where $g(n, K_i)$

is connected for $1 \leqq i \leqq m$. Since $K_i \subset K \subset G$ and $g(n, K_i)$ is connected, and since $g(n, K_i) \cap Bd(G) = \emptyset$, we must have $g(n, K_i) \subset G$ for $1 \leqq i \leqq m$. Note that $g(n, K) = \bigcup \{g(n, K_i): 1 \leqq i \leqq m\}$. It follows that $g(n, K) \subset G \subset W$, completing the proof that X is a γ-space. It is easy to show that a semi-metrizable γ-space is developable, e.g., use Theorem 3.3 of [3], and the proof of Theorem 4 is complete.

3. Open questions. In [2], Fletcher and Lindgren conjectured that every quasi-metrizable space admits a compatible non-Archimedean quasi-metric. Y. A. Kofner gave a counterexample to this conjecture [4]; however, Kofner's space is not developable, so the Fletcher-Lindgren conjecture remains open for developable spaces. Recall that a developable space admits a compatible non-Archimedean quasi-metric if and only if it is orthocompact [2]. We therefore ask:

QUESTION 1. Is the space X of Example 3 orthocompact?

Question 2 is motivated by Theorem 1 and Question 3 by Theorem 4 and other theorems in the literature.

QUESTION 2. Is every locally connected, semi-metrizable space necessarily spherically connected?

QUESTION 3. Is every regular semi-metrizable space necessarily K-semi-metrizable? If not, is every Moore space a K-semi-metrizable space?

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Received March 12, 1975 and in revised form August 17, 1975.

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