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LINEAR DIFFERENTIAL SYSTEMS WITH MEASURABLE COEFFICIENTS

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The general homogeneous first order linear differential system is considered. The principal result concerns a representation of the solution space as a direct sum of subspaces such that on each summand upper and lower bounds for the norms of the solutions can be given. The main tool in obtaining this decomposition is the method of fixed points of integral operators.

I. Introduction. Consider the homogeneous linear differential system

$$x'(t) = A(t)x(t) \qquad -\infty < t < \infty \tag{1}$$

where $A(t)$ denotes an nxn complex matrix whose entries are assumed only to be measurable functions of t which are summable on bounded intervals and it is understood that (1) holds almost everywhere. Here x denotes a complex n-vector and for $x = \text{col}(x_1, \cdots, x_n)$ we use $\|x\| = \max_{1 \leq i \leq n} |x|_i$ throughout.

In [4] the second author has shown that when $A(t)$ in (1) is continuous and satisfies a diagonal dominance condition the solution space of (1) admits a type of exponential dichotomy. This result is also discussed in the notes [2, pg. 126-135]. In [1] the first author has established an analogous result for the linear difference equation

$$x(m+1) = A(m+1)x(m) \qquad m = 0, \pm 1, \cdots \tag{2}$$

In §2 we give a more general and improved result for (1), assuming only measurability for $A(t)$, and then use this information to give estimates for upper and lower bounds for solutions to (1). Our estimates are comparative in that they give norm comparisons for solutions at any two values of the variable t. These estimates were obtained by Martin [5] in the continuous case and in a slightly weaker form were announced by the second author in [3]. However, the methods used here are completely different from those used in [4] and [5] and seem more transparent.

In §4 we show, under the additional assumption that $A(t)$ be bounded, that our technique of proof is constructive in that all solutions of (1) bounded on $[0, \infty)$ arise as fixed points of a family of contraction mappings. Finally, we indicate the appropriate analogy with our work concerning (1) for showing that the bounded solutions of (2) arise in a similar manner.

2. Statement of main theorems. For $A(t) = (a_{ij}(t))$ we define

$$r_i(t) = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}(t)|, \; 1 \leqq i \leqq n .$$

Our first main theorem is a strengthened form of the second author's original theorem.

THEOREM 1. *Let the entries of $A(t)$ in* (1) *be measurable and let α and β be measurable functions so that $\alpha(t) > \beta(t)$ almost everywhere. Let $\{1, \cdots, n\} = I_1 \cup I_2$ such that both the following hold on the whole real line.*

(i) $\operatorname{Re} a_{ii}(t) + r_i(t) \leqq \beta(t),\ i \in I_1$

(ii) $\operatorname{Re} a_{ii}(t) - r_i(t) \geqq \alpha(t),\ i \in I_2$.

Let k denote the cardinality of I_1 and L the solution space of (1). *Then there exist subspaces $L^-(\beta)$ and $L^-(\alpha)$ of L such that each of the following holds*:

(i) *$x \in L^-(\beta)$ iff for any $t_1 \leqq t_2$,*

$$||x(t_2)|| \leqq ||x(t_1)|| \exp \int_{t_1}^{t_2} \beta(t)dt$$

(ii) *$x \in L^+(\alpha)$ iff for any $t_1 \leqq t_2$,*

$$||x(t_2)|| \geqq ||x(t_1)|| \exp \int_{t_1}^{t_2} \alpha(t)dt$$

(iii) $L = L^-(\beta) \oplus L^+(\alpha)$

(iv) $\dim L^-(\beta) = k$.

Using Theorem 1 we shall establish the estimates for upper and lower bounds of solutions of (1) given by the following.

THEOREM 2. *Let $A(t)$ in* (1) *be measurable and let L denote the solution space of* (1). *For each $i = 1, \cdots, n$ let*

$$c_i(t) = \operatorname{Re} a_{ii}(t) - r_i(t) ,$$
$$d_i(t) = \operatorname{Re} a_{ii}(t) + r_i(t) .$$

Suppose $\{1, \cdots, n\} = \bigcup_{j=1}^{s} I_j$ where if

$$\alpha_j(t) = \min \{c_i(t) \mid i \in I_j\}$$
$$\beta_j(t) = \max \{d_i(t) \mid i \in I_j\}$$

then $\beta_j(t) < \alpha_{j+1}(t)$ holds almost everywhere, $1 \leqq j \leqq s - 1$. Finally let n_j denote the number of indicies i with $i \in I_j$, $j = 1, \cdots, s$. Then *there exist subspaces* L_j *of* L, $j = 1, \cdots, s$, *such that each of the following holds:*

(i) $x \in L_j$ *iff whenever* $t_1 \leqq t_2$,

$$||x(t_1)|| \exp \int_{t_1}^{t_2} \alpha_j(s)ds \leqq ||x(t_2)|| \leqq ||x(t_1)|| \exp \int_{t_1}^{t_2} \beta_j(s)ds$$

(ii) $L = L_1 \oplus L_2 \oplus \cdots \oplus L_s$

(iii) $\dim L_j = n_j$, $\qquad j = 1, \cdots, s$.

3. Proofs of the main theorems. Let K denote the complex field and let

$$S = \{\operatorname{col}(x_1, \cdots, x_n) \in K^n \,|\, x_j = 0 \text{ if } j \in I_2\}$$

where I_2 is as in the statement of Theorem 1. Let P be the projection in $\mathscr{L}(K^n)$ defined by $P(x) = \operatorname{col}(\varepsilon_1 x_1, \cdots, \varepsilon_n x_n)$ if $x = \operatorname{col}(x_1, \cdots, x_n)$ where $\varepsilon_j = \begin{cases} 1 \text{ if } j \in I_1 \\ 0 \text{ if } j \in I_2. \end{cases}$

The proof of Theorem 1 uses the following preliminaries.

PROPOSITION 1. *Let the entries of* $A(t)$ *be measurable and assume that both the following hold for all* $t \geqq 0$:

(3) (i) $$\operatorname{Re} a_{ii}(t) + r_i(t) \leqq -\delta(t),\ i \in I_1$$

(4) (ii) $$\operatorname{Re} a_{ii}(t) - r_i(t) \geqq \delta(t),\ i \in I_2$$

where $\delta(t)$ *is measurable,* $\delta(t) > 0$ a.e., *and* $\int_0^\infty \delta(s)ds = \infty$. *Then for each* $b \in S$ *there exists a unique solution* x *of* (1) *such that both the following hold:*

(i) $P(x(0)) = b$

(ii) $||x(t_2)|| \leqq ||x(t_1)|| \exp - \int_{t_1}^{s_1} \delta(s)ds$

whenever $0 \leqq t_1 \leqq t_2$.

Proof. We define two addends of $A(t)$ by

(5) $$D(t) = \operatorname{diag}(a_{11}(t), \cdots, a_{nn}(t)) \ , \quad \text{and}$$

(6) $$N(t) = A(t) - D(t)$$

and four additional matrix functions by

$$V_1(t) = \operatorname{diag}\left(\gamma_1 \exp \int_0^t a_{11}(s)ds, \cdots, \gamma_n \exp \int_0^t a_{nn}(s)ds\right) \tag{7}$$

where $\gamma_j = \begin{cases} 1 & \text{if } j \in I_1 \\ 0 & \text{if } j \in I_2, \end{cases}$

$$V_2(t) = \operatorname{diag}\left(\gamma_1 \exp \int_0^t a_{11}(s)ds, \cdots, \gamma_n \exp \int_0^t a_{nn}(s)ds\right) \tag{8}$$

where $\gamma_j = \begin{cases} 0 & \text{if } j \in I_1 \\ 1 & \text{if } j \in I_2, \end{cases}$

$$W_1(t) = \operatorname{diag}\left(\gamma_1 \exp - \int_0^t a_{11}(s)ds, \cdots, \gamma_n \exp - \int_0^t a_{nn}(s)ds\right) \tag{9}$$

where $\gamma_j = \begin{cases} 1 & \text{if } j \in I_1 \\ 0 & \text{if } j \in I_2, \end{cases}$ and

$$W_2(t) = \operatorname{diag}\left(\gamma_1 \exp - \int_0^t a_{11}(s)ds, \cdots, \gamma_n \exp - \int_0^t a_{nn}(s)ds\right) \tag{10}$$

where $\gamma_j = \begin{cases} 0 & \text{if } j \in I_1 \\ 1 & \text{if } j \in I_2. \end{cases}$

From (5)-(10) we observe that

$$\frac{dV_k}{dt} = D(t)V_k,\ k = 1, 2\ , \quad \text{and} \tag{11}$$

$$V_1(t)W_1(t) + V_2(t)W_2(t) = I = \text{Identity}\ . \tag{12}$$

For each fixed $b \in S$ we define the set M_b by

$$M_b = \Big\{x\colon [0, \infty) \longrightarrow K^n \,|\, x \text{ is continuous and } \\ \|x(t)\| \leqq \|b\| \exp - \int_0^t \delta(s)ds,\ t \geqq 0\Big\}$$

and the mapping T_b on M_b by $T_b(x) = x^*$ where

$$\begin{aligned} x^*(t) &= V_1(t)\left[b + \int_0^t W_1(s)N(s)x(s)ds\right] \\ &\quad - V_2(t)\int_t^\infty W_2(s)N(s)x(s)ds\ . \end{aligned} \tag{13}$$

For $i \in I_1$ we have from (7), (9), and (13) that

$$x_i^*(t) = b_i \exp \int_0^t a_{ii}(s)ds + \int_0^t \left(\exp \int_s^t a_{ii}(\sigma)d\sigma\right) \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}(s)x_j(s)ds\ .$$

Since $x \in M_b$, (3) and the above give

$$\begin{aligned}
|x_i^*(t)| &\leqq \|b\| \exp \int_0^t - (\delta(s) + r_i(s))ds \\
&\quad + \int_0^t \Big[\Big(\exp - \int_s^t (\delta(\sigma) + r_i(\sigma))d\sigma\Big) \sum_{\substack{j=1 \\ j\neq i}}^n |a_{ij}(s)| \, \|x(s)\|\Big] ds \\
&\leqq \|b\| \exp - \int_0^t \delta(s)ds \Big[\exp - \int_0^t r_i(s)ds \\
&\quad + \int_0^t \Big(\exp - \int_s^t r_i(\sigma)d\sigma\Big) r_i(s)ds\Big] \\
&\leqq \|b\| \exp - \int_0^t \delta(s)ds \Big[\exp \int_0^t - r_i(s)ds + \exp \int_s^t - r_i(\sigma)d\sigma \Big|_{s=0}^{s=t} \\
&= \|b\| \exp - \int_0^t \delta(s)ds \, .
\end{aligned}$$

Similarly, $i \in I_2$ implies

$$x_i^*(t) = -\int_t^\infty \Big(\exp - \int_t^s a_{ii}(\sigma)d\sigma\Big) \sum_{\substack{j=1 \\ j\neq i}}^n a_{ij}(s)x_j(s)ds \, ,$$

so

$$\begin{aligned}
|x_i^*(t)| &\leqq \int_t^\infty \Big(\exp - \int_t^s (r_i(\sigma) + \delta(\sigma))d\sigma\Big) r_i(s) \|b\| \Big(\exp - \int_0^s \delta(\sigma)d\sigma\Big) ds \\
&\leqq \Big(\|b\| \exp - \int_0^t \delta(s)ds\Big) \int_t^\infty \Big(\exp - \int_t^s (r_i(\sigma) \\
&\quad + \delta(\sigma))d\sigma)(r_i(s) + \delta(s)\Big) ds = \|b\| \exp - \int_0^t \delta(s)ds \, .
\end{aligned}$$

Thus $T_b(M_b) \subseteqq M_b$, and the set $T_b(M_b)$ is uniformly bounded. From the equality

$$\begin{aligned}
x^*(t_2) - x^*(t_1) &= [V_1(t_2) - V_1(t_1)] \Big[b + \int_0^{t_1} W_1(s)N(s)x(s)ds\Big] \\
&\quad + V_1(t_2) \int_{t_1}^{t_2} W_1(s)N(s)x(s)ds \\
&\quad - [V_2(t_1) - V_1(t_2)] \int_{t_2}^\infty W_2(s)N(s)x(s)ds \\
&\quad + V_2(t_1) \int_{t_1}^{t_2} W_2(s)N(s)x(s)ds
\end{aligned}$$

it follows that the restriction of $T_b(M_b)$ to any compact interval is equicontinuous. Since T_b is clearly continuous we have by the Schauder-Tychonoff theorem the existence of at least one $x_0 \in M_b$ so that $T_b(x_0) = x_0^* = x_0$. Rather than prove directly that T_b has a *unique* fixed point in M_b we shall prove the slightly stronger assertion that *there is at most one solution* $x_0(t)$ *of* (1) *such that* $T_b x_0 = x_0$ *and* $\|x_0(t)\| \to 0$ *as* $t \to \infty$. (To see that this is stronger than uniqueness recall that since

$\int_0^\infty \delta(s)ds = +\infty$, $x \in M_b \Rightarrow \|x(t)\| \to 0$ as $t \to \infty$.)

We therefore assume that x and y are two distinct solutions of (1) such that $T_b x = x$, $T_b y = y$, and that $\|x(t)\| \to 0$ and $\|y(t)\| \to 0$ as $t \to \infty$.

Let

$$\rho(x, y) = \sup_{t \geqq 0} \|x(t) - y(t)\| > 0 .$$

Since $\|x(t) - y(t)\| \to 0$ as $t \to \infty$ there exists a t_1 such that $\rho(x, y) = \|x(t_1) - y(t_1)\|$. For $i \in I_1$ we have

$$\begin{aligned}
|x_i(t_1) - y_i(t_1)| &= |x_i^*(t_1) - y_i^*(t_1)| \\
&\leqq \int_0^{t_1}\Big(\exp - \int_s^{t_1}(r_i(\sigma) + \delta(\sigma)\Big)d\sigma \sum_{\substack{j=1 \\ j\neq i}}^n |a_{ij}(s)|\,|x_j(s) - y_j(s)|\,ds \\
&\leqq \rho(x, y)\int_0^{t_1}\Big(\exp - \int_s^{t_1}(r_i(\sigma) + \delta(\sigma))d\sigma\Big)r_i(s)ds \\
&\leqq \rho(x, y)\int_0^{t_1}\Big(\exp - \int_s^{t_1}(r_i(\sigma) + \delta(\sigma)\Big)(r_i(s) + \delta(s))ds \\
&= \rho(x, y)\Big[1 - \exp - \int_0^{t_1}(r_i(\sigma) + \delta(\sigma))d\sigma\Big] \\
&< \rho(x, y).
\end{aligned}$$

Similarly if $i \in I_2$ then

$$\begin{aligned}
|x_i(t_1) - y_i(t_1)| &= |x_i^*(t_1) - y_i^*(t_1)| \\
&= \int_{t_1}^{\infty}\Big(\exp - \int_{t_1}^{s}(r_i(\sigma) + \delta(\sigma))d\sigma\Big)\sum_{\substack{j=1 \\ j\neq i}}^n |a_{ij}(s)|\,|x_j(s) - y_j(s)|\,ds \\
&\leqq \rho(x, y)\int_{t_1}^{\infty}\Big(\exp - \int_{t_1}^{s}(r_i(\sigma) + \delta(\sigma))d\sigma\Big)r_i(s)ds \\
&< \rho(x, y)\int_{t_1}^{\infty}\Big(\exp - \int_{t_1}^{s}(r_i(\sigma) + \delta(\sigma))d\sigma\Big)(r_i(s) + \delta(s))ds \\
&= \rho(x, y).
\end{aligned}$$

We have therefore arrived at the contradiction

$$\rho(x, y) = \max |x_i(t_1) - y_i(t_1)| < \rho(x, y) .$$

Hence there can exist at most one solution of $T_b x = x$ with $\|x(t)\| \to 0$ as $t \to \infty$.

We next observe that for any $x \in M_b$

$$\begin{aligned}
\frac{d}{dt}(x^*(t)) &= D(t)V_1(t)\Big[b + \int_0^t W_1(s)N(s)x(s)ds\Big] \\
&\quad - D(t)V_2(t)\int_t^\infty W_2(s)N(s)x(s)ds
\end{aligned}$$

$$+ [V_1(t)W_1(t) + V_2(t)W_2(t)]N(t)x(t)$$
$$= D(t)x^*(t) + N(t)x(t)$$

so the fixed point x_0 satisfies

$$x_0'(t) = A(t)x_0(t) .$$

Now note, for

$$V(t) = \text{diag}\left(\exp\int_0^t a_{11}(s)ds, \cdots, \exp\int_0^t a_{nn}(s)ds\right),$$

that

$$V'(t) = D(t)V(t), \; V(0) = I$$

so by the variation of parameters formula any solution of

$$x'(t) = D(t)x(t) + f(t)$$

must satisfy

$$x(t) = V(t)x(0) + V(t)\int_0^t V^{-1}(s)f(s)ds .$$

Thus, for $i \in I_2$

$$x_i(t) = \exp\int_0^t a_{ii}(s)ds\left[x_i(0) + \int_0^t\left(\exp - \int_0^s a_{ii}(\sigma)d\sigma\right)f_i(s)ds\right] \tag{15}$$

so if $x(t)$ is to be a solution of (1) with $||x(t)|| \to 0$ as $t \to \infty$ then

$$\lim_{t\to\infty}\left|x_i(0) + \int_0^t\left(\exp - \int_0^s a_{ii}(\sigma)d\sigma\right)f_i(s)ds\right| = 0$$

or

$$x_i(0) = -\int_0^\infty\left(\exp - \int_0^s a_{ii}(\sigma)d\sigma)\right)f_i(s)ds . \tag{16}$$

(15) and (16) now give that

$$x_i(t) = -\left(\exp\int_0^t a_{ii}(s)ds\right)\int_t^\infty\left(\exp - \int_0^s a_{ii}(\sigma)d\sigma\right)f_i(s)ds . \tag{17}$$

Requiring that $P(x(0)) = b$ gives for $i \in I_1$, that

$$x_i(t) = \exp\int_0^t a_{ii}(s)ds\left[b_i + \int_0^t\left(\exp - \int_0^s a_{ii}(\sigma)d\sigma\right)f_i(s)ds\right]. \tag{18}$$

(17) and (18) now give that

$$x(t) = V_1(t)\left[b + \int_0^t W_1(s)f(s)ds\right] - V_2(t)\int_t^\infty W_2(s)f(s)ds .$$

Letting $f(s) = N(s)x(s)$ from our above calculations we conclude that any solution of (1) satisfying $P(x(0)) = b$ and $||x(t)|| \to 0$ as $t \to \infty$ satisfies $T_b x = x$. Thus to complete the proof of the proposition we need only show that the unique fixed point x_0 of T_b on M_b satisfies the indicated inequality.

Fix $t_1 \geqq 0$. For $t \geqq 0$ define $B(t) = A(t + t_1)$ and let $\beta(t) = \delta(t + t_1)$. Then for $B(t) = (b_{ij}(t))$

$$\operatorname{Re} b_{ii}(t) + \sum_{j \neq i} |b_{ij}(t)| \leqq -\beta(t),\ i \in I_1$$

and

$$\operatorname{Re} b_{ii}(t) - \sum_{j \neq 1} |b_{ij}(t)| \geqq \beta(t),\ i \in I_2 .$$

Thus $B(t)$ satisfies the same hypotheses with respect to $\beta(t)$ as does $A(t)$ with respect to $\delta(t)$. Let $c = P(x_0(t_1)) \in S$. By what we have already proved there exists a unique $x_1 \in M_c$ so that

$$x_1'(t) = B(t)x_1(t),\ P(x_1(0)) = c .$$

Since $x_1(t) \in M_c$ we have that

$$||x_1(t)|| \leqq ||P(x_0(t_1))|| \exp - \int_0^t \beta(s)ds,\ t \geqq 0 . \tag{19}$$

Also we have that

$$x_0'(t + t_1) = B(t)x_0(t + t_1),\ t \geqq 0 ,$$
$$P(x_0(t + t_1))\Big|_{t=0} = P(x_0(t_1)) = c , \quad \text{and}$$
$$||x_0(t + t_1)|| \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty .$$

Threfore, from our above uniqueness result applied to $B(t)$ we conclude that $x_0(t + t_1) = x_1(t)$.

Thus (19) gives that

$$||x_0(t + t_1)|| \leqq ||P(x_0(t_1))|| \exp - \int_0^t \beta(s)ds , \quad t \geqq 0$$

so

$$||x_0(t_2)|| \leqq ||P(x_0(t_1))|| \exp - \int_{t_1}^{t_2} \delta(s)ds , \quad 0 \leqq t_1 \leqq t_2$$

from which the inequality in the statement of the proposition follows.

Our second preliminary is a direct generalization of the second author's original theorem to the case where $A(t)$ is measurable and $\delta(t) > 0$ is no longer assumed to be constant on $(-\infty, \infty)$.

PROPOSITION 2. *Let $A(t)$ in (1) satisfy the hypotheses of Proposition 1 on the whole real line except for the requirement that $\int_0^\infty \delta(s)ds = \infty$. Then there exist vector spaces L^- and L^+ of solutions of (1) such that each of the following holds:*

(i) $x \in L^-$ *iff whenever* $-\infty < t_1 \leqq t_2 < \infty$

$$\|x(t_2)\| \leqq \|x(t_1)\| \exp - \int_{t_1}^{t_2} \delta(s)ds \tag{20}$$

(ii) $x \in L^+$ *iff whenever* $-\infty < t_1 \leqq t_2 < \infty$

$$\|x(t_2)\| \geqq \|x(t_1)\| \exp \int_{t_1}^{t_2} \delta(s)ds \tag{21}$$

(iii) $\dim (L^-) = k$

(iv) $L = L^- \oplus L^+$.

Proof. First let us assume as in Proposition 1 that $\int_0^\infty \delta(s)ds = \infty$. Fix $t_0 \in (-\infty, \infty)$ and let $B(t) = A(t + t_0)$, $\beta(t) = \delta(t + t_0)$, $t \in [0, \infty)$. Then $B(t)$ satisfies the same conditions with respect to $\beta(t)$ as does $A(t)$ with respect to $\delta(t)$. Let x be any solution of (1) which satisfies inequality (20) for $0 \leqq t_1 \leqq t_2$. By Proposition 1 there exists a unique solution y of $y'(t) = B(t)y(t)$ so that $P(y(0)) = P(x(t_0)) \in S$ and

$$\|y(t_2)\| \leqq \|y(t_1)\| \exp - \int_{t_1}^{t_2} \beta(s)ds , \quad 0 \leqq t_1 \leqq t_2 .$$

If $z(t) = x(t_0 + t)$ then $z'(t) = B(t)z(t)$, $P(z(0)) = P(x(t_0)) = P(y(0))$, and

$$\|z(t)\| \longrightarrow 0$$

Thus by the uniqueness of Proposition 1 $x(t_0 + t) = z(t) = y(t)$, $t \in [0, \infty)$, so

$$\|z(t_2)\| \leqq \|z(t_1)\| \exp - \int_{t_1}^{t_2} \beta(s)ds , \quad 0 \leqq t_1 \leqq t_2 .$$

holds. Hence any solution of (1) which satisfies (20) for $0 \leqq t_1 \leqq t_2$ does so for $-\infty < t_1 \leqq t_2$.

Now for each $b \in S$ let y_b denote the unique solution of $x'(t) = A(t)x(t)$, $t \in [0, \infty)$, whose existence is established by Proposition 1 and let x_b denote the solution of $x'(t) = A(t)x(t)$, $t \in (-, \infty)$ determined by the initial condition $x_b(0) = y_b(0)$. By our above observations x_b satisfies (20) on the whole real line. Let $L^- = \{x_b \mid b \in S\}$. By the uniqueness of Proposition 1, formula (13), and the fact that $\dim S = k$ it follows that L^- is a vector space of dimension k.

Now let $C(t) = -A(-t)$, $t \in (-\infty, \infty)$. Then there are $n - k$ integers $i \in I_2$ such that

$$\operatorname{Re} c_{ii}(t) + \sum_{j \neq i} |c_{ij}(t)| \leqq -\delta(-t) < 0$$

and k integers $i \in I_1$ so that

$$\operatorname{Re} c_{ii}(t) - \sum_{j \neq i} |c_{ij}(t)| \geqq \delta(-t) > 0 .$$

Hence, by our preceding argument there exists an $n-k$ dimensional vector space R^- of solutions of $y'(t) = C(t)y(t)$ such that

$$||y(t_2)|| \leqq ||y(t_1)|| \exp - \int_{t_1}^{t_2} \delta(-s)ds, \ -\infty < t \leqq t_2 < \infty .$$

Let $L^+ = \{x \mid x(t) = y(-t), y \in R^-\}$. Then $x \in L^+$ implies that $x'(t) = A(t)x(t)$ and that

$$||x(t_2|| \geqq ||x(t_1)|| \exp \int_{t_1}^{t_2} \delta(s)ds , \quad -\infty < t_1 \leqq t_2 < \infty$$

which establishes (21).

Since $\int_{-\infty}^{\infty} \delta(s)ds > 0$ it follows that $L^- \cap L^+ = \{0\}$. Since $\dim L^+ = \dim R^- = n - k$ we have that $L = L^- \oplus L^+$.

We now remove the restriction that $\int_0^\infty \delta(s)ds = \infty$. For each integer $m = 1, 2, \cdots$ define the matrix E_m by

$$E_m = \operatorname{diag}\left(\frac{\varepsilon_1}{m}, \frac{\varepsilon_2}{m}, \cdots, \frac{\varepsilon_n}{m}\right)$$

where

$$\varepsilon_j = \begin{cases} -1 & \text{if} \quad j \in I_1 \\ \ \ 1 & \text{if} \quad j \in I_2 \end{cases}$$

and let $A_m(t) = A(t) + E_m$. Then for $\delta_m(t) = \delta(t) + 1/m$ we have that $\int_0^\infty \delta_m(s)ds = \infty$ so our preceding argument applies to the system

$$x'(t) = A_m(t)x(t) .$$

Denoting the solution space of this system by L_m we have the corresponding decomposition $L_m = L_m^- \oplus L_m^+$.

For each integer m we define a vector space V_m by

$$V_m = \{x(0) \mid x \in L_m^-\} .$$

Let $\{c_{1m}, \cdots, c_{km}\}$ be a basis for V_m which is orthonormal with respect to the complex inner product on K^n. By the compactness of the unit ball in K^n there exists a sequence of integers $\{m_j\}$ and vectors $c_1, \cdots,$

c_k such that $\lim_{j\to\infty} c_{im_j} = c_i$, $1 \leqq i \leqq k$. These vectors are orthonormal and hence independent. Let V^- be the k-dimensional space spanned by $c_1, \cdots, c_k$ and let

$$L^- = \{x \in L \mid x(0) \in V^-\}$$

(where L denotes the solution space of (1) as before). Then for $x \in L^-$ there exist scalers $\alpha_1, \cdots, \alpha_k$ such that

$$x(0) = \sum_{i=1}^{k} \alpha_i c_i \ .$$

Let $x_j(t)$ denote the solution of

$$x'(t) = A_{m_j}(t)x(t)$$

such that

$$x_j(0) = \sum_{i=1}^{k} \alpha_i \, c_{im_j} \ .$$

Then $\lim_{j\to\infty} x_j(0) = x(0)$ and by what we have already proved x_j satisfies the inequality

$$||x_j(t_2)|| \leqq ||x_j(t_1)|| \exp - \int_1^2 \Big(\delta(s) + \frac{1}{m_j}\Big) ds,\ t_1 \leqq t_2 \ .$$

Thus, since $\lim_{j\to\infty} A_{m_j}(t) = A(t)$ uniformly on $(-\infty, \infty)$ it follows that $x_j(t) \longrightarrow x(t)$ uniformly on compact intervals as $j \longrightarrow \infty$, and that

$$||x(t_2)|| \leqq ||x(t_1)|| \exp - \int_{t_1}^{t_2} \delta(s) ds \ , \quad t_1 \leqq t_2 \ .$$

This establishes the existence of L^- in the statement of Proposition 1. The existence of L^+ follows in a similar fashion. The proof that $L^+ \cap L^- = \{0\}$ follows as before.

Proof of Theorem 1. For each $t \in (-\infty, \infty)$ let

$$\gamma(t) = (1/2)(\alpha(t) + \beta(t)) \quad \text{and} \quad \rho(t) = (1/2)(\alpha(t) - \beta(t)) > 0 \ . \tag{22}$$

Define the matrix $B(t)$ by

$$B(t) = A(t) - \gamma(t)I = (b_{ij}(t)) \ , \quad t \in (-\infty, \infty) \ .$$

Then

$$\operatorname{Re} b_{ij}(t) + \sum_{j\neq i} |b_{ij}(t)| \leqq -\rho(t) \ , \quad i \in I_1 \ ,$$

and

$$\operatorname{Re} b_{ij}(t) - \sum_{j \neq i} |b_{ij}(t)| \geqq \rho(t) , \quad i \in I_2$$

hold almost everywhere. Let M denote the solution space of $y'(t) = B(t)y(t)$, $t \in (-\infty, \infty)$. Then by Proposition 2 there exist subspaces M^- and M^+ of M so that each of the following holds:

(i) $y \in M^-$ iff whenever $-\infty < t_1 \leqq t_2 < \infty$

$$||y(t_2)|| \leqq ||y(t_1)|| \exp - \int_{t_1}^{t_2} \rho(t)dt$$

(ii) $y \in M^+$ iff whenever $-\infty < t_1 \leqq t_2 < \infty$

$$||y(t_2)|| \geqq ||y(t_1)|| \exp \int_{t_1}^{t_2} \rho(t)dt$$

(iii) $\dim M^- = k$

(iv) $M = M^- \oplus M^+$.

Now y is a solution of $y'(t) = B(t)y(t)$ if and only if $y(t) = x(t) \exp - \int_0^t \gamma(s)ds$ for some solution of (1). Therefore, if we set

$$L^-(\beta) = \left\{x \,|\, x(t) = y(t) \exp \int_0^t \gamma(s)ds,\ y \in M^-\right\}$$

and

$$L^+(\alpha) = \left\{x \,|\, x(t) = y(t) \exp \int_0^t \gamma(s)ds,\ y \in M^+\right\}$$

the conclusions of Theorem 1 follow from those above and (22).

Proof of Theorem 2. Without loss of generality we may assume that

$$\alpha_1(t) < \beta_1(t) < \alpha_2(t) < \cdots < \alpha_s(t) < \beta_s(t)$$

holds almost everywhere. Let j be any integer so that $1 \leqq j \leqq s$. Then by Theorem 1 there exist subspaces $L^-(\beta_j)$ and $L^+(\alpha_j)$ of L such that each of the following holds:

(i) $x \in L^-(\beta_j)$ implies whenever $-\infty < t_1 \leqq t_2 < \infty$ that

$$||x(t_2)|| \leqq ||x(t_1)|| \exp \int_{t_1}^{t_2} \beta_j(t)dt \tag{23}$$

(ii) $x \in L^+(\alpha_j)$ implies whenever $-\infty < t_1 \leqq t_2 < \infty$ that

$$||x(t_2)|| \geqq ||x(t_1)|| \exp \int_{t_1}^{t_2} \alpha_j(t)dt \tag{24}$$

(iii) $\dim L^-(\beta_j) = n_1 + n_2 + \cdots + n_j$

(iv) $\dim L^+(\alpha_j) = n - (n_1 + n_2 + \cdots + n_{j-1})$.

Let $L_j = L^-(\beta_j) \cap L^+(\alpha_j)$. Then $\dim J_j \geqq n_j$ and if $x \in L_j$ x satisfies the inequality in the statement of Theorem 1 by (23) and (24). Since $\dim L = n_1 + \cdots + n_s = n$ it follows that $\dim L_j = n_j$ and that

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_s .$$

4. Applications. As our first application of the preceding techniques of proof we point out that mapping T_b actually gives an iterative scheme for computing the bounded solutions of (1) on $[0, \infty)$ when $A(t)$ satisfies stronger conditions than those of Theorem 1.

THEOREM 3. *Let $A(t)$ be as in Theorem 1. In addition let $A(t)$ be bounded and assume the existence of a fixed number $\delta > 0$ so that $\delta(t) \geqq \delta > 0$ holds for all $t \in [0, \infty)$. Then to each $b \in S$ corresponds a unique bounded solution x_b of*

$$x'(t) = A(t)x(t), \; t \in [0, \infty) \tag{25}$$

which is given by

$$x_b = \lim_{n \to \infty} T_b^n(x)$$

where x is any element of M_b. Furthermore, all solutions of (25) *bounded on $[0, \infty)$ arise in this manner.*

Proof. Since $A(t)$ is bounded and $\delta(t) \geqq \delta > 0$ there exists a constant γ with $0 < \gamma < 1$ such that

$$r_i(t) \leqq \frac{\gamma \delta(t)}{1 - \gamma}$$

holds for all $t \in (-\infty, \infty)$ and all $1 \leqq i \leqq n$. Thus

$$r_i(t) \leqq \gamma(t) + \delta(t)), \; -\infty < t < \infty, 1 \leqq i \leqq n .$$

Referring to the uniqueness proof of Proposition 1 we see that if $T_b x = x^*$, $T_b y = y^*$ for $x, y \in M_b$ then for $i \in I_1$

$$\begin{aligned} |x_i^*(t) - y_i^*(t)| &\leqq \rho(x, y)\int_0^t \Big(\exp - \int_s^t (r_i(\sigma) + \delta(\sigma))d\sigma\Big) r_i(s)ds \\ &\leqq \gamma\rho(x, y) \int_0^t \Big(\exp - \int_s^t (r_i(\sigma) + \delta(\sigma))d\sigma\Big)(r_i(s) + \delta(s))ds \\ &\leqq \gamma\rho(x, y) . \end{aligned}$$

Similarly, for $i \in I_2$ we see that

$$|x_i^*(t) - y_i^*(t)| \leqq \gamma\rho(x, y)\int_t^\infty\left(\exp - \int_t^s (r_i(\sigma) + \delta(\sigma))d\sigma\right)(r_i(s) + \delta(s))ds$$
$$= \gamma\rho(x, y) .$$

Hence

$$\sup_{t\geqq 0} \| T_b x(t) - T_b y(t) \| \leqq \gamma \sup_{t\geqq 0} \|x(t) - y(t)\|$$

so under our present hypotheses T_b is a contraction mapping on M_b. Theorem 3 now follows from our preceding work and the contraction mapping principle.

As our second application we indicate the analogues of our preceding technique for the problem of determining the bounded solutions of the linear difference equation

$$x(m + 1) = A(m + 1)x(m) \qquad m = 0, 1, 2, \cdots \tag{26}$$

THEOREM 4. *Let* $\{1, \cdots, n\} = I_1 \cup I_2$, *let* k *denote the cardinality of* I_1, *and assume that both the following hold for some* $\delta \in (0, 1)$ *and all* $m = 0, 1, 2, \cdots$

(i) $|a_{ii}(m)| + r_i(m) \leqq 1 - \delta < 1,\ i \in I_1$

(ii) $|a_{ii}(m)| - r_i(m) \geqq 1 + \delta > 1,\ i \in I_2$.

Let S *and* M_b, $b \in S$, *be as before. Then to each* $b \in S$ *corresponds a unique bounded solution of* (26) *which is the fixed point of the contraction mapping* F_b *defined on* M_b *by* (27). *Furthermore, every bounded solution of* (26) *arises in this manner.*

Indication of proof. For each $m = 0, 1, 2, \cdots$ we define

$$D(m) = \operatorname{diag}(a_{11}(m), \cdots, a_{nn}(m))$$

and

$$N(m) = A(m) - D(m) .$$

Let

$$V_1(m) = \operatorname{diag}(f_1(m), \cdots, f_n(m))$$

where

$$f_j(m) = \begin{cases} \prod_{i=2}^{m} |a_{jj}(i)| & \text{if } \ j \in I_1 \\ 0 \ \text{ if } \ j \in I_2 \end{cases}$$

and define $V_2(m)$, $W_1(m)$, and $W_2(m)$ by analogy between the above and (8)-(10).

Let S be as before and for $b \in S$ let

$$M_b = \{x\colon Z^+ \longrightarrow K^n \mid ||x(m)|| \leqq ||b||(1-\delta)^m,\ m \in Z^+\}$$

where $Z^+ = \{0, 1, 2, \cdots\}$. Define F_b on M_b by $F_b x = y$ where

$$y(m) = V_1(m)\Big[b + \sum_{s=1}^{m} W_1(s)N(s)x(s)\Big] - V_2(m)\sum_{s=m+1}^{\infty} W_2(s)N(s)x(s)\ . \tag{27}$$

The proof then follows by direct analogy with our preceding work for (1).

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