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ABSOLUTE SUMMABILITY OF FOURIER SERIES WITH FACTORS

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Kanno in 1969 and M. Izumi and S. Izumi in 1970 have obtained results concerning the absolute Nörlund summability of Fourier series with factors. The present paper contains theorems sharper than the aforementioned results.

1. Definitions and notations. Let $\{p_n\}$ be a given sequence of constants, real or complex, such that $P_n = \sum_{k=0}^{n} p_k \neq 0$ for $n \geqq 0$ and $p_n = P_n = 0$, for $n < 0$. A given series $\sum_{n=0}^{\infty} a_n$ is said to be summable (N, p_n), if t_n tends to a finite limit as $n \longrightarrow \infty$, where

$$t_n = \sum_{k=0}^{n} P_{n-k}\, a_k / P_n \ .$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|N, p_n|$, if $\sum_{n=1}^{\infty} |t_n - t_{n-1}| < \infty$. The $|N, p_n|$ method reduces to the $|C, \delta|$ method in the special case in which $P_n = A_n^{\delta}$, where A_n^{δ} is defined by the identity

$$\sum_{n=0}^{\infty} A_n^{\delta} x^n = (1-x)^{-\delta-1},\ |x| < 1,\ \delta \neq -1, -2, \cdots .$$

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$ and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) \ .$$

We shall use the following notations throughout.

$$\varphi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\};$$

$$R_n = (n+1)p_n/P_n;\ S_n = \sum_{k=1}^{n} \frac{P_k}{kP_n} \ .$$

Given a function $\lambda(t)$ and a sequence $\{\mu_n\}$, we write for $n = 1, 2, \cdots$,

$$\lambda(n) = \lambda_n;\ \Delta_n \lambda_n = \Delta\lambda_n = \lambda_n - \lambda_{n+1} \ ,$$

$$J_n(\mu) = \sum_{k=n}^{\infty} \frac{\mu_k \lambda_k}{kP_k} \ ,$$

$[x]$ denotes the greatest integer not greater than x, in particular we write $m = [n/2]$ and $\tau = [C/2t]$, for some fixed positive constant C. K denotes a positive constant which is not necessarily the same at each occurrence. $\sum_a^b$ will be taken as 0, if $a > b$.

2. Introduction and the main results. Concerning the $|C|$-summability of a Fourier series and a corresponding series with factors the following is known ([1], [8])[1].

THEOREM A. *For* $0 \leqq \alpha < 1$, *and* $\beta > \alpha$ *the series* $\sum_{n=1}^{\infty} n^{\alpha} A_n(x)$ *is summable* $|C, \beta|$, *if*

$$\int_0^{\pi} t^{-\alpha} |d\varphi(t)| \leqq K . \tag{2.1}$$

Theorem A for $\alpha = 0$ was proved by Bosanquet [1] who has also shown that the result is the best possible in the sense that β cannot be replaced by 0. For the other values of α the result was proved by Mohanty [8].

In the direction of Theorem A, M. Izumi and S. Izumi [4] have recently proved the following.

THEOREM B. *Let* $\{p_n\}$ *be a positive monotonic decreasing sequence and* $\lambda(t)$, $t > 0$, *be a positive increasing function, then the series* $\sum_{n=1}^{\infty} \lambda_n A_n(x)$ *is summable* $|N, p_n|$, *if the following conditions hold.*

$$J_n(1) \leqq K\lambda_n / P_n , \tag{2.2}$$

$$\{\lambda_n / n\} \text{ is monotonic decreasing} , \tag{2.3}$$

and

$$\int_0^{\pi} \lambda(1/t) |d\varphi(t)| \leqq K . \tag{2.4}$$

With a slight modification in the proof of Theorem B as contained in [4] it may be seen that the result continues to hold even if $\lambda(t)$ is a constant function. In view of this, we may obtain Theorem A from Theorem B by taking $\lambda(t) = t^{\alpha}$, $0 \leqq \alpha < 1$. Examining the hypotheses of Theorem B closely, it appears that the condition (2.2) is indispensable. For, if $\lambda(t) = K$, then the condition (2.2) is equivalent to the boundedness of the sequence $\{S_n\}$ (see [2], Lemma 3) which has been shown to be a necessary condition for the (N, p_n) summability of $\sum_{n=1}^{\infty} A_n(x)$ by Hille and Tamarkin ([3], Theorem II). The condition (2.4), of course corresponds to the condition (2.1) of Theorem A. The following theorem, which we prove in the present paper, shows that condition (2.3) is redundant in Theorem B.

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[1] We write $\int_0^{\pi}$ for $\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\pi}$.

THEOREM 1. *Let $\lambda(t)$, $t > 0$, be a positive nondecreasing function and $\{p_n\}$ be a positive monotonic nonincreasing sequence such that* (2.2) *holds and for some positive constant C*

$$\int_0^\pi \lambda(C/t)\,|d\varphi(t)| \leqq K\,, \tag{2.4'}$$

then $\sum_{n=1}^{\infty} \lambda_n A_n(x)$ is summable $|N, p_n|$.

Lemma 6 of the present paper shows the role that a specific choice of $\lambda(t)$ plays in (2.4′).

Another generalisation of Theorem A, in the form of the following theorem is due to Kanno [5].

THEOREM C. *Let $\{p_n\}$ and $\{\Delta p_n\}$ are both nonnegative and nonincreasing and $\lambda(t)$, $t > 0$, be a positive nondecreasing function such that $\{\lambda_n/P_n\}$ is nonincreasing. Then the series $\sum_{n=1}^{\infty} R_n\lambda_n A_{n+1}(x)$ is summable $|N, p_n|$, if* (2.4′) *holds and*

$$J_n(R) \leqq K\lambda_n/P_n\,. \tag{2.5}$$

Dropping the condition that $\{\Delta p_n\}$ is nonincreasing and replacing the condition: '$\{\lambda_n/P_n\}$ is nonincreasing' by the condition: '$\{\lambda_n p_n/P_n\}$ is nonincreasing' which appears to be lighter than the former, we obtain the following more refined result than Theorem C whenever $P_n \to \infty$.

THEOREM 2. *Let $\{p_n\}$ be nonnegative nonincreasing sequence with $P_n \to \infty$ as $n \to \infty$ and $\lambda(t)$, $t > 0$, be a positive nondecreasing function such that $\{\lambda_n p_n/P_n\}$ is nonincreasing and* (2.5) *holds. Then the series $\sum_{n=1}^{\infty} R_n\lambda_n A_{n+1}(x)$ is summable $|N, p_n|$, if* (2.4′) *holds.*

In order to consider the remaining case of Theorem C in which $P_n \to l$ (finite) as $n \to \infty$, we observe that in this case (2.5) implies that

$$\sum_{k=1}^{\infty} \frac{R_k\lambda_k}{k} < \infty\,.$$

Further, since $\lambda(t)$ is positive nondecreasing, (2.4′) implies that

$$\int_0^\pi |d\varphi(t)| \leqq K\,.$$

Thus, applying Lemma 7 with $\varepsilon_k = R_k\lambda_k$, we obtain the absolute convergence of $\sum_{k=1}^{\infty} R_k\lambda_k A_{k+1}(x)$ which *a fortiori* implies its $|N, p_n|$ summability, since by virtue of Lemma 3, the method is absolutely regular whenever $\{p_n\}$ is nonnegative nonincreasing. Thus, a sharper result under lighter assumptions is obtained for this case.

It may be observed that Theorem 2 and Theorem 1, which include Theorem C when $P_n \to \infty$ and Theorem B respectively, are established by a unified proof shorter than the existing proofs of Theorem B and Theorem C. For some interesting corollaries of Theorem C and *a fortiori* of Theorem 2 reference may be made to [5].

3. Preliminary results. We need the following lemmas for the proof of our theorems.

LEMMA 1. *Let $\{a_n\}$ be a given sequence, then for any x, we have*

$$(1-x)\sum_{k=r}^{s} a_k x^k = a_r x^r - a_s x^{s+1} - \sum_{k=r}^{s-1} \Delta a_k x^{k+1},$$

where r and s are integers such that $s \geqq r \geqq 0$.

LEMMA 2. *If $\{q_n\}$ is a nonnegative nonincreasing sequence such that $Q_n = \sum_{k=0}^{n} q_k$, then for $0 \leqq a \leqq b \leqq \infty$ and any n,*

$$\left|\sum_{k=a}^{b} q_k \exp i(n-k)t\right| \leqq KQ_\tau,$$

uniformly in $0 < t \leqq \pi$.

Lemma 2 follows from the proof of Lemma (5.11) of [7], when we take $\tau = [C/2t]$ in place of $[1/t]$.

LEMMA 3. *If $\{p_n\}$ is nonnegative nonincreasing, then for all $k \geqq 0$ and $1 \leqq a \leqq b \leqq \infty$,*

$$\sum_{n=a}^{b} P(n, k) = \sum_{n=a}^{b}\left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}}\right) \leqq 1$$

and for any $n > 0$, $P(n, k) \geqq 0$ so that the (N, p_n) method is absolutely regular.

Proof. We have

$$\sum_{n=a}^{b}\left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}}\right) = \frac{P_{b-k}}{P_b} - \frac{P_{a-k-1}}{P_{a-1}} \leqq 1.$$

For, $n > 0$ we observe that

$$P(n, k) = \frac{p_{n-k}P_n - p_n P_{n-k}}{P_n P_{n-1}} \geqq 0,$$

since $\{p_n\}$ is nonnegative nonincreasing. It is known [6] that necessary and sufficient conditions for the absolute regularity of the (N, p_n) method are that

$$p_{n-k}/P_n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \quad \text{and} \quad \sum_{n=1}^{\infty} |P(n, k)| \leqq K$$

for all k. The latter of these conditions is already proved while the former follows from the fact that $(k+1)p_k \leqq P_k$.

LEMMA 4. *If $\{p_n\}$ is nonnegative nonincreasing, then*
(i) *for any positive integer r, $P_{rn} \leqq rP_n$ and, if in addition $\{\lambda_n\}$ is nonnegative nondecreasing such that* (2.2) *holds, or $P_n \rightarrow \infty$ with n and* (2.5) *holds, then*
(ii) $\lambda_{2n} \leqq K\lambda_n$.

Proof. Using the hypothesis that $\{p_n\}$ is nonnegative and nonincreasing, we write

$$P_{rn} \leqq \sum_{\nu=1}^{r} \sum_{k=(\nu-1)n}^{\nu n} p_k \leqq \sum_{\nu=1}^{r} P_n = rP_n .$$

This proves (i). Next, if $\{\mu_n\}$ is a nonnegative sequence, then for $\rho \geqq 2n$,

$$J_n(\mu) \geqq J_{2n}(\mu) \geqq \lambda_{2n} \sum_{k=2n}^{\rho} \frac{\mu_k}{kP_k} ,$$

since $\{\lambda_n\}$ is nondecreasing. Taking $\rho = 3n$ and $\mu_n = 1$ for all n and observing that $\{P_n\}$ is positive nondecreasing, we have

$$J_n(1) \geqq \frac{(n+1)\lambda_{2n}}{3nP_{3n}} \geqq \frac{1}{9} \frac{\lambda_{2n}}{P_n} , \tag{3.1}$$

by virtue of the result (i). Similarly, since

$$\frac{R_k}{kP_k} \geqq \frac{p_{k+1}}{P_k P_{k+1}} = \frac{1}{P_k} - \frac{1}{P_{k+1}} \geqq 0 ,$$

taking $\rho \rightarrow \infty$ and $\mu_n = R_n$, we get

$$J_n(R) \geqq \frac{\lambda_{2n}}{P_{2n}} \geqq \frac{1}{2} \frac{\lambda_{2n}}{P_n} . \tag{3.2}$$

In view of the inequalities (3.1) and (3.2) and the hypotheses (2.2), (2.5), we have

$$K \frac{\lambda_n}{P_n} \geqq 9J_n(\mu) \geqq \frac{\lambda_{2n}}{P_n} ,$$

where $\mu_n = 1$ or $\mu_n = R_n$. This gives the result (ii) and thus we complete the proof of the lemma.

LEMMA 5. *If $\{p_n\}$ is nonnegative nonincreasing and $\{\lambda_n\}$ is positive nondecreasing such that* (2.2) *holds, then for all $N \geqq n$,*

$$n \sum_{k=n}^{N} |\Delta(\lambda_k/k)| \leqq K\lambda_n , \qquad n = 1, 2, \cdots .$$

Proof. We first observe that $\{n/P_n\}$ is nondecreasing. For,

$$\Delta\left(\frac{P_n}{n}\right) = \frac{1}{n(n+1)}\{P_n - np_{n+1}\} \geqq 0 ,$$

since $\{p_n\}$ is nonnegative nonincreasing. Thus, under the hypothesis (2.2), we have

$$K\lambda_n \geqq P_n \sum_{k=n}^{\infty} \frac{\lambda_k}{kP_k} \geqq n \sum_{k=n}^{\infty} \frac{\lambda_k}{k^2} . \tag{3.3}$$

Further, if $N \geqq n$, then

$$n\frac{\lambda_{N+1}}{N+1} = n\lambda_{N+1} \sum_{k=N+1}^{\infty} \frac{1}{k(k+1)} \leqq n \sum_{k=n}^{\infty} \frac{\lambda_k}{k(k+1)} , \tag{3.4}$$

since $\{\lambda_n\}$ is positive and nondecreasing. Thus,

$$\begin{aligned} n \sum_{k=n}^{N} \left|\Delta\left(\frac{\lambda_k}{k}\right)\right| &\leqq n \sum_{k=n}^{N} \frac{\lambda_{k+1} - \lambda_k}{k+1} + n \sum_{k=n}^{N} \frac{\lambda_k}{k(k+1)} \\ &= 2n \sum_{k=n}^{N} \frac{\lambda_k}{k(k+1)} + n\frac{\lambda_{N+1}}{N+1} - \lambda_n \\ &\leqq 3n \sum_{k=n}^{\infty} \frac{\lambda_k}{k(k+1)} \leqq K\lambda_n \end{aligned}$$

by virtue of (3.3) and (3.4). This completes the proof of the lemma.

LEMMA 6. *Let $\theta(t)$ and $\lambda(t)$ be two nonnegative nondecreasing functions such that $\theta(n) = \lambda(n) = \lambda_n$. Let a and b be two positive numbers such that*

$$I(\theta, \varepsilon) = \int_{\varepsilon}^{\pi} \theta(a/t)\,|d\varphi(t)| \quad \text{and} \quad I(\lambda, \varepsilon) = \int_{\varepsilon}^{\pi} \lambda(b/t)\,|d\varphi(t)|$$

exist, for every $\varepsilon > 0$. If $\lambda_{2n} \leqq K\lambda_n$, then $I(\theta, 0) < \infty$ if and only if $I(\lambda, 0) < \infty$.

Proof. We assume without loss of generality that $a \leqq b$. Thus, for $0 < t \leqq b$, we have

$$a/t \leqq b/t \leqq 2[b/t]$$

and, therefore, using the hypotheses that $\theta(t)$ and $\lambda(t)$ are nondecreasing, we have

$$\theta(a/t) \leqq \lambda(2[b/t]) \leqq K \quad \lambda(b/t) \, , \tag{3.5}$$

since $\theta(n) = \lambda(n)$ and $\lambda_{2n} \leqq K\lambda_n$. Now taking $0 < t \leqq a$, we have for some fixed integer r

$$b/t \leqq 2^r[a/t]$$

and, therefore,

$$\lambda(b/t) \leqq K\lambda([a/t]) = K\theta(a/t) \, . \tag{3.6}$$

The lemma readily follows from (3.5) and (3.6).

LEMMA 7. *If* $\sum_{n=1}^{\infty}(|\varepsilon_n|/n) < \infty$ *and* $\int_0^{\pi}|d\varphi(t)| \leqq K$, *then*

$$\sum_{n=1}^{\infty} |\varepsilon_n A_{n+1}(x)| < \infty \, .$$

Proof. Since $\int_0^{\pi}|d\varphi(t)| \leqq K$, we have by integration by parts,

$$|A_n(x)| = \left|\frac{2}{\pi}\int_0^{\pi}\frac{\sin nt}{n}d\varphi(t)\right| \leqq \frac{1}{n}\int_0^{\pi}|d\varphi(t)| \leqq \frac{K}{n} \, .$$

The desired result now follows directly.

4. Proof of Theorem 1 and Theorem 2. If t_n denotes the nth (N, p_n) mean of $\sum_{n=1}^{\infty} A_n(x)\mu_n\lambda_n$, then

$$t_n - t_{n-1} = \frac{2}{\pi}\int_0^{\pi}\varphi(t)g(n, t)dt$$

where

$$g(n, t) = \sum_{k=1}^{n} P(n, k)\mu_k\lambda_k \cos kt \, ,$$

$\mu_k = 1$ or R_k and $P(n, k)$ is defined by Lemma 3.

Integrating by parts, we get

$$\int_0^{\pi}\varphi(t)g(n, t)dt = -\int_0^{\pi}\left(\int_0^t g(n, u)du\right)d\varphi(t)$$

and, therefore,

$$\sum_{n=1}^{\infty} \left| t_n - t_{n-1} \right| \leqq \sum_{n=1}^{\infty} \int_0^{\pi} \left| \int_0^t g(n, u) du \right| \left| d\varphi(t) \right| .$$

Thus, using the hypothesis (2.4′), we observe that in order to prove the $|N, p_n|$ summability of $\sum_{n=1}^{\infty} \mu_n \lambda_n A_n(x)$, it is sufficient to show that uniformly in $0 < t \leqq \pi$,

$$\Sigma = \sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} P(n, k) \lambda_k \mu_k k^{-1} \sin kt \right| \leqq K\lambda(C/t) . \tag{4.1}$$

We write

$$\begin{aligned} \Sigma &= \sum_{n=1}^{2\tau+1} \left| \sum_{k=1}^{n} P(n, k) k^{-1} \lambda_k \mu_k \sin k\mathrm{t} \right| \\ &\quad + \sum_{n=2\tau+2}^{\infty} \left| \left(\sum_{k=1}^{\tau} + \sum_{k=\tau+1}^{n} \right) P(n, k) k^{-1} \lambda_k \mu_k \sin kt \right| \\ &\leqq \Sigma_1 + \Sigma_2 + \Sigma_3 , \end{aligned} \tag{4.2}$$

say. Now, we observe that

$$\lambda(2\tau + 1) \leqq K\lambda(C/t) . \tag{4.3}$$

For, if $\tau \geqq 1$, then $2\tau + 1 < 2^2\tau$ and (4.3) follows from the result (ii) of Lemma 4 and the hypothesis that $\lambda(t)$ is a positive nondecreasing function. The latter also implies (4.3) directly when $\tau = 0$.

Since $|\sin kt| \leqq kt$, we have by a change in order of summations and Lemma 3

$$\begin{aligned} \Sigma_1 &\leqq t \sum_{k=1}^{2\tau+1} \mu_k \lambda_k \sum_{n=k}^{2\tau+1} P(n, k) \\ &\leqq t \sum_{k=1}^{2\tau+1} \mu_k \lambda_k \leqq K\lambda(C/t) , \end{aligned} \tag{4.4}$$

by virtue of (4.3). Again writing $|\sin kt| \leqq kt$ and applying the result of Lemma 3, we get

$$\begin{aligned} \Sigma_2 &\leqq t \sum_{n=2\tau+2}^{\infty} \sum_{k=1}^{\tau} P(n, k) \mu_k \lambda_k \\ &= t \sum_{k=1}^{\tau} \mu_k \lambda_k \sum_{n=2\tau+2}^{\infty} P(n, k) \leqq K\lambda(C/t) . \end{aligned} \tag{4.5}$$

In order to estimate Σ_3, we consider the following sum and write for a sufficiently large N,

$$\begin{aligned} \Sigma^* &= \sum_{n=2\tau+2}^{N} \left| \sum_{k=\tau+1}^{n} P(n, k) k^{-1} \lambda_k \mu_k \exp(ikt) \right| \\ &= \sum_{n=2\tau+2}^{N} \left| \left(\sum_{k=\tau+1}^{m} + \sum_{k=m+1}^{n} \right) P(n, k) k^{-1} \lambda_k \mu_k \exp(ikt) \right| \\ &\leqq \Sigma_1^* + \Sigma_2^* , \end{aligned} \tag{4.6}$$

say. Applying first, Lemma 1 with $x = \exp(it)$ and then effecting suitable changes in order of summations, we get

$$\begin{aligned}
\Sigma_1^* &\leqq Kt^{-1} \sum_{n=2\tau+2}^{N} \sum_{k=\tau+1}^{m-1} \left| \{\Delta_k(P(n, k))\mu_k k^{-1}\lambda_k + P(n, k+1)\Delta(k^{-1}\mu_k\lambda_k)\} \right| \\
&\quad + Kt^{-1} \sum_{n=2\tau+2}^{N} P(n, m)\mu_m\lambda_m m^{-1} + K \sum_{n=2\tau+2}^{N} P(n, \tau+1)\mu_{\tau+1}\lambda_{\tau+1} \\
&\leqq Kt^{-1} \sum_{k=\tau+1}^{N} k^{-1}\mu_k\lambda_k \sum_{n=2k+1}^{N} \left| \Delta_k(P(n, k)) \right| \\
&\quad + Kt^{-1} \sum_{k=\tau+1}^{N} \left| \Delta(k^{-1}\mu_k\lambda_k) \right| \sum_{n=2k+1}^{N} P(n, k+1) \\
&\quad + Kt^{-1} \sum_{n=2\tau+2}^{N} m^{-1}\mu_m\lambda_m \left(\frac{p_{n-m}}{P_n} - \frac{p_n P_{n-m-1}}{P_n P_{n-1}} \right) \\
&\quad + K\lambda_{\tau+1} \sum_{n=2\tau+2}^{N} P(n, \tau+1) \\
&= \Sigma_{11}^* + \Sigma_{12}^* + \Sigma_{13}^* + \Sigma_{14}^* ,
\end{aligned} \tag{4.7}$$

say. Since, due to nonnegative nonincreasing nature of $\{p_n\}$ — $\Delta_k P(n, k) \geqq 0$, for relevant values of k, we have

$$\begin{aligned}
\Sigma_{11}^* &= Kt^{-1} \sum_{k=\tau+1}^{N} k^{-1}\mu_k\lambda_k \sum_{n=2k+1}^{N} \left(\frac{p_{n-k-1}}{P_{n-1}} - \frac{p_{n-k}}{P_n} \right) \\
&\leqq Kt^{-1} \sum_{k=\tau+1}^{N} k^{-1}\mu_k\lambda_k \frac{p_k}{P_{2k}} \\
&\leqq Kt^{-1} p_{\tau+1} J_{\tau+1}(\mu) \leqq K\lambda(C/t) ,
\end{aligned} \tag{4.8}$$

by virtue of (4.3), the hypothesis (2.2) or (2.5) and the fact that $\{R_n\} \in B$.

First taking $\mu_n = 1$, we have by Lemma 3 and Lemma 5 with (4.3)

$$\Sigma_{12}^* \leqq Kt^{-1} \sum_{k=\tau+1}^{N} \left| \Delta(\lambda_k/k) \right| \leqq K\lambda(C/t) . \tag{4.9}$$

Next, when $\mu_n = R_n$, that is, in the case of Theorem 2, we have

$$\begin{aligned}
\Sigma_{12}^* &\leqq Kt^{-1} \sum_{k=\tau+1}^{N} \left| \Delta\left(\frac{k+1}{k} \frac{\lambda_k p_k}{P_k} \right) \right| \\
&\leqq KR_{\tau+1}\lambda(C/t) \leqq K\lambda(C/t) ,
\end{aligned} \tag{4.9$'$}$$

by virtue of the hypothesis that $\{\lambda_n p_n/P_n\}$ is nonincreasing and the result (4.3). Now,

$$\begin{aligned}
\Sigma_{13}^* &\leqq Kt^{-1} \sum_{n=2\tau+2}^{N} m^{-1}\mu_m\lambda_m p_m \frac{1}{P_m} \\
&\leqq Kt^{-1} p_{\tau+1} J_{\tau+1}(\mu) \leqq K\lambda(C/t) ,
\end{aligned} \tag{4.10}$$

by virtue of (4.3), the hypothesis (2.2) or (2.5) and the fact that $\{R_n\} \in B$. Further applying Lemma 3 and using (4.3), we directly get

$$\Sigma_{14}^* \leqq K\lambda(C/t) . \tag{4.11}$$

Writing

$$P(n, k) = \frac{p_{n-k}}{P_n} - \frac{p_n P_{n-k-1}}{P_n P_{n-1}} ,$$

we have

$$\Sigma_2^* \leqq \sum_{n=2\tau+2}^{N} \frac{1}{P_n} \left| \sum_{k=m+1}^{n} p_{n-k}\lambda_k k^{-1}\mu_k \exp(ikt) \right| + \sum_{n=2\tau+2}^{N} \frac{p_n}{P_n P_{n-1}} \left| \sum_{k=m+1}^{n} P_{n-k-1}\lambda_k k^{-1}\mu_k \exp(ikt) \right| .$$

Observing that $\{k^{-1}\mu_k\}$ is nonincreasing and applying Abel's lemma, we obtain by Lemma 2

$$\begin{aligned} \Sigma_2^* &\leqq K \sum_{n=2\tau+2}^{N} \frac{1}{P_n} \lambda_n m^{-1}\mu_m \max_{m<\nu\leqq n} \left| \sum_{k=m+1}^{\nu} p_{n-k} \exp(ikt) \right| \\ &\quad + K \sum_{n=2\tau+2}^{N} \frac{p_n}{P_n P_{n-1}} P_m \lambda_n m^{-1}\mu_m \max_{m<\nu\leqq n} \left| \sum_{k=m+1}^{\nu} \exp(ikt) \right| \\ &\leqq K(P_\tau + (\tau+1)p_{2\tau+2})J_{\tau+1}(\mu) \leqq K\lambda(C/t) , \end{aligned} \tag{4.12}$$

by virtue of Lemma 4 and the hypothesis (2.2) or (2.5) with (4.3).

Combining (4.6)-(4.12), we prove that $\Sigma^* \leqq K\lambda(C/t)$ which in its turn implies that $\Sigma_3 \leqq K\lambda(C/t)$. The last result combined with (4.2), (4.4) and (4.5) shows that (4.1) is valid and we have, thus proved the $|N, p_n|$ summability of $\sum_{n=1}^{\infty} \mu_n\lambda_n A_n(x)$. Observing that the above proof remains unaffected, if $A_n(x)$ is replaced by $A_{n+1}(x)$, we conclude the $|N, p_n|$ summability of $\sum_{n=1}^{\infty} \mu_n\lambda_n A_{n+1}(x)$.

This completes the proof of our theorems.

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